Fundamental Particles — Fundamental Physics

Ronald Kleiss
IMAPP, Radboud University, Nijmegen

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1.1 Divisions of the universe

Doing physics is possible because we can divide phenomena into classes, for instance by distinguishing between ‘slow’ and ‘fast’ movements, ‘small’ and ‘large’ happenings, and ‘light’ and ‘heavy’ bodies. This is only possible if there exist ‘yardsticks’: these are the fundamental dimensionful natural constants. Note that these constants are not human constructs: they are *built into the fabric of the universe*. Their numerical values change, of course, under a change of the system of units, but in any system of units, they are *there*. Any alien civilization ought to arrive at (a combination of) the same fundamental constants. We present them below.

1.2 Fundamental quantities

Physics of the type comprising elementary-particle physics deals with the three fundamental concepts of mass, length, and time. These, and various combinations of them, are the objects that are measured in experiments, and predicted by theories. In the table below we present the most important ones.

Quantity	Symbol	Unit
Length	L	m
Time	T	s
Mass	M	kg
Velocity	v	m/s
Accelereation	a	m/s^2
Momentum	p	kg m/s
Energy	E	kg m^2/s^2
Angular Momentum/ Action	J	kg m^2/s
Force	F	kg m/s^2

1.3 The speed of light

The speed of light (*in vacuo*) is known exactly:

$$c = 2.99792458\ \frac{\mathrm{m}}{\mathrm{s}}\quad . \tag{1}$$

It is an *empirical fact* that this is a constant: indeed, the realization that this is so leads one directly to postulate special relativity. The fact that it is a *velocity* tells us that, in a sense, time intervals can be expressed as distances[1], and vice versa. Note that the value quoted is *exact*: in fact, since we can measure times very much more accurately than distances, the value of c is used as the current *definition* of the meter.

Although the constant is called 'the speed of light', it is actually not at all relevant that there is a material object (light) that moves around at that speed: the relevant fact is that the constant exists. As we shall see, *any* massless material object necessarily moves at that speed, and no material object or message can move faster. Even if no-one had ever seen the light, c would still be there. It might be more appropriate[2] to call it 'Einstein's constant'.

1.4 Planck's constant

This has the dimension of *action*, which is equivalent to energy times time, or momentum times distance: it is also the dimension of angular momentum. Its numerical value is given by[3]

$$\hbar = 1.05457266(\pm 63)\ 10^{-34}\ \frac{\text{kg m}^2}{\text{s}}\quad . \tag{2}$$

Quite often, an *interaction* taking place between subatomic systems involves the transfer of an amount of action of order $\hbar$ from one part of a system to another part: for instance, in an atomic transition the atom's angular momentum usually changes by precisely $\hbar$: this amount is taken away by the photon emitted by the atom.

Analogously to the situation in which the constantcy of c leads to special relativity, the realization that $\hbar$ is (apparently) a universal constant will lead one to postulate quantum mechanics.

[1]The notion of 'lightyear' embodies this beautifully; note that in the first Star Wars movie (bar scene in the town of Mos Eisely, on the planet Tatooine) the word is deplorably misused as a measure of *time*.

[2]But embarassingly pedantic.

[3]Actually, Planck's constant is larger by precisely a factor 2π, and denoted by h: the notation $\hbar$ is, strictly speaking, known as *Dirac's constant*, but since it is this combination that is almost always used, one usually calls $\hbar$ Planck's constant. In the numerical value given, the number in brackets denotes the experimental uncertainty in the last digits.

1.5 Newton's constant

Newton's law describing the gravitational attractive force between two pointlike bodies of mass M_1 and M_2 separated by a distance r reads

$$|\vec{F}| = G_N \, \frac{M_1 M_2}{r^2} \quad . \tag{3}$$

By comparing the units on either side of the equation one immediately sees that some dimensionful constant G_N *must* be involved, and so it is: we have

$$G_N = 6.67259(\pm 85)\ 10^{-11}\ \frac{\mathrm{m}^3}{\mathrm{kg\ s}^2} \quad . \tag{4}$$

Newton's constant is not known as precisely as either c or $\hbar$, since gravitation is such a weak force[4] that is is difficult to measure. Also, its universality is not equally well established[5]. Nevertheless, we assume it to be also universal, and this opens the way to the the general theory of relativity.

1.6 Dimensionless and nonfundamental constants

In addition to the dimensionful constants $c, \hbar$ and G_N, there are several *dimensionless numbers* that are also important. In the first place, Coulomb's law describes the electrostatic force between two stationary charges Q_1 and Q_2 separated by a distance r. In Gaussian units, its reads very much like Newton's law:

$$|\vec{F}| = \frac{1}{4\pi}\ \frac{Q_1 Q_2}{r^2} \quad . \tag{5}$$

No dimensionful constant is inserted here, since this law serves as *the definition of charge*[6]. It implies that $Q_1 Q_2$ has unit kg $\mathrm{m}^3/\mathrm{s}^2$. Note that this is also precisly the unit of the combination $\hbar c$. Taking the positive unit charge (the proton charge)

[4]Weak, that is, compared to the other known fundamental forces: that it *appears* to dominate everyday life is due to the fact that gravitational effects are always accumulative since graviational forces are always attractive and long-range, whereas the other forces have either a short range (such as the weak and strong force), or can be both attractive and repulsive (the electromagentic forces) so that their effects tend to cancel out.

[5]Notice that we cannot measure the mass of distant objects directly. Instead, their masses are usually *inferred* from assuming Newton's constant to be, indeed, constant.

[6]Quite simply: take two protons, hold them apart by one meter, and measure the force with wchich they repel each other. In practice it is, of course, more involved.

e, we can therefore construct a universal dimensionless number, known as the *electromagnetic fine structure constant*. Its measured value is

$$\alpha \equiv \frac{e^2}{4\pi\,\hbar c} = \frac{1}{137.0359895(\pm 61)} \quad , \tag{6}$$

or, equivalently,

$$e = 0.3028221315(67) \quad . \tag{7}$$

The number α describes the strength of the electromagnetic interactions in a universal way[7] , since no change of our system of units is going to change the value of α. The *reason* why α has this value is not really known[8]. A good explanation for the size of this number would be extremely interesting!

The unit charge has also been measured in the more standard unit C (Coulomb). We have

$$e = 1.60217733(\pm 49)\ 10^{-19}\ \mathrm{C} \quad . \tag{8}$$

This ratio is not fundamental, since it is actually just the definition of what constitutes one Coulomb of charge. More directly relevant to elementary particle physics, it gives us the ratio between the particles' typical energy scale, that of *electronvolts*, to the more macroscopic Joule:

$$1\ \mathrm{eV} = 1.60217733(\pm 49)\ 10^{-19}\ \mathrm{J} \quad , \tag{9}$$

which is the typical energy scale for atomic transitions and processes[9] In the practice of elementary-particle physics, the *Giga-electronvolt* GeV is more useful still[10], and we have of course

$$1\ \mathrm{GeV} = 1.60217733(\pm 49)\ 10^{-10}\ \mathrm{J} \quad . \tag{10}$$

[7]One should realize one important fact here: saying that α is universal amounts to stating that *all protons have* exactly *the same electric charge* — a fundamental empirical fact about the elementary particles!

[8]Or, rather, there are very many speculations about it: so far, none of them lead to testable predictions for other observable facts, and may therefore not be totally relevant at this moment. One of the more embarrasing examples is the 'derivation' of the British astronomer Eddington of the value $1/\alpha \approx 137$. This involves the following argument: the metric (see later) has $4 \times 4 = 16$ components, out of which $3 \times 3 + 1 \times 1 = 10$ do *not* mix space and time, and 6 do. Repeating this, the 'second power' of the metric contains $10 \times 10 + 6 \times 6 = 136$ components of 'unmixed' type (and $10 \times 6 + 6 \times 10 = 120$ of 'mixed type'). This gives us the 'estimate' $1/\alpha = 136$, and "...Eddington later offered reasons for adding 1. These carried little conviction".

[9]This is not surprising, since the Volt has originally been introduced as a useful quantity for 'electrochemistry', where the typical processes involved are precisely these atomic transitions.

[10]Because of the value of the proton mass — see later.

Another link between the microscopic and the macroscopic world is provided by the insight that heat and temperature are due to the motion of an object's constituent parts. Therefore, there is a relation between the kinetic energy of an object's constituents and the temperature, known as *Boltzmann's constant*:

$$k = 1.380658(\pm 12)\ 10^{-23}\ \frac{\text{J}}{\text{K}} = 8.617385(\pm 73)\ 10^{-5}\ \frac{\text{eV}}{\text{K}}\ \ . \tag{11}$$

Also, this is not fundamental since it serves as the definition of one degree Kelvin. Since temperature is a notion that is only well-defined for systems of infinitely many particles, it is not very important in many areas of elementary-particle physics where interactions between only a few particles are discussed[11].

A final non-fundamental but useful number is *Avogadro's number*, that is, the number of molecules of a substance in one mol[12]. of that substance:

$$N_A = 6.0221367(\pm 36)\ 10^{23} \tag{12}$$

This may be seen as the definition of the kilogram in units of the nucleon mass. Since it is not known extremely precisely, it is not used for that purpose, but it is not a fundamental number.

1.7 Excercises

Excercise 1 Counting constants
This is a discussion item. In 'fundamental physics' we have the three most important quantities of length, mass and time. We also have three fundamental constants. Is this **(a)** too few, **(b)** too many, **(c)** just right? Try to imagine what situation would develop if another fundamental constant were found.

Excercise 2 The proton mass
Recall the definition of 'one mol' of a substance. What is the mass of one mol of hydrogen gas? From this, compute the mass of a proton. Assume that the electron is very light compared to a proton (which is true).

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[11]Noteable exceptions are collisions between very heavy atomic nuclei, and the very very early universe.

[12]By definition, one mol of a certain molecule is just so many of those molecules that their combined weight in grams is equal to the atomic weight of the molecule. One mol of pure helium gas therefore weighs 4 grams.

Excercise 3 The neutron mass
Give an argument to show that the mass of a neutron must be quite close to that of a proton.

Excercise 4 Counting nucleons in the earth
Assume the earth is made mostly of iron (which is true), with a specific weight of about 5 (or look it up in BINAS). Given that the circumference of the earth is 40,000 km, estimate the number of nucleons (protons and neutrons) in the earth. Use the fact that iron has atomic number 26, and atomic weight about 56.

Excercise 5 Counting electrons in the earth
Estimate the number of electrons in the earth.

Excercise 6 Counting particles in yourself
You are not made up from iron but mostly of light elements (carbon, oxygen, hydrogen). Repeat excercises 4 and 5 for yourself rather than for the earth.

Excercise 7 Repulsion
The totality of electrons in the earth excercises a repulsive force on the totality of electrons in your body. Compute this force, and compare to your actual weight (you may assume the electrons of the earth to be concentrated in its center). Why are you not shot into outer space?

Excercise 8 Fundamental sizes
From the three fundamental constants c, $\hbar$ and G_N we can construct a quantity with the dimension of length. Assuming no arbitrary dimensionless numbers are added into the mixture, compute this length, which is called the *Planck length*. Do the same in order to arrive at a quantity with the dimension of time, wich is called the *Planck time*. And, of course, there is a *Planck mass*: compute this as well. Compare the Planck mass to the proton mass

2 Elements of Relativity

The constancy of c has implications for the structure of the space-time arena of elementary-particle physics, which we now briefly review.

2.1 Minkowski space and its metric

If we neglect the effects of gravitation (which are extremely weak in any case), space can be described by three Cartesian coordinates $\vec{x} = (x^1, x^2, x^3)$, expressed in meters, and one time cordinate t, expressed in seconds. Using c, we can transform t into a coordinate with the dimension of a distance: $x^0 \equiv ct$. We may then combine these four coordinates into one four-dimensional *spacetime*, called Minkowski space. In this space a point (called an *event*) is indicated by

$$x^\mu = (x^0, x^1, x^2, x^3) \quad .$$

Let two events A and B have coordinatfes x^μ_A and x^μ_B. Between A and B we then have a four-vector

$$x^\mu = x^\mu_A - x^\mu_B \quad , \quad \mu = 0, 1, 2, 3 \quad . \tag{13}$$

The fundamental postulate of special relativity is that the *real* distance between A and B, that is, *the distance that matters in the description of physics*, is given by

$$x \cdot x = (x^0)^2 - |\vec{x}|^2 \tag{14}$$

We can write this also as

$$x \cdot x = \sum_{\mu,\nu=0}^{3} g_{\mu\nu}\, x^\mu\, x^\nu \equiv g_{\mu\nu}\, x^\mu\, x^\nu \quad , \tag{15}$$

where we have introduced the Einstein convention, in which repeated indices (one upper, one lower) are to be summed over; and

$$g_{\mu\nu} = \begin{cases} +1 & \text{if } \mu = \nu = 0 \\ \text{-1} & \text{if } \mu = \nu \in \{1, 2, 3\} \\ 0 & \text{otherwise} \end{cases} \quad . \tag{16}$$

This object is called the *metric*: it defines Minkowski space. We can also define a second metric, with upper indices, as its inverse:

$$g_{\mu\alpha}\, g^{\alpha\nu} = \sum_{\alpha=0}^{3} g_{\mu\alpha}\, g^{\alpha\nu} = \delta_\mu{}^\nu \quad , \tag{17}$$

where the Kronecker symbol is given by

$$\delta_\mu{}^\nu = \begin{cases} 1 & \text{if } \mu = \nu \\ 0 & \text{otherwise} \end{cases} \quad . \tag{18}$$

Numerically, $g^{\mu\nu}$ equals $g_{\mu\nu}$, but this is an accidental property of Minkowski space and does not hold for more general metrics[13]. Using these two metrics we can associate to any quantity with upper indices a counterpart with lower indices, and *vice versa*: for instance, for any K^μ we have

$$K_\mu = g_{\mu\nu}\, K^\nu \quad \Rightarrow \quad K_0 = K^0 \ \ , \ \ K_n = -K^n \ \ (n = 1, 2, 3) \ \ ; \tag{19}$$

and similarly we have $K^\mu = g^{\mu\nu} K_\nu$.

For any two four-vectors p^μ and q^μ, their inner product can now be written in a number of equivalent forms:

$$p \cdot q = p^\mu q_\mu = p_\mu q^\mu = p^0 q^0 - \vec{p} \cdot \vec{q} = g_{\mu\nu} p^\mu q^\nu = g^{\mu\nu} p_\mu q_\nu \ \ . \tag{20}$$

Note that $p \cdot p$ *can* be negative: this happens if $|p^0| < |\vec{p}|$. The real distance between two events can therefore be *imaginary*.

2.2 Symmetries of Minkowski space

Given a notion of the real distance between two events, it stands to reason that a good description of physics ought to be independent of coordinate transformations that leave the real distance between any two events invariant: these are the *symmetries* of Minkowski space, which we shall now investigate.

2.2.1 Coinciding origins, continuous transformations

First, we shall consider only coordinate systems whose origins (both in space and time) coincide. That is, if a four-vector has components x^μ in the one system, and components x'^μ in the other system, then $x^\mu = 0 \iff x'^\mu = 0$. If the inner product is to be conserved for all four-vectors, then the transformations must be linear, and can therefore be written as matrices[14]:

$$x'^\mu = \Lambda^\mu{}_\nu\, x^\nu \ \ . \tag{21}$$

[13]These are encountered in the general theory of relativity, not discussed in these notes.

[14]It is convenient to picture x^μ as a *column vector*, in which case the *upper index* enumerates the *rows*. Then, x_μ is a *row vector*, in which the *lower index* enumerates the *columns*. So, $\Lambda^\mu{}_\nu$ is a matrix, with rows enumerated by the upper index, and columns enumerated by the lower index.

If the real distance is to be preserved, we must have

$$x' \cdot x' = x \cdot x \quad \forall\, x^\mu \quad , \tag{22}$$

and hence

$$\Lambda^\mu{}_\alpha\, \Lambda^\nu{}_\beta\; g_{\mu\nu} = g_{\alpha\beta} \quad . \tag{23}$$

First, assume that the coordinate transformation is infinitesimal, so that Λ differs but very little from the identity transformation: that is, we write

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \epsilon\; T^\mu{}_\nu \quad , \tag{24}$$

with ϵ very small In this approximation, we find for Eq.(23):

$$\begin{aligned} &\Lambda^\mu{}_\alpha\, \Lambda^\nu{}_\beta\; g_{\mu\nu} = \\ &= \left(\delta^\mu{}_\alpha + \epsilon\; T^\mu{}_\alpha\right)\left(\delta^\nu{}_\beta + \epsilon\; T^\nu{}_\beta\right) g_{\mu\nu} \\ &= g_{\alpha\beta} + \epsilon\; T_{\alpha\beta} + \epsilon\; T_{\beta\alpha} + \epsilon^2\; T^\mu{}_\alpha T_{\mu\beta} \quad , \end{aligned} \tag{25}$$

so that if ϵ is small enough that we may neglect ϵ^2, the condition for the T is

$$T_{\alpha\beta} = -T_{\beta\alpha} \quad . \tag{26}$$

We can readily find 6 linearly independent matrices satisfying this:

$$\begin{aligned} T^{(01)\mu}{}_\nu &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad T^{(02)\mu}{}_\nu = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \\ T^{(03)\mu}{}_\nu &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad , \quad T^{(12)\mu}{}_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \\ T^{(13)\mu}{}_\nu &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad , \quad T^{(23)\mu}{}_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad . \end{aligned} \tag{27}$$

These matrices are called the *generators of the Lorentz group*. Non-infinitesimal transformations can be obtained by 'exponentiating' these generators: For instance, we have

$$\left[\exp\left(\omega T^{(12)}\right)\right]^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & \sin\omega & 0 \\ 0 & -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad , \tag{28}$$

which is a rotation over an angle ω in the 1,2 plane; and

$$\left[\exp\left(\eta T^{(01)}\right)\right]^{\mu}{}_{\nu} = \begin{pmatrix} \cosh\eta & \sinh\eta & 0 & 0 \\ \sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad , \tag{29}$$

which is a Lorentz boost along the x-direction, with a relative speed v between the two coordinate systems determined by

$$\eta = \frac{1}{2}\log\left(\frac{1+v/c}{1-v/c}\right) \quad . \tag{30}$$

The number η is called the *rapidity*[15]. It plays the rôle of an angle in the sense that two boosts along the same axis lead to a rapidity that is the sum of the rapidities of the separate boosts.

The above transformations (and any combination of them) make up a group of coordinate tranformations called the *proper Lorentz transformations*. To the best of our knowledge, the laws of physics are indeed unchanged under proper Lorentz transformations.

2.2.2 Coinciding origins, discrete transformations

More transformations are possible that leave the origin of coordinate systems unchanged. In the first place, the real distance is unchanged if for every vector x^μ we replace $\vec{x}$ by $-\vec{x}$: the is the so-called *parity transformation* P. In matrix form, it reads

$$P^{\mu}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad . \tag{31}$$

Similarly, the real distance remains unchanged if x^0 is replaced by $-x^0$: this is the *time-reversal transformation* T:

$$T^{\mu}{}_{\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad . \tag{32}$$

[15] A term coined by Lorentz himself, who distinguished between ‘snelheid’ (velocity) and ‘rapheid’ (rapidity).

The proper Lorentz transformations combined with P and T make up the group of *Lorentz transformations*. The discrete transformations P and T were for a long time assumed to also be symmetries of nature, in the sense that the laws of physics remain the same under a P or T transform: however, since half a century it has been established that physics is *changed* under either a P or a T transformation, or their combination PT (see excercise 14).

2.2.3 Changes of origin

Since the vector spanned by two events is independent of the choice of coordinate origin, a change in this origin is also an invariance of the real distance. Such a *translation* can, of course, not be written in matrix form since it is not linear, but nevertheless it is an observed symmetry of nature. The group of translations combined with that of Lorentz transformations makes up the group of *Poincaré transformations*.

2.3 Invariances and conservation laws

For the purposes of illustration, let us briefly revert to non-relativistic classical mechanics. Consider a very simple system, of two particles moving in one dimension. In the Lagrangian formulation of this mechanics, the evolution of the system is governed by the Lagrangian:

$$L = L(x_1, x_2, \dot{x}_1, \dot{x}_2) \quad , \tag{33}$$

where $x_{1,2}$ are the particles' coordinates, and $\dot{x}_{1,2}$ their velocities. The equations of motion are then given by the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial}{\partial \dot{x}_j}L = \frac{\partial}{\partial x_j}L \quad , \quad j = 1, 2 \quad . \tag{34}$$

Let us now assume that the Lagrangian is *translation invariant*, that is, a shift of the origin does not affect the physics. This means that L can only depend on the difference of the coordinates:

$$L = L(x_1 - x_2, \dot{x}_1, \dot{x}_2) \quad . \tag{35}$$

In that case, we find immediately that

$$\frac{\partial}{\partial x_1}L + \frac{\partial}{\partial x_2}L = 0 \quad , \tag{36}$$

and therefore we have

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}_1}L + \frac{\partial}{\partial \dot{x}_2}L\right) = 0 \;; \tag{37}$$

in other words, for the system there exists a quantity that is time-independent, or *conserved*. In this case, this quantity is

$$\frac{\partial}{\partial \dot{x}_1}L + \frac{\partial}{\partial \dot{x}_2}L$$

Usually, $\dot{x}$ will enter in the Lagrangian in the combination $\dot{x}^2/2m$, which is the kinetic energy: in that case, the conserved quantity is the total momentum of the particles.

This idea can be extended to more dimensions, and to the case of relativistic field theories[16]. The general law is: *if* there is a continuous transformation that does not change the Lagrangian (or usually, therefore, the Hamiltonian, the total energy of the system), *then* there exists a property of the system that is *conserved*. An important remark is in order here. We have stated that the *laws of physics* are invariant under Poincaré transformations; whether or not a given *system* is invariant under any of these transformations is quite a different matter! Below, we give some invariances that can occur, and their related conserved quantities.

system symmetry	**conserved quantity**
space translation	momentum
time translation	energy
space rotation	angular momentum
complex phase shift	electric charge
complex colour rotation	total colour charge

The last two symmetries involve quantum mechanics, and will be discussed later. Note the absence of Lorentz boosts as invariances: this is because most often the total energy of a system *does* change under Lorentz boosts, because the particles making up the system will pick up kinetic energy under a boost.

2.4 Relativistic Kinematics

Newtonian mechanics is not suitable (nor valid) in the relativistic world, and we have to update it. Consider a point particle traversing an observed space distance

[16]But not here, since that would take too long.

$d\vec{x}$ in an observed time interval dt. Its observed velocity, therefore, is

$$\vec{v} = \frac{d\vec{x}}{dt} \quad . \tag{38}$$

This, however, is not a four-vector. To obtain a four-vector, we note that $dx^\mu = (c\ dt, d\vec{x})$ is indeed a four-vector, and that $d\tau$, defined by

$$c^2\ (d\tau)^2 = dx^\mu\ dx_\mu = (c\ dt)^2 - |d\vec{x}|^2 \tag{39}$$

is invariant under Poincaré transformations. It is called the *proper time interval*[17]. Therefore the *relativistic velocity*, or *four-velocity*:

$$u^\mu = \frac{dx^\mu}{d\tau} \tag{40}$$

is a good four-vector to be used in physical theories. We can also define the particle's *four-momentum*:

$$p^\mu = m\ u^\mu \quad , \tag{41}$$

where m is the particle's mass[18]. The total four-momentum of a system of non-interacting particles is simply the sum of the four-momenta of the individual particles[19]. The three space components, $\vec{p}$, of p^μ correspond to the relativistic improvement of the Newtonian momentum; the fourth component, p^0, is therefore most reasonably interpreted as the relativistic analogue of the Newtonian kinetic energy, divided by c for dimensional reasons.

A useful consequence of the use of four-momentum is the following fact: if four-momentum conservation holds in a certain process in one given coordinate system, it will automatically hold in other coordinate systems that are related to the first one by Poincaré transformations. This makes it at least *possible* that conservation of energy and momentum may be universal laws.

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[17]In the coordinate system in which the particle happens to be at rest, $d\vec{x}$ is zero and therefore $d\tau = dt$; this explains the name.

[18]In nonrelativistic mechanics, the momentum is linear in the velocity, and the kinetic energy is quadratic. Other measures of the amount of motion are of course possible, such as the combination $m^{3/2}|\vec{v}|^4\vec{v}$. Such alternatives are not used in nonrelativistic mechanics since they do not obey any obvious conservation law. In relativistic mechanics, the situation is more æsthetically pleasing: since $u^\mu u_\mu = c^2$ which is just a constant, the linear momentum is, in fact, the *only* possibility!

[19]For interacting particles the situation is of course different, since two interacting particles have, by definition, a total energy that is different from the sum of their individual energies — binding energy and so on, right?

In analogy to Newton's second law, we may also define the relativistic, or Minkowski, force acting on a particle, by equating it to the momentum change:

$$K^\mu = \frac{d}{d\tau}p^\mu \quad . \tag{42}$$

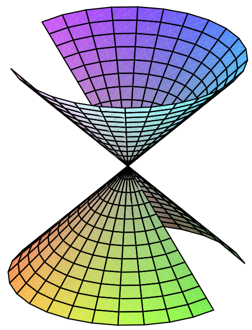
The concept of 'force' is, however, embedded in quite a different way in the relativistic quantum field theories describing elementary particles, and the Minkowski force plays essentially no rôle.

2.5 Causality

The notion of cause and effect is a fundamental one in physics, and relativity theory has to say the following about it. Consider two events A and B, separated by a four-vector $x^\mu = x_B^\mu - x_A^\mu$. It is easy to see that, if $x \cdot x > 0$, then the *sign* of x^0 is invariant under rotations, boosts, and translations (and also under P, but of course not under T). In that case, $x > 0$ means that B happens *later than* A in every accessible[20] coordinate system, and therefore A *can* be a cause of what happens at B; if $x^0 < 0$, the same holds with A and B interchanged. On the other hand, if $x \cdot x < 0$, the sign of x^0 can be changed by a Lorentz boost, and therefore in some coordinate systems A precedes B, while in other coordinate systems A comes later than B. In that case, no causal link between A and B can be possible that is compatible with our preconceptions about what cause and effect *mean*.

As follows from excercise 10, $x \cdot x > 0$ means that it is possible to travel between A and B at less than the speed of light; if $x \cdot x < 0$ one has to exceed the speed of light when travelling between the two events. The spacetime around a given event E can therefore be decomposed into three distinct regions. Consider all light signals that can start at E: this is the *forward light cone* of E. Similarly, the *backward light cone* of E is made of the path of all light signals that can arrive at E. The region inside the forward like cone is the *future* of E, the region inside the backward light cone is the *past* of E, and the region outside the lightcones is the *elsewhere* of E.

[20]Remember that P and T are not physically possible as far as we know.



Forward and backward lightcones. The point E sits where the cones (with pieces cut out for clarity) touch.

Finally, we see that any particle that transmits information (linking cause to effect) through spacetime must do so with velocities not larger than c. This last statement is not absolute, however, since it only holds under the conditions where non-quantummechanical special relativity holds, that is, for particles travelling freely during macroscopic time intervals[21].

2.6 $E = Mc^2$: High-Energy Physics

In terms of the observed space and time intervals, the relativistically correct definition of the kinetic energy is seen to be

$$E = \frac{mc^2}{\sqrt{1 - |\vec{v}|^2/c^2}} \quad . \tag{43}$$

For particles at rest, this leads to the famous formula $E = mc^2$. Note that, apart from the numerical factor, the *possibility* of such a relation is immediately implied by the very existence of the natural constant c! It suggests the possibility that (but not the mechanism by which), under the right circumstances, an amount of matter may be converted in a lesser amount of matter and kinetic energy or, *vice versa*, particles colliding with sufficient kinetic energy may result in heavier partticles being produced. At any rate, in relativistic theories mass as such is *not* conserved: the law of 'conservation of mass' well-known from chemistry is an approximation valid because in chemical reactions the energies involved are of the order of eV,

[21] We shall see that the quantum-mechanical description of a freely moving particle indeed forbids superluminal speeds in the appropriate limit.

the order of magnitude of chemical binding energies, and hence very much smaller than the relevant rest energies, of the order of GeV, the typical mass of nucleons like protons and neutrons.

The idea of colliding particles at relativistic speeds in order to produce other, possibly new, states of mattter is one of the reasons why the field of elementary-particle physics is also called *high-energy physics*.

2.7 Excercises

Excercise 9 Properties of the real distance
Show that the four-vector defined in (13), and hence also the real distance (14), are independent of the choice of origin of the Cartesian coordinate system. Also show that the real distance (14) is independent of a rotation of the coordinate axes, without referring to Lorentz transformations.

Excercise 10 Timelike, spacelike and lightlike
Consider two events A and B separated both in space and time. Show the following results.

- The real distance between A and B is real and nonzero if and only if a signal travelling uniformly between A and B does so with a speed *smaller than* c; in that case, A ad B are said to have a *timelike* separation.
- The real distance between A and B is imaginary and nonzero if and only if a signal travelling uniformly between A and B does so with a speed *greater than* c; in that case, A ad B are said to have a *spacelike* separation.
- The real distance between A and B is zero if and only if a signal travelling uniformly between A and B does so with a speed *equal to* c; in that case, A ad B are said to have a *lightlike* separation.

Excercise 11 Metric or matrix?
Referring to footnote 14, show that it is inappropriate to write the metric as a matrix.

Excercise 12 Lorentz transformations: algebraic results

- Prove that the matrices of Eq.(27) have the correct property (26).

- Return to Eq.(23), and this time assume that ϵ^2 is *not* negligble, but that ϵ^3 *is*. Prove that in that case we can use the same T's as before, but now with

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \epsilon\ T^{\mu}{}_{\nu} + \frac{1}{2}\epsilon^2\ T^{\mu}{}_{\lambda}T^{\lambda}{}_{\nu} \quad .$$

- The exponent of a matrix A is defined by

$$\exp(A) = 1 + \sum_{n\geq 1} \frac{1}{n!}A^n \quad , \tag{44}$$

where 1 stands for the unit matrix. For the matrices $T^{(12)}$ and $T^{(01)}$, compute their second, third, and fourth powers, and use this to prove Eqs.(28) and (29).

- Consider the Lorentz transformation of Eq.(29). Express the position of the one coordinate system in terms of the other coordinate system, and use this to prove Eq.(30).

- Prove the statement in the text about the rapidity of a combination of boosts being the sum of the rapidities of the separate boosts.

Excercise 13 Discrete transformations
Using the notion of determinants, prove that there is *no* combination of proper Lorentz transformations that equals either P or T.

Excercise 14 Parity violation
Imagine George W. Bush launching a fleet of ICBMs from the continental U.S.: these missiles, therefore, emerge from the northern hemisphere. Now, apply a P transformation to the earth and everything in it: prove that now the missiles come from the *southern* hemisphere. In 1956, this experiment was performed with polarized Cobalt-60 nuclei, which are observed to undergo radioactive decay by emitting positrons preferentially into one hemisphere, and less often into the other one. Prove that this observation implies that, *whatever* the physical interaction responsible for this emission, it violates the invariance of physics under P .

Excercise 15 A useful expression
The *minimal Lorentz transformation* from p^μ to q^μ is that proper Lorentz transformation that takes a four-vector p^μ over into q^μ, while leaving any vector r^μ invariant for which $p \cdot r = q \cdot r = 0$. We define

$$\Lambda(p \to q)^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} - \frac{2(p+q)^{\mu}(p+q)_{\nu}}{(p+q)\cdot(p+q)} + \frac{2}{p\cdot p}q^{\mu}p_{\nu} \tag{45}$$

to be this minimal Lorentz transform. Show that it obeys the following:

$$
\begin{aligned}
&\Lambda(p \to q)^{\mu}{}_{\nu}\, p^{\nu} = q^{\mu} \\
&\Lambda(p \to q)^{\mu}{}_{\nu}\, \left(2\frac{p\cdot q}{p^2}p^{\nu} - q^{\nu}\right) = p^{\mu} \\
&q_{\mu}\, \Lambda(p \to q)^{\mu}{}_{\nu} = p_{\nu} \\
&\Lambda(p \to q)^{\mu}{}_{\nu}\, r^{\nu} = r^{\mu} \\
&a'^{\mu}\, a'_{\nu} = a^{\mu}\, a_{\nu} \quad , \quad a'^{\alpha} = \Lambda(p \to q)^{\alpha}{}_{\nu}\, a^{\nu} \quad ,
\end{aligned}
\tag{46}
$$

where the last holds for any four-vector a.

Excercise 16 Proper time

Prove that a particle's proper time interval is real if it moves slower than c. In that case, prove that the proper time and the real time are related by

$$
d\tau = dt\,\sqrt{1 - |\vec{v}|^2/c^2} \quad , \tag{47}
$$

thereby proving the time dilatation effect.

Excercise 17 Four-velocity and four-momentum

Prove the following facts:

- The four-velocity u^{μ} obeys $u^{\mu}u_{\mu} = c^2$, and the four-momentum of a particle of mass m therefore obeys $p^{\mu}p_{\mu} = m^2c^2$.

- In terms of the observed space and time intervals, we can write

$$
\begin{aligned}
\vec{p} &= \frac{m\vec{v}}{\sqrt{1 - |\vec{v}|^2/c^2}} \quad , \\
p^0 &= \frac{mc}{\sqrt{1 - |\vec{v}|^2/c^2}} \quad .
\end{aligned}
\tag{48}
$$

- In the 'Newtonian limit' $c \to \infty$, we have

$$
\begin{aligned}
\vec{p} &= m\vec{v} + \mathcal{O}\left(|\vec{v}|^3/c^2\right) \quad , \\
p^0 &= mc + \frac{1}{2}m|\vec{v}|^2/c + \mathcal{O}\left(|\vec{v}|^5/c^3\right) \quad .
\end{aligned}
\tag{49}
$$

Excercise 18 Creating particles
Imagine the following situation: there are Π-particles, of mass M, two of which can create a third one provided the correct amount of kinetic energy is present. Consider one Π sitting still, and another one impingeing on it with a total energy E. Compute the minimal energy E necessary to create a third Π particle; also the minimal energy E necessary to create $2, 3, 4, \ldots$ new Π particles. Do the same computation, but now for the situation where the two original Π particles have equally large but opposite momentum: what is now the minimal energy required to create $1, 2, 3, \ldots$ additional Π particles?

3 Elements of Quantum Mechanics

Here we give a brief discussion of quantum mechanics. This is a vast subject of which we cannot do more than scratch the surface. Quantum mechanics is *much* more counter-intuitive even than relativity!

3.1 Quantum States

In quantum mechanics, a system[22] is described by a *quantum state*, denoted by a 'ket' $|L\rangle$, where L is a list of the known properties of the system. For instance, a single particle located at $\vec{x}$ might be denoted by $|\vec{x}\rangle$; two particles with momenta $\vec{p}_1$ and $\vec{p}_2$ might be denoted by $|\vec{p}_1, \vec{p}_2\rangle$, and so on. This could of course also be used in a classical, non-quantum description of systems, but in quantum mechanics the states have particular properties.

1. The *superposition principle*: any linear combination of quantum states of a system is *also* a quantum state of the system. That is, if $|L_1\rangle$ and $|L_2\rangle$ are states, then
$$|\psi\rangle = z_1|L_1\rangle + z_2|L_2\rangle \tag{50}$$
is also a state for arbitrary *complex* numbers $z_{1,2}$ (not both zero).

2. $|L\rangle$ and $z|L\rangle$ describe *the same* physical state for any complex numer z. The only exception is the null state $0 = 0|L\rangle$ which is not a physical state.

3. For any quantum state $|\psi\rangle$ there is a conjugate state, denoted by a 'bra' $\langle\psi|$, with the following properties:
$$|\psi\rangle^\dagger = \langle\psi| \quad , \quad \langle\psi|^\dagger = |\psi\rangle \quad , \quad (z_1|L_1\rangle + z_2|L_2\rangle)^\dagger = z_1^*\langle L_1| + z_2^*\langle L_2| \quad . \tag{51}$$
There exists an inner product of states, denoted by $\langle\cdot|\cdot\rangle$, called a *bracket*[23]. It has the following properties:
$$\begin{aligned} 1: &\quad |\psi\rangle = z_1|L_1\rangle + z_2|L_2\rangle \quad , \quad |\phi\rangle = z_3|L_3\rangle + z_4|L_4\rangle \quad \Rightarrow \\ &\quad \langle\psi|\phi\rangle = z_1^* z_3\langle L_1|L_3\rangle + z_1^* z_4\langle L_1|L_4\rangle + z_2^* z_3\langle L_2|L_3\rangle + z_2^* z_4\langle L_2|L_4\rangle \\ 2: &\quad \langle\psi|\psi\rangle \geq 0 \\ 3: &\quad \langle\psi|\psi\rangle = 0 \quad \Longleftrightarrow \quad |\psi\rangle = 0 \end{aligned} \tag{52}$$

[22] A system is idealized to be isolated from the rest of the universe. This is of relevance for the 'measurement problem', discussed below.

[23] 'Bra" times 'ket' = 'bracket' — get it? Dirac, who introduced these notations, was *not* world-famous for his sense of humour.

We see that any physical state $|\psi\rangle$ (except the null state) can be multiplied by an appropriate complex number such that $\langle\psi|\psi\rangle = 1$. These are called *normalized* physical states (NPSs).

3.2 Probabilistic interpretation

We still have to give an interpretation to a superposition of physical states. Let $|x_1\rangle$ be the NPS of a particle in which we know with certainty that the particle is at position x_1, and $|x_2\rangle$ that NPS in which we know with certainty that the particle is at position x_2. Now consider the NPS

$$|\psi\rangle = z_1|x_1\rangle + z_2|x_2\rangle \quad , \quad |z_1|^2 + |z_2|^2 = 1 \quad , \tag{53}$$

where the last condition is necessary to have an NPS. The interpretation of this state is the following: if the position of the particle in this state is measured, *either* of x_1 or x_2 will be found. The *probability* to find the particle at x_1 is given by $|z_1|^2$, and the probability to find it at x_2 is $|z_2|^2$.

If the particle is in state $|x_1\rangle$, the probability to find it at x_2 is zero: we write this as

$$\langle x_1|x_2\rangle = 0, \tag{54}$$

and such states are called *orthogonal*[24]. This means that, in the present case.

$$z_1 = \langle x_1|\psi\rangle \quad , \quad z_2 = \langle x_2|\psi\rangle \quad . \tag{55}$$

The complex numbers $z_{1,2}$ are not probabilities, but *probability amplitudes*. Whereas probabilities are non-negative and additive, probability amplitudes interfere with each other in much more complicated ways.

3.3 Hilbert space, bases and wave functions

The quantum states of a system are seen to form a (complex) vector space, called *Hilbert space*. Quite often, this space has an infinite number of dimensions. As usual in vector spaces, one can find (or *try* to find) an orthonormal basis of states, that is an set of mutually orthogonal NPSs $|L_n\rangle$, such that *every* state $|\psi\rangle$ can be written as a unique superposition of the basis states[25]:

$$|\psi\rangle = \sum_n z_n\,|L_n\rangle \quad . \tag{56}$$

[24] Actually they are *orthonormal* since also $\langle x_1|x_1\rangle = \langle x_2|x_2\rangle = 1$.

[25] Here, we use a discrete sum. Depending on the situation, one can also have a *continuous* sum, that is, an integral.

Since we have (from orthonormality)

$$\langle L_m | L_n \rangle = \delta_{m,n} \quad , \tag{57}$$

the basis states must correspond to physical states that are completely distinct from each other. A (continuous) example is the set of all possible positions of a particle, denoted by $|\vec{x}\rangle$. We then have the following relations:

$$\begin{aligned} \langle \vec{x} | \vec{x}' \rangle &= \delta^3(\vec{x} - \vec{x}') \quad , \\ |\psi\rangle &= \int \psi(\vec{x}) \; |\vec{x}\rangle \; d^3\vec{x} \quad , \\ \psi(\vec{x}) &= \langle \vec{x} | \psi \rangle \quad . \end{aligned} \tag{58}$$

The complex function (not state!) $\psi(\vec{x})$ is called the *wave function* of the system. In non-relativistic quantum mechanics, it obeys the so-called *Schrödinger equation*.

3.4 The 'measurement problem'

Suppose we know that the particle of section (3.2) is in the state

$$|\psi\rangle = z_1 |x_1\rangle + z_2 |x_2\rangle \quad , \quad z_{1,2} \neq 0 \quad . \tag{59}$$

Further, suppose that we measure the particle's position, and find it to be at x_1. Imagine, now, that we *immediately* repeat the position measurement: of course, we can then be certain[26] that we shall still find it at x_1, even without actually doing the second measurement. But this certainty implies that after the first measurement the state of the particle is not $|\psi\rangle$, but rather $|x_1\rangle$. The very act of observation has changed the state of the system!

This is the 'measurement problem', also known as the 'collapse of the wave function'. Despite decennia of discussion, the mechanism of how this change of state happens is not clear, although proposals abound[27]. It must be relevant that for perfectly isolated systems, that have no interaction with the outside world, no measurement is of course possible[28]. Any measurement involves transfer of some

[26] We assume the second measurement to follow so closely upon the first one that the particle has no time to move by any physical process.

[27] The measurement problem is a black-hole topic for physicists: many of those who enter it never emerge from it again...

[28] If you are completely invisible, you are also *blind*.

amount of action[29], and the minimum amount of action is $\hbar$. The idealization of an isolated quantum system is therefore *not* valid precisely at the moment the measurement takes place[30].

3.5 Observables and expectation values

We have discussed quantum states, but not yet the way that physical information about them is represented: the physically measureable objects, called *observables*. Of course, a measurement performed on a quantum system in a given state leaves the quantum system in *some* state (possibly a different one, as discussed). The natural choice, therefore, is to represent an observable by a linear operator (a 'matrix') acting on the states. That is, for every observable X there is a linear operator $\hat{X}$ that acts on states. If the system happens to be in a state $|x\rangle$ in which the outcome of the measurement of X is, with certainty, going to be x, the state is an *eigenstate* of $\hat{X}$ with eigenvalue x:

$$\hat{X}\,|x\rangle = x\,|x\rangle \quad . \tag{60}$$

Furthermore, the linear operator is *self-adjoint*, or Hermitian:

$$\hat{X}^\dagger = \hat{X} \quad . \tag{61}$$

Now, consider a superposition of such eigenstates:

$$|\psi\rangle = \sum_n z_n\,|x_n\rangle \quad , \quad \langle x_m|x_n\rangle = \delta_{m,n} \quad , \quad \sum_n |z_n|^2 = 1 \quad . \tag{62}$$

Measuring the value of X on the system in this state will, as discussed before, give x_1 with probability $P(x_1) = |z_1|^2$, x_2 with probability $P(x_2) = |z_2|^2$, and so on. The *expectation value* $\langle X\rangle$ is defined in the usual manner:

$$\langle X\rangle = \sum_n P(x_n)\,x_n \quad . \tag{63}$$

[29]Even just *seeing* an object is only possible if the object scatters light rays, that is, changes their momentum.

[30]Possibly the weirdest idea is the *many-worlds* picture of measurement: for the quantum state we have discussed this says that upon measurement of the position with the two alternative outcomes x_1 and x_2, the universe splits into two parallel realities, one in which x_1 has been found, and one in which x_2 has been found. You, as the observer, have only a single conscience in each alternative world, and therefore see only a single outcome.

Notice that we have

$$\hat{X}\,|\psi\rangle = \sum_n \; z_n\, x_n\, |x_n\rangle \quad , \tag{64}$$

and therefore the expectation value can be written as

$$\langle X\rangle = \langle\psi|\left(\,\hat{X}\,|\psi\rangle\,\right) \equiv \langle\psi|\,\hat{X}\,|\psi\rangle \quad . \tag{65}$$

3.6 The Schrödinger equation

One of the most important observables of a system is its total energy E, represented by the (Hermitian!) Hamiltonian operator $\hat{H}$: in fact, choosing $\hat{H}$ amounts to *defining* the system. The time evolution of quantum states is governed by the Schrödinger equation[31]:

$$i\hbar\,\frac{\partial}{\partial t}|\psi\rangle = \hat{H}\,|\psi\rangle \quad . \tag{66}$$

This is where Planck's constant $\hbar$ enters. The Schrödinger equation as such cannot be *proven*: it is a postulate, on the same footing as Newton's law of gravity. On the other hand, whenever nonrelativistic quantum mechanics is appropriate, no phenomenon whatever is in disagreement with it, just as Newtonian gravity rules supreme whenever it might be expected to do so.

We can obtain a Schrödinger equation for expectation values as well. From Eq.(66) we find

$$i\hbar\,\frac{\partial}{\partial t}\langle\psi| = -\,\langle\psi|\,\hat{H} \quad , \tag{67}$$

and, making the usual assumption that the Hamiltonian itself does not contain any *explicit* time dependence, we find for the expectation value of any observable A with operator $\hat{A}$:

$$\frac{\partial}{\partial t}\,\langle A\rangle = \frac{\partial}{\partial t}\langle\psi|\hat{A}|\psi\rangle = \frac{1}{i\hbar}\langle\psi|\,[\hat{A},\hat{H}]\,|\psi\rangle = \frac{1}{i\hbar}\left\langle[\hat{A},\hat{H}]\right\rangle \quad , \tag{68}$$

where we have introduced the *commutator* of two operators:

$$[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} \quad . \tag{69}$$

An immediate consequence follows from the fact that $\hat{H}$ commutes with itself:

$$\frac{\partial}{\partial t}\,\langle E\rangle = 0 \quad . \tag{70}$$

This is energy conservation in quantum language.

[31]That is, for non-relativistic quantum mechanics. In the relativistic case other equations are relevant, such as the Klein-Gordon, Dirac, Proca or Maxwell equations.

3.7 The fundamental commutator $[\hat{x}, \hat{p}]$

The simplest quantum system is a free particle in one dimension. From the classical kinetic energy,

$$E = \frac{1}{2m}\, p^2 \quad , \tag{71}$$

where m is the particle's mass, we choose for the Hamiltonian

$$\hat{H} = \frac{1}{2m}\, \hat{p}^2 \quad , \tag{72}$$

where $\hat{p}$ is the operator of the particle momentum. There is of course also the operator $\hat{x}$ for the particle's position. Since classically we have

$$m\, \frac{\partial}{\partial t} x = p \quad , \tag{73}$$

we want to find the quantum relation

$$m\, \frac{\partial}{\partial t} \left\langle x \right\rangle = \left\langle p \right\rangle \quad . \tag{74}$$

But, from Eq.(68), we see that this implies

$$\left\langle p \right\rangle = \frac{1}{2i\hbar} \left\langle \hat{x}\hat{p}^2 - \hat{p}^2\hat{x} \right\rangle \quad , \tag{75}$$

and therefore the Schrödinger equation can only lead to good physics if we have

$$[\hat{x}, \hat{p}] = i\hbar \quad . \tag{76}$$

In the three-dimensional case, we have[32]

$$[\hat{x}_j, \hat{x}_k] = [\hat{p}_j, \hat{p}_k] = 0 \quad , \quad [\hat{x}_j, \hat{p}_k] = i\hbar\, \delta_{j,k} \quad , \quad j,k = 1,2,3 \quad . \tag{77}$$

[32] At least, this is what is mostly used. There are alternative models of the universe in which the various components of $\hat{\vec{x}}$ do not commute: this is the field of *non-commutative geometry*, which we shall not discuss in these notes.

3.8 The uncertainty relation

Since $\hat{x}$ and $\hat{p}$ have a constant, nonzero commutator, there is *no* basis of quantum states that are simultaneously eigenstates of $\hat{x}$ and $\hat{p}$: it is not possible to known both position and momentum with certainty. We can make this statement even more precise. Consider an arbitrary quantum state $|\psi\rangle$. Consider two observables A and B with operators $\hat{A}$ and $\hat{B}$. For simplicity we shall assume that in the state $|\psi\rangle$ it so happens that $\langle A\rangle = \langle B\rangle = 0$. Now, we may build another quantum state depending on a parameter λ:

$$|\lambda\rangle \equiv \left(\hat{A} + i\lambda\hat{B}\right)|\psi\rangle \quad . \tag{78}$$

Since $|\lambda\rangle$ is a quantum state, we know that

$$\begin{aligned} 0 &\leq \langle\lambda|\lambda\rangle \\ &= \langle\psi|\left(\hat{A} + i\lambda\hat{B}\right)^{\dagger}\left(\hat{A} + i\lambda\hat{B}\right)|\psi\rangle \\ &= \langle\psi|\left(\hat{A}^2 + \lambda^2\hat{B}^2 + i\lambda[\hat{A},\hat{B}]\right)|\psi\rangle \quad . \end{aligned} \tag{79}$$

This gives us the inequality

$$\sigma(A)^2 + \lambda^2\ \sigma(B)^2 + \lambda\ \left\langle i[\hat{A},\hat{B}]\right\rangle \geq 0 \quad . \tag{80}$$

Note that the combination $i[\hat{A},\hat{B}]$ is itself hermitian. The left-hand side of the inequality is *minimized* if we choose

$$\lambda = \lambda_0 = -\frac{\left\langle i[\hat{A},\hat{B}]\right\rangle}{2\sigma(B)^2} \quad , \tag{81}$$

so the most precise form of the inequality can be written as

$$\sigma(A)^2\ \sigma(B)^2 \geq \frac{1}{4}\left\langle i[\hat{A},\hat{B}]\right\rangle^2 \quad . \tag{82}$$

This is the Heisenberg uncertainty relation. If $\hat{A}$ and $\hat{B}$ commute, then the uncertainties $\sigma(A)$ and $\sigma(B)$ can both be zero in principle. For the position and momentum operators, however, we find

$$\sigma(x)\ \sigma(p) \geq \frac{\hbar}{2} \quad . \tag{83}$$

The smaller $\sigma(x)$, that is, the smaller the statistical spread in the possible position values, the larger $\sigma(p)$ has to be, that is, the larger is the statistical spread in possible momentum values[33].

[33]It is often said that 'position and momentum cannot be measured simultaneously'. This is not completely correct, since in principle nothing forbids one to measure momentum so quickly after

3.9 Conservation laws

Suppose that there is an observable Z, with a corresponding Hermitian operator $\hat{Z}$ that commutes with the Hamiltonian $\hat{H}$. We then have that for every possible quantum state of the system,

$$\frac{\partial}{\partial t}\langle Z\rangle = 0 \quad , \tag{84}$$

that is, the quantity Z is *conserved*. As an application consider (in one dimension) the effect on the wave-function of changing the origin of the coordinate system over a distance a:

$$\begin{aligned}\psi(x) &\to \psi(x+a)\\ &= \psi(x) + a\psi'(x) + \frac{a^2}{2!}\psi''(x) + \frac{a^3}{3!}\psi'''(x) + \cdots\\ &= \exp\left(a\,\frac{\partial}{\partial x}\right)\,\psi(x) \quad .\end{aligned} \tag{85}$$

The generator of an infinitesimal shift is seen to be the operator

$$\frac{\partial}{\partial x} \quad = \quad \frac{i}{\hbar}\hat{p} \quad : \tag{86}$$

Translations are generated by the momentum operator. If the Hamiltonian is invariant under translations, we must then have

$$[\hat{p},\hat{H}] = 0 \quad \Rightarrow \quad \frac{\partial}{\partial t}\langle p\rangle = 0 \quad . \tag{87}$$

Translation invariance leads to momentum conservation, as in the classical case. We find that *the generator of every infinitesimal transformation that commutes with $\hat{H}$ is a conserved observable of the system.*

3.10 What can we compute?

An important consequence of the way quantum mechanics is constructed is the restriction it imposes on what can be predicted. Since the act of observation on

having measured position that the intervening time becomes negligible. Rather, the fact that the position and momentum operators have no common eigenstates means that after the position measurement which brings the system in a position eigenstate that is a superposition of momentum eigenstates, an momentum measurement will single out a momentum eigenstate that is a superposition of position eigenstates. Therefore, measuring p *destroys* the information we had obtained by measuring x.

a general state picks out one of its eigenstate components, it is in general not possible to predict the outcome of an observation. The best one can do is to compute the *probability* for a particular outcome.

As an illustration, consider a collision process between two particles. The quantum state starts out (at $t = -\infty$, say) as $|\text{in}, t = -\infty\rangle$ a state consisting of two particles approaching each other. As time passes, the quantum state evolves, possibly in a very complicated way. At late times, we have therefore

$$|\text{in}, t = -\infty\rangle \quad \to \quad |\text{in}, t = +\infty\rangle = \hat{S}\ |\text{in}, t = -\infty\rangle \ . \tag{88}$$

Here we have introduced the *S*-matrix which governs the time evolution from $t = -\infty$ to $t = +\infty$. But $|\text{in}, t = +\infty\rangle$ is not what is observed: it is usually a complicated superposition of the states describing all possible outcomes of the collision. Upon observation, *one* of these is found; let us denote it by $|\text{out}, t = +\infty\rangle$. The probability amplitude for this to happen is what we can compute, and it is given by

$$\langle\text{out}, t = +\infty|\text{in}, t = +\infty\rangle = \langle\text{out}, t = +\infty|\hat{S}|\text{in}, t = -\infty\rangle \ . \tag{89}$$

This is called the *S*-matrix element for this particular process. It is important to realize that conservation of probability requires

$$\langle\text{in}, t = -\infty|\text{in}, t = -\infty\rangle = \langle\text{in}, t = +\infty|\text{in}, t = +\infty\rangle \ . \tag{90}$$

Therefore, the matrix $\hat{S}$ must be *unitary*[34]: $\hat{S}\ \hat{S}^\dagger = \hat{S}^\dagger\ \hat{S} = 1$. As a consequence, the individual elements of the *S* matrix cannot be arbitrarily large: in fact they can at most be 1 in absolute value[35]. Any model of elementary particles and their interactions must respect this *unitarity bound*. This is a quite strict requirement which, to a large extent, determines the structure of the Standard Model.

3.11 Waves and high-energy physics

Let us consider a freely moving particle with a well-determined momentum $\vec{p}$. In the language of wavefunctions, this implies that

$$-i\hbar\,\frac{\partial}{\partial x_k}\psi(\vec{x}, t) = p_k\ \psi(\vec{x}, t) \quad , \quad k = 1, 2, 3 \ . \tag{91}$$

[34]It could also be *anti*unitary, but we shall not consider this.

[35]This holds for a discrete matrix; for matrices with continuous indices a more complicated but analogous bound holds.

This differential equation is simply solved: we find

$$\psi(\vec{x},t) \propto \exp\left(\frac{i}{\hbar}\vec{x}\cdot\vec{p}\right) \quad . \tag{92}$$

But we also know that the particle's energy must equal $E = \vec{p}^2/(2m)$: the Schrödinger equation therefore tells us

$$i\hbar\,\frac{\partial}{\partial t}\psi(\vec{x},t) = E\,\psi(\vec{x},t) \tag{93}$$

so that the full solution reads

$$\psi(\vec{x},t) \propto \exp\left(-\frac{i}{\hbar}(Et - \vec{x}\cdot\vec{p})\right) \quad : \tag{94}$$

the particle's wavefunction is, indeed, a plane wave. This is the (in)famous *wave-particle duality*. Note that the combination $Et-\vec{x}\cdot\vec{p}$ is precisely the Minkowskian product $x_\mu p^\mu$, so connecting up with special relativity is possible.

A final word: the wavelength of the plane wave indicated above is $2\pi\hbar/|\vec{p}|$: this is precisely the *de Broglie* wavelength. The angular frequency ω of the wave is related to the energy E:

$$E = \hbar\,\omega \quad . \tag{95}$$

If we want to use particle waves rather than light waves to investigate matter at small distances, the wavelengths of the waves involved must be of the order of the details studied[36]. In the study of subnuclear structures, the momenta involved are therefore necessarily very large: this is the second reason why the field of elementary-particle physics is also called high-energy physics.

3.12 Excercises

Excercise 19 An alternative interpretation
An alternative interpretation of the state given in Eq.(53) might be that the particle is found with certainty at position $x_3 = z_1x_1+z_2x_2$. This would, of course, require the $z_{1,2}$ to be real numbers. Even conceding this, show that this interpretation cannot be correct, by considering a translation of the origin of the coordinate system.

[36] Or, rather, one cannot resolve details much smaller than the wavelength.

Excercise 20 The wavefunction and its interpretation

1. Prove the third equation in Eq.(58).
2. Prove, from the probabilistic interpretation discussed in the text that $|\psi(\vec{x})|^2$ represents the probability (or, rather, the probability density) to find the particle at position $\vec{x}$.
3. Prove that for a NPS we have
$$\int |\psi(\vec{x})|^2 \, d^3\vec{x} = 1 \quad . \tag{96}$$

Excercise 21 Expectation value and variance

1. Prove Eq.(65).
2. Prove that, for any $p > 0$, $\left\langle \hat{X}^p \right\rangle = \langle\psi| \hat{X}^p |\psi\rangle$ if the system is in NPS $|\psi\rangle$.
3. Prove that the *variance* of the observable X in that case can be written as
$$\sigma(X)^2 \equiv \left\langle (X - \langle X \rangle)^2 \right\rangle = \langle\psi|\hat{X}^2|\psi\rangle - \langle X \rangle^2 \quad . \tag{97}$$
4. Prove the following: if $\langle X \rangle = x$ and $\sigma(X)^2 = 0$, then the system's state is an eigenstate of $\hat{X}$: $\hat{X} |\psi\rangle = x |\psi\rangle$.

Excercise 22 Properties of Hermitian operators: linear algebra

1. Consider a Hermitian operator $\hat{A} = \hat{A}^\dagger$. Prove that its eigenvalues are real numbers.
2. Prove that it is possible to find an orthonormal basis of eigenstates for $\hat{A}$.
3. Consider, in addition to $\hat{A}$, another Hermitian operator $\hat{B}$. Suppose that there is a basis of orthonormal NPSs that are simultaneous eigenstates of $\hat{A}$ and $\hat{B}$:
$$\hat{A} |a_n, b_m\rangle = a_n |a_n, b_m\rangle \quad , \quad \hat{B} |a_n, b_m\rangle = b_m |a_n, b_m\rangle \quad . \tag{98}$$
Show that this is only possible if $\hat{A}$ and $\hat{B}$ commute:
$$[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A} = 0 \quad . \tag{99}$$

Excercise 23 Time evolution of states
Prove Eqs.(67), (68), (75) and (76).

Excercise 24 Operators for the wavefunction
The quantum state of the system can be represented by the wavefunction as given in Eq.(58):

$$|\psi\rangle \quad \Longleftrightarrow \quad \psi(x) \ . \tag{100}$$

Show that, in the same way, we have the representation of the operators for position and momentum:

$$\begin{aligned} \hat{x}\ |\psi\rangle \quad &\Longleftrightarrow \quad x\ \psi(x) \ , \\ \hat{p}\ |\psi\rangle \quad &\Longleftrightarrow \quad -i\hbar\ \frac{\partial}{\partial x}\ \psi(x) \ . \end{aligned} \tag{101}$$

Excercise 25 Schrödinger equations in wavefunction language
For many systems, the classical energy has the form

$$E = \frac{1}{2m}p^2 + V(x) \ , \tag{102}$$

where $V(x)$ is some potential energy. Show that, in the language of wavefunctions, the Schrödinger equation then takes the form

$$i\hbar\ \frac{\partial}{\partial t}\ \psi(x) = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{(\partial x)^2} + V(x)\right)\ \psi(x) \ . \tag{103}$$

The *time-independent Schrödinger equation* is the eigenvalue equation for the Hamiltonian operator:

$$\hat{H}\ |\psi\rangle = E\ |\psi\rangle \ . \tag{104}$$

Show that, in the language of wavefunctions, the time-independent Schrödinger equation takes the form

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{(\partial x)^2} + V(x)\right)\ \psi(x) = E\ \psi(x) \ . \tag{105}$$

This is the Schrödinger equation as given in Serway etc.

Excercise 26 One-dimensional box
Consider a one-dimensional system of a particle ‘in a box of size L’, that is, the

particle can move freely between 0 and L but is forbidden to be outside this interval. Argue that the wavefunction with the *largest* spread in position is that one that concentrates the probabilities at the endpoints, that is, a position measurement will either result in $x = 0$ or $x = L$, both with 50% probability, and no other possible results. Show that in that case $\langle x \rangle = L/2$ and $\sigma(x) = L/2$. Use the Heisenberg relation to show that for any quantum state of this system we must have

$$\sigma(p) \geq \sigma_0(p) \equiv \frac{\hbar}{L} \ . \tag{106}$$

Explain why it is possible for the system to *not* be in an eigenstate of $\hat{p}$, but still be in an eigenstate of $\hat{H} = \hat{p}^2/(2m)$ (hint: standing waves). Find the lowest possible energy (from Serway!) and compare with $\sigma_0(p)^2/(2m)$.

4 Paths and diagrams

Here we give a brief and schematic introduction into the ideas of path integrals and Feynman diagrams, as well as into the properties of propagating particles.

4.1 Natural units

In what follows we shall adopt the usual convention of high-energy physics. To wit, we shall take our units of length, time and mass in such a way that c and $\hbar$ attain the numerical value *one*. They then effectively drop out from our equations, which simplifies things a lot. One more condition will specify precisely what these 'natural units' are: this is to take for the unit of energy the GeV. Masses can then be expressed in GeV, lengths as well as times in unuits of $1/\text{GeV}$. This may seem weird at first but one quickly gets used to it. If one wants to revert to the more familiar meter, second and kilogram one simply adds the appropriate factors of c and $\hbar$ to give the object under consideration its correct dimensionality, and then inserts the known values for c and $\hbar$ (see excercise 27).

4.2 Feynman's Ultimate Multi-Slit Experiment

Consider the two-slit experiment, which can be used to demonstrate the wavelike nature of light, but *also* that of particles such as electrons. Take the slits to be a distance L from the screen, and a distance d apart from each other. Take a point located at a vertical distance x from the point on the screen located in the middle of the geometric projections of the slits on the screen. The distance from the point x to the two slits is then given by

$$r_{\pm} = \sqrt{L^2 + \left(\frac{d}{2} \pm x\right)^2}\ . \tag{107}$$

Supposing that the waves come out of the slits in perfect phase equal to zero, the number of wavelengths from the point x to either slit is then $r_{\pm}/\lambda$, where λ is the wavelength of the photon or electron or whatnot. The phases of the waves upon arrival at the projection screen are therefore $\exp(ir_{\pm}/\lambda)$. The combined wave intensity [37] therefore reads

$$I = |\exp(ir_+/\lambda) + \exp(ir_-/\lambda)|^2 = 2 + 2\cos\left(\frac{r_+ - r_-}{\lambda}\right) \approx 2 + 2\cos\left(\frac{2dx}{L\lambda}\right) \quad , \tag{108}$$

where we have assumed that L is very large compared to x and d. Whenever this difference reaches a multiple of 2π the waves will interfere constructively, and when it is π plus this multiple they interfere destructively. We see that both paths have to be taken into account to explain the observed behaviour of the particles.

Now, imagine that we use not two but N slits: obviously we then have to consider N different paths from the particle source to the point x. An extension is also obvious: if we use two screens, one with N_1 slits behind the other one with N_2 slits, we shall have to take into account $N_1 N_2$ different paths.

The ultimate in this game is to use an *infinite* numer of slitted screens, each with an *infinite* number of slits — so many slits, in fact, that every single atom of every screen has been drilled away. The particle then propagates through empty space, *but we still have to take into account all possible paths of the particle between the source and the projection screen!* This is the idea of the *Feynman sum over paths*[38]: each path carries a phase $\exp(iS/\hbar)$, where S (called the *action*) is prescribed by the theory.

4.3 The Schrödinger equation from the Feynman sum

As an illustration, we consider the following example, for a nonrelativistic point particle moving in one dimension. At time t, the wave function of the particle is given by $\psi(y,t)$, where y is the position. At a later time $t+dt$, the wave function is given by $\psi(x, t+dt)$, where now x is the position. Feynman's idea is now to write

$$\psi(x, t+dt) = N \int_{-\infty}^{\infty} \text{d(paths)} \; \exp\left(i\frac{S}{\hbar}\right) \; \psi(y,t) \quad , \tag{109}$$

[37] Here we do not take into account the attenutation of the waves as they spread out behind the slits.

[38] Actually, the path *integral* since the paths form of course a continuous set.

where we consider all paths starting at y at time t and ending at x at time $t + dt$. The action contains the Lagrangian of the particle:

$$S = \int_{t}^{t+dt} d\tau \left[\frac{1}{2} m \, \dot{\xi}(\tau) - V(\xi(\tau))\right] \quad , \tag{110}$$

where $\xi(\tau)$ is the path giving position ξ as a function of time τ. The $\dot{\xi}$ is therefore the velocity. We now assume that dt is very small. Also we assume that the only paths that realy count are the straight lines between the given end points:

$$\xi(\tau) = y + (x - y)\frac{\tau - t}{dt} \quad . \tag{111}$$

The action is then

$$S = \frac{m}{2 \; dt}(y - x)^2 - V(y) dt \quad . \tag{112}$$

The factor N is an overall normalization of the paths. We now expand, assuming that for small dt the values of y that really contribute must be quite close to x (otherwise the velocity of the particle, entering in S, will be very large):

$$\begin{aligned} \psi(x, t + dt) &\rightarrow \psi(x, t) + dt \; \frac{\partial}{\partial t}\psi(x, t) \quad , \\ \psi(y, t) &\rightarrow \psi(x) + (y - x) \; \frac{\partial}{\partial x}\psi(x, t) + \frac{(y - x)^2}{2} \; \frac{\partial^2}{(\partial x)^2}\psi(x, t) \quad , \\ \exp\left(i\frac{S}{\hbar}\right) &\rightarrow \exp\left(\frac{im(y - x)^2}{2\hbar \; dt}\right)\left(1 - \frac{i}{\hbar}V(x) dt\right) \quad , \end{aligned} \tag{113}$$

To zeroth order in dt, we therefore have

$$\psi(x, t) = N \int_{-\infty}^{\infty} dy \; \exp\left(\frac{im(y - x)^2}{2\hbar \; dt}\right) \; \psi(x, t) \quad , \tag{114}$$

so that we find that N must satisfy

$$N \int_{-\infty}^{\infty} dy \; \exp\left(\frac{im(y - x)^2}{2\hbar \; dt}\right) = 1 \quad . \tag{115}$$

From standard rules about Gaussian integrals (see excercise (29), this then also implies that

$$N \int_{-\infty}^{\infty} dy \; (y-x)^2 \; \exp\left(\frac{im(y-x)^2}{2\hbar \; dt}\right) = i\frac{\hbar \; dt}{m} \quad , \tag{116}$$

which in fact justifies the approximation that y and x must be close to have an appreciable contribution from the path. This allows us to write out the terms of first order in dt:

$$\frac{\partial}{\partial t}\psi(x,t) = \frac{i\hbar}{2m} \; \frac{\partial^2}{(\partial x)^2}\psi(x,t) - \frac{i}{\hbar}V(x)\psi(x,t) \quad , \tag{117}$$

and multiplying by $i\hbar$ we then obtain

$$i\hbar \; \frac{\partial}{\partial t}\psi(x,t) = \left[-\frac{\hbar^2}{2m} \; \frac{\partial^2}{(\partial x)^2} + V(x)\right]\psi(x,t) \quad : \tag{118}$$

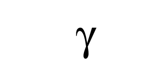
the Schrödinger equation! We see that the Feynman sum, with an appropriate choice of the action, gives us the correct quantum physics. Extending to four dimensions, and using relativistically formulated actions, we describe the behaviour of relativistic quantum particles.

4.4 The Feynman diagrams

Feynman diagrams are a practical application of the idea of Feynman sums. We start with a specified number of particles with well-defined momenta (say) in the initial state, and a likewise specified situation in the final state. The particles can do the following things.

- Each particle can *propagate*, that is, move freely without disturbance through spacetime from some spacetime point to another. This is described by lines, called *propagators*. Two examples are the propagator of an electron and that of a photon:

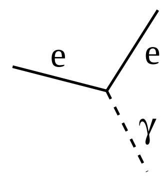
  

  and
  

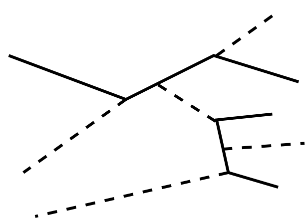
  The precise type of line is of course irrelevant, it is more important that they are used in a consistent manner[39]. The lines may also be curved without changing their meaning.

[39]In fact, a photon line is usually wavy; my graphical package can't do that, unfortunately.

- Three or more particles can meet in some spacetime point. This is denoted by a *vertex*, a place in a diagram where three or more lines come together. An example is a vertex where an electron an a photon meet, the photon gets absorbed, and the electron moves away on its own. The *same* vertex describes an electron coming in, emitting a photon and moving off. Here it is:



The rule is now to draw *all possible* diagrams leading from the initial state (which we draw on the left-hand side) to the final state (which we draw on the right-hand side). These correspond to the possible paths of the system from the initial to the final state.

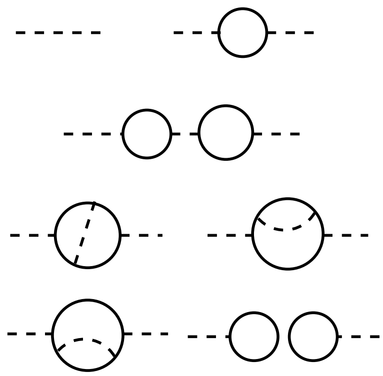


One of the 1680 possible diagrams for the process $e\gamma\gamma \to eee\gamma\gamma$ that have the minimum possible number of vertices.

Each diagram is actually just a complex number, to be added in the transition probability amplitude[40]. The numerical value of each diagram is made up out of a product of the several propagators and the *coupling constants* associated with each vertex. For the electron-electron-photon interactions discussed, the coupling constant is just ie, where e is the *electric charge* of the electron. Assuming the coupling constants to be small, we may be allowed to keep only those diagrams that contain the *minimum number* of vertices, and this is what is quite often done. For higher precision in the predictions, diagrams with more vertices are also included[41].

[40]The fact that we add the various contributions, and not multiply their exponentials, has to do with *perturbation theory*, in which we assume that the non-trivial transition amplitudes with at least one interaction are small compared to the trivial one, in which nothing happens.

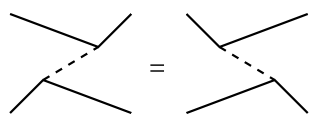
[41]Computing these ‘higher-order’ diagrams forms the basis of quite a number of careers in particle physics.



The diagrams for the process $\gamma \to \gamma$ with less than for vertices. In view of what is mentioned below, the last, disconnected diagram must, in fact, not be included.

Whatever the order, we must in all cases write down *all* the contributing diagrams (see excercise (30)). There is one other condition, namely that we are dealing with a *single process*. That means that every incoming particle must be connected to every outgoing particle in *some* way, and the Feynman diagram accordingly consists of a single piece (this is called a *connected* Feynman diagram). Diagrams consisting of two or more unconnected pieces really describe two processes taking place 'at once'.

The initial state, from which the propagators come into the interactions, and the final state towards which the propagators travel off after the interactions have taken place, are in some sense 'at infinity'. The intermediate interaction vertices can be anywhere they like[42]. We therefore have to sum over their positions in spacetime. Therefore, the two diagrams below are completely identical:



Whether an internal line *appears* to be moving from past to future, or backwards, is completely unimportant. The sum over all positions of the interaction vertices amounts to summing (or integrating) over all possible momenta travelling in the intermediate propagators. However, since we consider only physical theories with translation invariance, we have momentum conservation, also valid at each vertex. If a diagram does *not* contain any closed loops, the intermediate momenta are therefore usually fixed to one particular value[43]. After satisfying all momentum conservation restrictions, there is always one overall Dirac delta

[42] Sum over *all possible paths* — right?

[43] Diagrams with closed loops contain momenta that are not constrained by momentum conservation, and over which we therefore have to integrate. These integrals are not only very compli-

function of momentum conservation left. If we denote the incoming momenta by p_i^μ $(i = 1, 2, \ldots, n_{\text{in}})$ and the outgoing momenta by q_j^μ $(j = 1, 2, \ldots, n_{\text{out}})$, the probability amplitude for this process can therefore be written as

$$\text{amplitude} = (2\pi)^4\, \delta^4 \left(\sum_{i=1}^{n_{\text{in}}} p_i - \sum_{j=1}^{n_{\text{out}}} q_j \right) \mathcal{M} \quad , \tag{119}$$

where $\mathcal{M}$ contains all the other ingredients, the propagators and couplings involved in the process: it is called the *matrix element*, after the S-matrix.

4.5 Defining a particle world

If we want to understand the particle world, we have to specify (or conjecture) the particles' properties, in order to compute scattering amplitudes. That is, we have to specify the *propagators* of the theory, which describe the particles when they are free of any interactions. In addition we have to give the interactions we think are possible between the particles. Such a list of *Feynman rules* effectively tells us all we know about the particles — it is equivalent to specifying the particle world.

4.6 The Feynman propagator

The interaction vertics inside a Feynman diagram can, in the simplest theories, be represented by a single numerical factor. The propagators are different, since they must embody the particle's intrinsic properties such as its mass, spin, and for unstable particles also their lifetime, and moreover describe how they move through spacetime: ideally, the propagator, linked to *free* paricles, should imply Newton's First Law (in fact it does!).

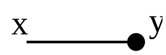
The propagator is most easily viewed as a *Green's function*. Let us describe the *quantum field* (the relativistic generalization of the wave function' of a particle) by $\phi(x)$, where $x^\mu = (ct, \vec{x})$ is the spacetime point considered. Moreover, suppose that a 'source' of particles is present, which we describe by $J(x)$. The quantum field must respond to the presence of the source, much as the electromagnetic field must respond to the presence of charged particles. This is done by the propagator

cated but also, usuallly, divergent. This problem can be handled by *regularization* and *renormalization*, but this would lead us too far afield.

Δ:

$$\phi(x) = \int_{-\infty}^{\infty} dy\ \Delta(x-y)\ J(y)\ \ . \tag{120}$$

In terms of Feynman diagrams this can be written simply as

$\phi(x) \quad = $ 

,

where we have indicated with a dot the interaction with the source (to be integrated over all y as discussed, since it is a vertex). Note the fact that this equation looks very similar to Eq.(109). The choice commonly made for the propagator of the simplest type of particle is

$$\Delta(x-y) = \frac{i}{(2\pi)^4} \int_{-\infty}^{\infty} d^4k\ \frac{e^{-i(x-y)\cdot k}}{k\cdot k - m^2 + im\Gamma}\ \ . \tag{121}$$

The propagator takes its simplest form in the above Fourier-integral expression. If we express the Feynman diagrams in terms of momenta (as is usual), the propagator corresponding to a line with momentum p in the diagram then carries the factor

$$\frac{i}{p\cdot p - m^2 + im\Gamma}\ \ .$$

The physical interpretation of m and Γ will be discussed next.

4.7 An important integration result

We shall use the following integral, which can actually be quite simply proven using complex analysis. Let ω be a complex number with *negative* imaginary part. Then, for a real number x the integral

$$H(x) \equiv \frac{i}{2\pi} \int_{-\infty}^{\infty} dk\ \frac{e^{ix\cdot k}}{k^2 - \omega^2}\ \ , \tag{122}$$

where the integral over k runs over the real axis, is

$$H(x) = \exp(-i|x|\omega)\ \ , \tag{123}$$

where $|x|$ denotes the absolute value of x

4.8 Unstable particles

First, let us consider a source that produces particles all over space, precisely at the moment $t = 0$:

$$J(y) = \delta(y^0) \quad . \tag{124}$$

Some straightforward integration now leads us along the following path:

$$\begin{aligned}
\phi(x) &= \int_{-\infty}^{\infty} d^4y \; \Delta(x-y) \; J(y) \\
&= \frac{i}{(2\pi)^4} \int_{-\infty}^{\infty} dy^0 \; d^3\vec{y} \; dk^0 \; d^3\vec{k} \; \frac{e^{-i(x^0-y^0)k^0} e^{i(\vec{x}-\vec{y})\cdot\vec{k}}}{(k^0)^2 - |\vec{k}|^2 - m^2 + im\Gamma} \; \delta(y^0) \\
&= \frac{i}{(2\pi)^4} \int_{-\infty}^{\infty} d^3\vec{y} \; dk^0 \; d^3\vec{k} \; \frac{e^{-ix^0k^0} e^{i\vec{x}\cdot\vec{k}} (2\pi)^3 \delta^3(\vec{k})}{(k^0)^2 - |\vec{k}|^2 - m^2 + im\Gamma} \; \delta(y^0) \\
&= \frac{i}{2\pi} \int_{-\infty}^{\infty} dk^0 \; \frac{e^{-ix^0k^0}}{(k^0)^2 - m^2 + im\Gamma} \quad ,
\end{aligned} \tag{125}$$

where we have used the identity

$$\int_{-\infty}^{\infty} d^3\vec{y} \; \exp(i\vec{y}\cdot\vec{k}) = (2\pi)^3 \; \delta^3(\vec{k}) \quad . \tag{126}$$

The integral expression for $\phi(x)$ is precisely of the form discussed above, provided we take

$$\omega = \left(m^2 - im\Gamma\right)^{1/2} \approx m - i\frac{\Gamma}{2} \quad , \tag{127}$$

where we have assumed $\Gamma \ll m$. We therefore obtain

$$\phi(x) = \exp\left(-im|x^0| - \frac{\Gamma}{2}|x^0|\right) \quad . \tag{128}$$

The *probability* to find particles at time $x^0 > 0$ is then of course

$$\text{probability} = |\phi(x)|^2 = \exp(-\Gamma x^0) \quad ,: \tag{129}$$

the probability decreases exponentially! This is precisely what we would expect for particles that disappear by *decaying*. The average lifetime of these particles is the that case $1/\Gamma$. Γ is called the *decay width* of the particle: it is small for quite stable particles, and large for unstable particles.

What about stable particles? We treat them by letting Γ become very very small indeed, in fact *infinitesimally small*[44]. For stable patricles, the propagator in momentum language is therefore taken to be

$$\frac{i}{p\cdot p - m^2 + i\epsilon} \ ,$$

where ϵ is taken to be infintesimally small, but *positive*, since our integral formula only works if ω has a *negative* imaginary part.

we now also see the meaning of m: the propagator (4.8) peaks enormously whenever $p\cdot p$ is close to m^2. Particles therefore propagate almost exclusively when the square of their momentum is extremely close to m^2. The number m is therefore nothing but the *mass* of the particle!

The two constants m and Γ specify whatever there is to know about a particle without information on its interactions: to wit, mass and lifetime. Of course, particles may have more properties such as *spin*. These properties are implemented by using a more complicated propagator: the denominator of the form (4.8) is, however, always there since it governes such basic properties. It is generalizing the *numerator* that includes spin. In view of the more complicated mathematics involved, we shall not discuss this here.

4.9 Backwards in time: antimatter

Let us consider the Feynman propagator for a stable particle. Define

$$E(\vec{k}) = \sqrt{\vec{k}^2 + m^2} \ , \tag{130}$$

which is a positive number. It can be shown that if the particle propagates with momentum k^μ, not only does the main contributing paths come from those momenta that satisfy $k\cdot k = m^2$ which implies $(k^0)^2 = E(\vec{k})^2$, but even more: if $x^0 > y^0$, then the dominant contribution comes from those momenta that have $k^0 = E(\vec{k})$, and for $x^0 < y^0$ the dominant contribution comes from momenta that have $k^0 = -E(\vec{k})$. In fact we may write Eq.(121) for stable particles as

$$\phi(x) \ = \ \frac{1}{(2\pi)^3}\int\limits_{y^0<x^0} d^4y \ d^3\vec{k}\ \frac{1}{2E(\vec{k})}\exp\left(-i(x^0-y^0)E(\vec{k}) + i(\vec{x}-\vec{y})\cdot\vec{k}\right)\ J(y)$$

[44]In our physics, there should not be any discernible difference between strictly stable particles and particles with a lifetime of, say, 10^{314} years.

$$+ \quad \frac{1}{(2\pi)^3} \int\limits_{y^0>x^0} d^4y \; d^3\vec{k} \; \frac{1}{2E(\vec{k})} \exp\left(+i(x^0-y^0)E(\vec{k}) + i(\vec{x}-\vec{y})\cdot\vec{k}\right) \; J(y) \quad . \tag{131}$$

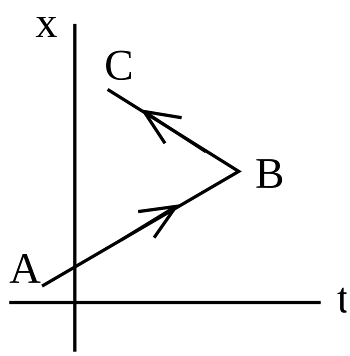
Here we recognize precisely the plane waves of quantum mechanics! The source 'sends out' plane waves, on their *mass shell* $k \cdot k = m^2$, and these build up the quantum field everywhere. We also see that, of these pane waves, the ones with *positive energy* ($k^0 = E(\vec{k})$) propagate into the *future* ($x^0 > y^0$), while the ones with *negative energy* ($k^0 = -E(\vec{k})$) propagate into the *past* ($x^0 < y^0$).

But what is this: particles with *negative* kinetic energy, propagating into the *past*? Doesn't this violate what we think we know about nature? No. In fact, a *negative* energy moving towards the *past* has exactly the same effects as en equally large *positive* energy moving towards the *future*. There is, of course, *some* difference. Suppose we consider an electron with negative energy, and negative charge, moving to the past. This corresponds to a particle with *positive* charge and energy moving towards the future! This particle is well-known: it is the *positron*.

Here we have a fundamental prediction arrived at by combining quantum theory with relativity: *for any particle type with given mass and lifetime, there exists an* **anti***particle type with the same mass and lifetime, but with opposite charge (or other such quantum numbers such as colour)*. The particle and antiparticle come from the same Feynman propagator, and therefore they must have identical mass and lifetime. For the electron, the positron: for the proton, the antiproton: for the neutron, the antineutron, and for the quark the antiquark, and so on. Some particles, such as the photon, happen to be their own antiparticle: but in that case they must be electrically neutral.

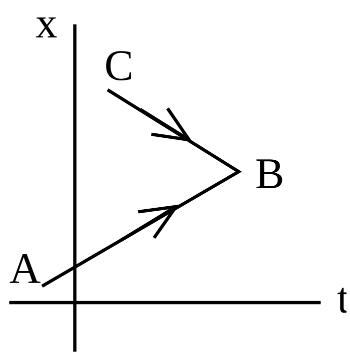
4.10 Annihilation and pair production

Consider the following history for a particle moving through space and time that is depicted in the following spacetime diagram:



The particle starts at spacetime point A and then moves, with positive energy E_1, forward to point B. There, for some reason, its energy gets the *negative* value $-E_2$, so that it will move backwards in time, now to point C. The particles 'direction' is given by the arrows. The particle must have dumped an amount of energy $E_1 + E_2$ at B for this to happen.

Now consider the physically indistinguishable history depicted below:

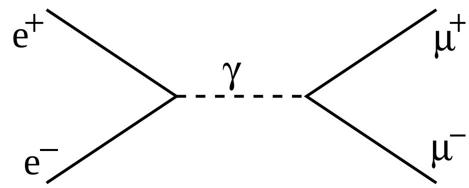


In *this* formulation, there is a particle starting at A, with energy E_1; and there is an *anti*particle, starting at C, with energy E_2. They both trave forward in time (arrows!) and meet at B, where they 'appear' to *annihilate* one another, dumping a total amount of energy $E_1 + E_2$.

In both cases, other particles or antiparticles carrying this energy must emerge. This is the celebrated annihilation of matter and antimatter. Note that this can only happen for a particle and *its own* antiparticle. For instance, there exists the *muon* particle μ^-. This particle is essentially the same as an electron, having the same charge, spin, and other quantum numbers: only its mass is different, the muon being some 200 times heavier than the electron. In spite of the similarities, a muon μ^- can never annihilate a positron e^+, but only its own antiparticle, the antimuon μ^+.

We may also reverse the pictures in time, and then we see that the energy released in annihilation may as well emerge as pairs of particles and antiparticles, the so-called *pair creation*.

As a further illustration, we can combine the annihilation of a positron-electron pair into a photon, followed by the photon's pair-producing a muon and its antimuon, into the following spacetime picture, where we have left out the space and time axes:



This is, *of course*, precisely the (lowest-order) Feynman diagram for this process!

4.11 Feynman rules with antiparticles

Particles and their antiparticles are described by the *same* Feynman propagator: in fact, that is where the antiparticles come from. There are therefore no new rules. We simply have additional possible processes. The only restriction on these in the case of QED is that not only momentum but also electric charge must be conserved in each vertex. We can therefore write down vertices in which an electron an positron annihilate into a photon, but not one where two electrons or two positrons combine; likewise, an electron, after emitting a photon, cannot continue as a positron but must live on as an electron.

4.12 Excercises

Excercise 27 Natural units

1. A particle has a mass of 1 GeV in natural units. What is its mass in kilograms?
2. An object has an area of $1/\text{GeV}^2$. what is its area in centimeter squared?
3. A process takes a time of $1/\text{GeV}$. What is this time in seconds?

Excercise 28 Some sizes
For the two-slit experiment, prove Eq.(108). Argue that an interference pattern will only be observable if the distance d between the slits is *not* much smaller than the wavelength of the object considered. Assume that the photons making up red laser light have an energy of about 2 eV, compute the relevant minimum value of d. Do the same for electrons, using the de Broglie wavelength of electrons with an energy of 50000 eV. Argue that this makes it reasonable to send the electrons not through carved-out slits in material but rather right through the atomic lattice of a solid.

Excercise 29 Gaussian integrals
Consider the integral

$$K = \int_{-\infty}^{\infty} dx \ \exp\left(-\frac{x^2}{2\sigma^2}\right) \quad . \tag{132}$$

Evaluate this integral by doing the famous 'doubling trick'.

1. Show that K^2 can be written as

$$K^2 = \int_{-\infty}^{\infty} dx \ dy \ \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right) \quad .$$

2. Go over to polar coordinates: $x = r\sin(\phi)$, $y = r\cos(\phi)$. Show that

$$K^2 = \pi \int_{0}^{\infty} dr \ r \ e^{\frac{-r^2}{2\sigma^2}} \quad .$$

3. Go over to a new variable t, defined by $r = \sqrt{2\sigma^2 t}$. Show that

$$K^2 = 2\pi\sigma^2 \int_{0}^{\infty} dt \ e^{-t} = 2\pi\sigma^2$$

and therefore that

$$K = \sqrt{2\pi\sigma^2} \quad . \tag{133}$$

By differentiating the result (133), prove that

$$\int_{-\infty}^{\infty} dx \ x^2 \ \exp\left(-\frac{x^2}{2\sigma^2}\right) = \sigma^2 \ \sqrt{2\pi\sigma^2} \quad . \tag{134}$$

Now put $\sigma^2 = -i\hbar \ dt/m$ and prove Eq.(116).

Excercise 30 Writing down diagrams
We consider diagrams for the relativistic quantum theory of electrons and photons. This is called Quantum electrodynamics (QED), and contains: one propagator for electrons, one propagator for photons, and one vertex where two electron propagators and one photon propagator meet. There are no other vertices in QED. Denote

electrons by e, and photons by γ. Write down the simplest diagrams for the process $e \to e\gamma$ and $e\gamma \to e$: show that in each case there is precisely *one* diagram with one vertex. Show that there are precisely *four* diagrams with three vertices. Prove that there can be *no* diagrams with two vertices for these processes. Also, consider the process $ee \to ee$. Show that there are *two* diagrams with two vertices, and *fourteen* with four vertices. Show that there are in this case *no* diagrams with three vertices. Write out the *eight* diagrams for the process $ee \to ee\gamma$ in lowest order: show that these have three vertices.

Excercise 31 Lifetime of the muon
The muon particle has a mass of about 0.1 GeV and a lifetime of some $2\ 10^{-6}$ seconds. Compute its decay width in GeV. Is it indeed small compared to the mass?

Excercise 32 Electron-positron annihilation
We shall denote electrons by e^- and positrons by e^+. Write down the simplest diagram that describes $e^+e^- \to \gamma$. The mass of the electron (and of the positron!) is about $5\ 10^5$ eV. Prove that an electron and a positron can *not* annihilate together into a single photon (hint: total mass). Prove that it *is* possible for them to form *two* photons: $e^+e^- \to \gamma\gamma$, and display the two diagrams that describe this in lowest order. Assuming the electron and the positron to annihilate at rest: what is then the energy of each photon?

Suppose you are injected with a radioactive substance that emits positrons, what will happen?

Excercise 33 Crossing diagrams
Crossing Feynman diagrams means moving particles from the initial state to the final state, or vice versa. Write the two lowest-order diagrams describing Compoton scattering: $e^-\gamma \to e^-\gamma$, and show how they are related by crossing to those for $e^+e^- \to \gamma\gamma$.

Excercise 34 Bhabha and muon production
We can extend QED by also allowing for muons (see above).. The muons have a $\mu\mu\gamma$ vertex just like the electrons, but there is *no* $e\mu\gamma$ vertex. Show that for Bhabha scattering: $e^+e^- \to e^+e^-$, there are *two* diagrams at lowest order (and give them), but that $e^+e^- \to \mu^+\mu^-$ is, in the same order, described by only *one* diagram (and give it).

5 Particles of the Standard Model

We present a brief discussion of those known particles believed[45] to be truly elementary. These have been very successfully collected into what has become known as the *Standard Model* (SM). It should be stressed at this point that the notion of 'model' employed here has its own interpretation. Although the current SM, when it was proposed, was only one out of a number of competing (and possibly inappropriate) ideas of what the elementary particles are, and how they interact, by now the amount of experimental evidence is so overwhelming that essentially nobody believes that the SM is getting things fundamentally wrong. It is fair to say that at present[46] there is no experimental result that seriously contradicts its predictions. In particular, the particles that the SM presents as established *really exist*, in the same way that molecules and atoms really exist.

It goes without saying that new eperimental information may become available to indicate the existence of either additional classes of particles (such as in supersymmetry) or of actual substructure of the known particles. The SM will then have to be either extended (in the first case), or replaced by something more fundamental (in the second case). Nevertheless, any successor to the SM will have to reproduce all its successes.

5.1 Spin, and statistics

The Feynman propagator discussed so far pertains, strictly speaking, to particles that do not transmit any information through spacetime than momentum: this involves the mass m and, if relevant, the total decay width Γ.

Slight generalizations of the Feynman propagator[47] allow the description of particles that carry, in addition, the *intrinsic* angular momentum known as *spin*. The Feynman propagator as discussed so far describes *spin-0* particles.

The simplest generalization of the propagator leads to particles with intrinsic spin equal to $\hbar/2$, or *spin-1/2 particles*. These are also known as *fermions* because they obey Fermi statistics. Perhaps the simplest example is that of the electron. Consider a state consisting of two electrons, with momenta p_1^μ and p_2^μ. Due to

[45] Anno 2006

[46] Anno 2006, again!

[47] Because of the various important consequences derivable from the Feynman propagator, such as Newton's first law, finite lifetimes with exponential decay, and the mass shell, such generalizations do not affect the *denominator* of the propagator but rather its *numerator*.

Fermi statistics, the quantum state

$$|p_1, p_2\rangle$$

does not, actually, exist: the physically existing combination is the antisymmetric one:

$$\frac{1}{\sqrt{2}}\Big(|p_1, p_2\rangle - |p_2, p_1\rangle\Big) \quad ,$$

so that any of the two momenta is not actually assignable to any specific electron: we always have to take into account this antisymmetric combination. In terms of Feynman diagrams, this means that two Feynman diagrams in which two electrons have their rôles interchanged have a relative minus sign. In fact, the rule is more general: whenever one Feynman diagram is transformed into another by the interchange of *any two fermions*, their relative signs are opposite. It may be possible that the rôles of, say, an electron and a quark in a diagram cannot be interchanged because there is no interaction vertex available in any of the two cases, but then the alternative diagram simply has the value zero. The antisymmetry also explains the Pauli principle: two electrons cannot be in the same state. In the above example, putting $p_1 = p_2$ makes the state vanish[48], and it is also seen from the Feynman minus-sign rule that it is not possible to produce two fermions in exactly the same state.

The next generalization of the Feynman propagator describes particles with intrinsic angular momentum equal to $\hbar$, or *spin-1 particles*. Now, consider the fact that from *two* spin-1/2 states can be combined into either a spin-0 or a spin-1 state. Therefore, interchanging either two spin-0 or two spin-1 particles amounts to interchanging *two pairs* of spin-1/2 particles, and hence no minus sign arises: the spin-0 and spin-1 particles obey *Bose* statistics, and as many of them as you like can be in the same quantum state.

The above connection between the spin of particles and their statistics is known as the *spin-statistics theorem*. It is seen to follow directly from the single prescription that spin-1/2 particles must obey Fermi statistics.

[48] A word of caution is in order here. Since the electrons have also a spin quantum number, their combined state will only vanish if also the spin quantum numbers are identical. If the spins happen to be opposite, they can make up a physical state. Indeed, the Pauli principle in its old form states that *at most two* electrons can share the same quantum numbers: but this dates from before the discovery of their spin.

Particles with higher spin are, in principle, possible, although their description becomes increasingly complicated with increasing spin. Particles with spin-2 are predicted (the graviton), and in supersymmetric theories also spin-3/2 particles (the gravitino), but spins higher than 2 are not expected to be seen anytime soon[49].

A final, but important remark: just as momentum must be conserved in every vertex in Feynman diagrams because of the translation invariance of our physics, so *spin* must be conserved as well[50] because of the rotation invariance of our physics. This puts restrictions on the possible couplings between particles. In any vertex, the number of half-integer spins participating must be *even*.

5.2 Other quantum numbers

In addition to mass, lifetime, and spin particles have other intrinsic properties, namely the strengths of their couplings to other particles. For instance, the strength with which a particle couples to the photon is the *electric charge* of that particle. The strength with which a quark couples to gluouns is the *colour charge* of that particle. Fermions couple to W and Z bosons with various *weak charges*, and so on. We shall describe them in due course.

5.3 The spin-1/2 particles

'up'-type quarks $Q = +2e/3$ 3 colours	**u:** up $m \approx 0.005$	**c:** charm $m \approx 1.45$	**t:** top $m \approx 175$
'down'-type quarks $Q = -e/3$ 3 colours	**d:** down $m \approx 0.007$	**s:** strange $m \approx 0.15$	**b:** bottom $m \approx 4.5$
neutral leptons $Q = 0$ no colour	ν_e**:** neutrino $m \approx 0$	ν_μ**:** neutrino $m \geq 0$	ν_τ**:** neutrino $m \geq 0$
charged leptons $Q = -e$ no colour	e**:** electron $m \approx 0.0005$	μ**:** muon $m \approx 0.1$	τ**:** tau $m \approx 1.78$

[49] In the theory of strings, states (not particles!) with arbitrarily high spin occur, but these states are expected to have masses of the order of the Planck mass M_P, hence the 'not anytime soon'...

[50] Since the vertices are *local*, that is, the particles must meet in one spacetime point, their relative *orbital angular momentum* must vanish. In each vertex, therefore, the total angular momentum involved is just the spin of the particles.

The spin-1/2 particles are here grouped in three 'families' consisting in four fermion types each. There are, at present, no indications of a fourth family. The first family comprises the 'normal' fermions; but this is solely due to the fact that these are the lightest ones, and so they do not easily decay. These fermions are, therefore, the ones encountered 'in the wild'. The reason for grouping the fermions into the families as shown here is the *weak interaction*. The W particle couples to the pair of (u,d) quarks much more readily than to the pair of (u,s) quarks, and so on. The (e,ν_e) pair couples very very much more readily than the pair (e,ν_μ), and so on. Apart from that, no *a priori* classifying argument exists. Of course, the observed pattern has led to a great number of speculations about a deeper-lying pattern for its explanation — so far without much of an experimental verification or falsification.

Another, and much thought-about, pattern is that of the masses. Even neglecting the neutrino masses (which appear to be so small as to demand a separate explanation), the 350000-times difference between the top quark mass and the electron mass seems to be outside the bounds of normal 'family diversity'. Again, any solid explanation is missing...

5.4 The spin-1 particles

Interaction	Particle	Symbol	Mass	Charge	# colours
Strong	gluon	g	0	0	8
Electromagnetic	Photon	γ	0	0	0
Weak	Charged boson	$W^\pm$	≈ 80.2	$\pm e$	0
	Neutral boson	Z^0	≈ 91.17	0	0

The fact that the spin-1 particles are bosons means that (energy and momentum permitting) they are readily created and annihiliated. Therefore, *these* particles are considered to 'mediate the interactions' — although, as we have seen, particles interact only when they *meet*.

The gluons couple to coloured systems. These may be single quarks, or other gluons, or various combinations of quarks and gluons. They do not couple to systems without colour, such as leptons or photons or weak-interaction bosons. Also particular combinations of quarks that are *colourless* are free from gluonic interactions.

The photons couple to any object with electric charge: quarks, charged leptons, W particles. Some systems, such as the proton-electron combination known as the hydrogen atom, are to some extent free from photon interactions. However, this statement must be qualified: photons with low energy and (hence) large wavelength are unable to resolve the details of the hydrogen atom, and therefore the hydrogen atom is *neutral* as far as static, or almost-static electromagnetic fields are concerned; photons with short wavelengths and (hence) higher energy *can* resolve the detailed structure of the hydrogen atom, and at such small scales the atom is not purely neutral. Indeed, photons of 13.6 electronvolts can break the atom up completely.

The weak bosons couple to *all* fermions (as well as to each other). Indeed, of the three fundamental ineractions mentioned, the neutrinos feel only the weak interaction — this is the reason why they are so difficult to detect. In some deep way, the weak ineractions appears to be related to the family structure of the SM, but nobody can really say much more than this[51].

5.5 Gravitons and Higgses

A final theory of the fundamental particles and their interactions would remain stunted without the inclusion of gravitation. Indeed, relativistic quantum theory indicates that also the gravitational interactions might give rise to a boson, the *graviton*. This, then, would be a massless particle of spin-2. Unfortunately, gravity is such a weak force that even the *classical* waves in the gravitational field have so far eluded direct detection[52]. A theoretically painful problem is that a relativistic quantum theory that involves gravity can be shown to be incurably inconsistent[53]. String theory may provide a way out here, but this is beyond the scope of these lectures.

The more accessible missing part of the SM is the so-called Higgs sector. This is intimately linked with, secondly, our perplexity about the patterns of fermion and boson masses, and, firstly, the queston why these particles have mass in the first place. In relativistic quantum theory, the inclusion of nonzero masses of the particles will make the theory inconsistent unless at least *one* type of *neutral, spin-0* particle, having no strong interactions, is also present. This is the 'minimal'

[51]Anno 2006, once more...

[52]The famous Hulse-Taylor binary pulsars lose energy steadily as they rotate about one another, and this energy loss corresponds precisely to what general relativity predicts; therefore the only imaginable way for them to lose this energy is through the emission of gravitational waves. These waves are, therefore, very probable, but so far no-one has actually *detected* such a wave.

[53]Technically, such a theory is *non-renormalizable*.

Higgs particle. It couples to other particles proportionally to their masses (and also to itself, proportionally to its own mass). Finding the Higgs particle involves studying the weak interactions at the scale of several hundreds of GeV: the perfect job for the LHC.

5.6 Excercises

Excercise 35 Fermi minus signs
Consider QED to lowest order. Draw the two Feynman diagrams for $e^-e^- \to e^-e^-$ and determine their relative sign. Do the same for the two diagrams involved in Bhabha scattering, $e^+e^- \to e^+e^-$. Write down the twelve diagrams for the process $e^+e^- \to e^+e^+e^-e^-$, and determine their relative signs.

Excercise 36 Thinking up vertices
Taking into account conservation of spin, charge and colour in each vertex, write down all possible vertices involving: (a) two fermions and a boson; (b) three bosons; (c) four bosons.

Excercise 37 The R number
We study the process of e^+e^- annihilation into a fermion-antifermion pair.

1. Write down the single Feynman diagram for $e^+e^- \to \mu^+\mu^-$ by the mediation of a photon.

2. Write down the corresponding single Feynman diagram for $e^+e^- \to q\bar{q}$, where q stands for any quark type and $\bar{q}$ for its antiquark.

3. From now on, assume that the coupling of muons to the photon is identical to that of the quarks to the photon, apart from the charges and colours (this is true!) Explain why the probability amplitude of the production of up-antiup quarks of any given colour type is, in absolute value, 2/3 of that of muon-antimuon production.

4. Considering that the *probability* of a given transition is the absolute values squared of the *probability amplitude*, explain why the probability to produce $u\bar{u}$ is 4/3 of that of producing a muon-antimuon.

5. Do the same for the other quark types.

6. Imagine the following process: we collide e^+ and e^- beams in their common centre-of-mass frame, steadily increasing the total energy. Explain why the ratio

$$R_{\text{hadronic}} = \frac{\text{Probability of producing } \textit{any } q\bar{q} \text{ pair of } \textit{any} \text{ colour}}{\text{Probability of producing muon-antimuon pairs}}$$

would take one the successive values 0,4/3, 5/3, 3, 10/3, 11/3, 5, as the energy increases from twice the electron mass (the obvious minimum) to more than twice the top quark mass.

7. Explain why the same ratios are *not* expected when we compare the quark-antiquark production to electron-positron production (Hint: see excercise 35).

6 Quantum Chromodynamics

Electromagnetism is one of the best-understood fundamental forces. The theory of the strong interaction bears resemblance to it, but there are also important differences.

6.1 Quarks and their colour

The strongly-interacting sector of the SM is governed by the existence of a quantum number which is commonly called 'colour'[54]. The important fact about this 'colour' is that quarks can come in three types, conventionally denoted by red (r), green (g) and blue (b), or whatever[55] . This quantum number plays the same rôle in the theory of Quantum Chromo-Dynamics (QCD) as does electric charge in that of Quantum electro-Dynamics, with the difference that there is only one type of electric charge[56], whereas the three colour charges are distinct from one another.

It must be remarked that the *antiquarks* posses the opposite colcour charges, that is, they have *anticolours* antired ($\overline{r}$), antiblue ($\overline{b}$) and antigreen ($\overline{g}$).

Finally, our world being quantum mechanical, we must realize that strongly interacting particles will usually occur in a *superposition* of colour states, for instance, a general state $|\psi\rangle$ for a quark can be written as

$$|\psi\rangle = z_r|r\rangle + z_b|b\rangle + z_g|g\rangle \quad , \quad |z_r|^2 + |z_b|^2 + |z_g|^2 = 1 \ . \tag{135}$$

6.2 Strong interactions, whiteness, and confinement

Another difference between QED and QCD is in the strength of the interactions. From excercise (38) it is seen that the strong interaction is many times more powerful that the electromagnetic one (hence the name). It has to be if it is to overcome the electric repulsion between the protons in nuclei, and between the same-charge quarks inside nucleons. As a rule of thumb one might say that any two systems

[54]To the best of our knowledge, quarks are pointlike particles, with zero diameter: but even if they have some size, this size is so very very much smaller than the wavelength of visible light, that quarks do not have a colour in the commonsense sense.

[55]Insert here your favourite three colours. The very idea of calling these quantum numbers *colours* comes from the theory of colour perception, in which red, green and blue combine to produce white: as is indeed proved by any colour TV.

[56]In the sense that the charges of differently charged particles may be added to give the net charge of the whole system.

with net colour charge are very strongly attracted to one another, and only 'white' systems, with no net colour charge, can occur freely. Note here a qualitative difference between QED and QCD. In QED, one can overcome the Coulomb attraction between particles by inserting enough kinetic energy into the system (so that they reach the 'escape velocity'). In QCD this appears *not* to be possible[57]: by hitting a proton hard enough, one can smash it into colourless fragments such as pions, kaons, and protons (*sic*), but one cannot break it up into three single quarks.

6.3 Hadrons

What, then, are the 'white' objects that can be constructed from coloured quarks? These should be quantum states that do not change (or change only by a trivial overall complex phase) if we decide to change the orthonormal basis of colour states. Excercise 39 shows that it is possible to construct such quantum states containing *three* quarks: these are the *baryons*, of which the proton and neutron are the best-known specimens[58]. Using antiquarks rather than quarks one can also construct *antibaryons* such as antiprotons and antineutrons[59]. It is also possible to combine a quark with an antiquark into a white combination: these are called *mesons*. It is *not* possible to combine, say, two quarks, or two quarks and one antiquark into a white quantum state, and indeed the *hadrons*, the strongly interacting particles encountered in nature or experiment are always either (anti)baryons or mesons. It is possible to make more exotic states, for instance a state consisting of six quarks, or of four quarks and one antiquark: but not much keeps such states from rapidly falling apart into (in the first case) two baryons, or (in the second case) one baryon and one meson.

The hadrons, being 'white', are free from the most powerful strong interactions, and therefore have a possibility of propagating through spacetime: there are, however, residual strong interactions following from the fact that *internally* they are very coloured indeed[60]. Therefore, they interact with one another. Although 'residual', it is this interaction that keeps nuclei together, so it is still quite strong when compared to the electromagnetic force.

[57] This has not yet been proven rigorously.

[58] Lots of other baryons are known and can be produced in the laboratory, but their lifetimes are very small.

[59] Neutrons are neutral, but antineutrons are *not* equal to neutrons.

[60] This is the analogue of the van der Waals force, by which two neutral atoms feel a slight electromagnetic interaction because of the difference in distribution of their internal positive and negative charges.

6.4 Examples of hadrons

Many 'white' hadronic states are known, and it is instructive to discuss the flavour content of the more familiar ones. In here, we shall leave aside the detailed dynamical description of hadron structure, but only focus on their gross 'large-scale' buildup[61].

The mesons are bound states of a quark and an antiquark. We have the *pions*, made up from the lightest quarks u and d:

$$
\begin{aligned}
|\pi^+\rangle &= |u\bar{d}\rangle \quad \text{mass about 140 MeV} \;, \\
|\pi^-\rangle &= |d\bar{u}\rangle \quad \text{mass about 140 MeV} \;, \\
|\pi^0\rangle &= \frac{1}{\sqrt{2}}\left(|u\bar{u}\rangle - |d\bar{d}\rangle\right) \quad \text{mass about 135 MeV} \;; \qquad (136)
\end{aligned}
$$

There is also the combination

$$
|\eta\rangle = \frac{1}{\sqrt{2}}\left(|u\bar{u}\rangle + |d\bar{d}\rangle\right) \;, \qquad (137)
$$

with a larger mass, of about 550 MeV. If we also include strange quarks, we have the *kaon* particles

$$
\begin{aligned}
|K^+\rangle &= |u\bar{s}\rangle \quad \text{mass about 494 MeV} \;, \\
|K^-\rangle &= |s\bar{u}\rangle \quad \text{mass about 494 MeV} \;, \\
|K^0\rangle &= |d\bar{s}\rangle \quad \text{mass about 498 MeV} \;, \\
|\bar{K}^0\rangle &= |s\bar{d}\rangle \quad \text{mass about 498 MeV} \;. \qquad (138)
\end{aligned}
$$

Note that, whereas the neutral pion is its own antiparticle, the neutral kaon is *not* equal to its own antiparticle[62]. These bound states all have their excited analogues, with higher angular momentum. Of course, many other meson combinations can

[61] Quantum mechanics tells us that, in order to study hadrons in details much smaller than their sizes, we have to irradiate them with particles of momentum equal to the hadron's mass or, preferably, even quite a bit more. In such high-energetic collisions, it is easy to excite particles and antiparticles from the vacuum, and the gross structure of the hadron tends to become washed out. Examined at a scale much smaller than its radius a proton, for instance, does not appear as three quarks held together by gluons, but rather as a rich soup containing many quarks, gluons, and antiquarks. As an illustration, it is therefore possible (and it happens in scattering processes) that we find (anti)strange quarks inside a proton.

[62] The neutral kaons form a very important system for the study of the particle-antiparticle relationship.

also occur, such as the $\phi \approx |s\bar{s}\rangle$. If we include even heavier quarks, we can for instance build up D mesons:

$$\begin{aligned} |D^+\rangle &= |c\bar{d}\rangle \quad \text{mass about 1870 MeV} \quad , \\ |D^-\rangle &= |d\bar{c}\rangle \quad \text{mass about 1870 MeV} \quad , \\ |D^0\rangle &= |c\bar{u}\rangle \quad \text{mass about 1865 MeV} \quad , \\ |\bar{D}^0\rangle &= |u\bar{c}\rangle \quad \text{mass about 1865 MeV} \quad . \end{aligned} \tag{139}$$

Of particular interest are the following mesons:

$$\begin{aligned} |\psi\rangle &= |c\bar{c}\rangle \quad \text{mass about 3097 MeV} \quad , \\ |\Upsilon\rangle &= |b\bar{b}\rangle \quad \text{mass about 9460 MeV} \quad . \end{aligned} \tag{140}$$

Of the baryons, bound states of three quarks, the most familiar are of course the proton and the neutron:

$$\begin{aligned} |p\rangle &= |uud\rangle \quad \text{mass about 938.3 MeV} \quad , \\ |n\rangle &= |udd\rangle \quad \text{mass about 939.6 MeV} \quad . \end{aligned} \tag{141}$$

These are the lightest baryons: heavier ones such as the $|\Delta^{++}\rangle = |uuu\rangle$ abound. Also baryons with strange, charm, and even bottom quarks are known.

6.5 Gluons and quark-gluon vertices

As an analogue to the electromagnetic interaction, the strong interactions must have their own ‘photon’: these particles are called the *gluons*[63]. Just like there is a vertex where two charged particles and a photon meet (in QED), there must therefore be a vertex in which two quarks and a gluon meet, with one important difference: the quarks meeting may have *different colours*. Since we want colour conservation to hold at the vertices[64], the gluon must therefore *carry away* the colour of the incoming quark, and *supply* the colour of the outgoing quark (in other words, supply the *anticolour* of the *incoming antiquark*: gluons, therefore

[63]The name arises from the fact that the quarks are supposed to be ‘glued’ together in the nucleons: another example of the brilliance of nerd-wit.

[64]This is not simply an *ad hoc* idea. After all, the universe at scales larger than the subnucleonic is made up from colourless, ‘white’ ingredients such as leptons and hadrons: therefore, it is invariant under colour basis changes. Translating this to the individual vertices, colour conservation at each vertex appears the only way to preserve this large-scale ‘whiteness’.

carry one colour and one anticolour. There are therefore red-antiblue gluons, and the like. In total, for 3 colours, we then arrive at 9 different gluon types. However, there is one important remark to be made here. Since the guons are (as everything in the universe) quantum systems, they can occur in superpositions of colour states. Now, the gluon colour state

$$|\text{‘singlet gluon’}\rangle = \frac{1}{\sqrt{3}}\left(|r\bar{r}\rangle + |g\bar{g}\rangle + |b\bar{b}\rangle\right) \tag{142}$$

has exactly the same colour structure as a meson (see again excercise (39)), and is therefore white: this particular state does *not* enter into strong interactions. There are, therefore, not 9 but rather 8 different relevant gluon types[65].

6.6 Gluon vertices

The fact that gluons carry combinations of colour and anticolour leads to an important corollary. *The gluons are coloured*. This is in contrast to the case of QED, where the photons themselves are electrically neutral, and therefore to not couple to one another[66]. Therefore, gluons themselves are subject to strong interactions: the theory of QCD therefore contains interaction vertices in which three, and even four, gluons meet at the same spacetime point[67]. Looking back, we can see that these gluon-self interactions arise necessarily from the fact that there are *several* distinct types of ‘colour’ quantum number. Indeed, any theory in which particles exchange one type of quantum number for another in a vertex leads to boson self-interactions[68].

[65]The red-antiblue gluon is what you think it is; but the red-antired gluon is not simply red-antired, but has the ‘mesonic’ state subtracted out. The red-antired, green-antigreen and blue-antiblue gluons together actually form not three, but only two strongly interacting states between them.

[66]This is only approximative. At lowest order, two photons colliding pass straight through one another without scattering: in higher orders, however, two photons *can* scatter off one another due to the occurrence of loop diagrams containing charged particles such as electrons: this is called *Euler* (no relative of the more famous one, I believe) scattering.

[67]Vertices in which five or more gluons meet are absent. This has to do with the more technical, detailed mathematical structure of the theory; in essence, three- and four-gluon vertices are required for the mathematical consistency of the theory, whereas more-gluon vertices would spoil it. So far, experiment has borne this out.

[68]Another example is provided by the ‘iso-spin’ quantum number governing the *weak* interactions. An electron (isospin -1/2) can turn into a neutrino (isospin +1/2) by emitting a W boson: the W bosons and their ilk therefore carry isospin themselves, and have to have three- and four-boson interactions, as is indeed the case.

It is interesting to realize that the gluons may scatter by themselves, without the need for any quarks entering in the Feynman diagrams[69]. Indeed, even bound states of gluons can be envisaged. These *glueballs* may be among the many species of hadrons in the few-GeV mass range, but have so far not been identified.

6.7 Strings fragmentation and jets

As we have seen, QCD 'forbids' quarks to become free from one another. On the other hand, consider the process $e^+e^- \to q\bar{q}$, taking place at a scattering energy of, say, 10 GeV, much larger than the quark masses. The quark and the antiquark, having been produced, will fly apart rapidly. If the 'one-GeV-per-Fermi' strong force (see excercise (38)) holds between them, they will fly apart to a distance of about 10 Fermi, turn back, and oscillate rapidly to form a 'meson' with a radius of 10 Fermi — or will they? In fact, no. To understand this, we may compare the strong interactions with the electromagnetic ones.

Consider two electric charges of opposite sign. The electric field between them can be pictured with field lines running from one charge to the other one, in the familiar pattern. If we now move the charges further apart, the shape of the field line pattern remains unchanged, but it becomes simply enlarged. As a consequence the *number* of field lines passing through a unit area between the charges decreases, and hence the force between the charges decreases, in the familiar $1/r^2$ manner.

The case of QCD is qualitatively different in the that the gluons feel each other's presence. We may picture this as as attraction between the field lines. This causes the pattern of field lines between opposite colour charges to be much more 'squeezed together': in fact the gluon field has the shape of a tube, or string[70], running from the (r, say) quark to the ($\bar{r}$) antiquark. As the quarks move apart, the tube hardly broadens out, and the number of field lines through a unit area between the quarks remains unchanged: hence the distance-independent force between the particles.

[69] Diagrams with only gluons as the incoming or outgoing particles can contain qarks lines in closed loops, *i.e.* in higher orders.

[70] Not to be confused with the, nowadays more famous, strings described by 'string theory'. In fact, string theory was originally conceived in the 60's to describe coloured objects undergoing strong interactions. This was not very successful, and string theory lay dormant until it was revived as a much more fundamental theory during the 80's.

We now come to what happens as the $q\bar{q}$ system flies apart. The colour string is actually a complicated dynamical system, containing about 1 GeV per Fermi of its length. It is therefore relatively easy for a quark-antiquark pair to pop into existence somewhere along the string. In many cases the colours of the appearing $q\bar{q}$ pair will be such that no 'white' subsystem forms. But sometimes it happens that precisely an $\bar{r}$ antiquark and a r quark appear. At that moment the original r quark and the occurring $\bar{r}$ antiquark form a 'white' system, and the other two particles also form a 'white' system. What this means is that, at this moment, the original string has 'snapped' into two smaller strings, that are both white and hence become essentially independent and hence easily fly away from one another. Of course, in general the 'original' quark and the 'new' antiquark will not have the same velocity, and the string pieces fragment further and further: this is called *string fragmentation*. What is observed in a particle collision in which a $q\bar{q}$ pair is produced at sizeable invariant mass is therefore not two particles, but rather two 'sprays' of (possibly many) hadrons particles: these are called *jets*. Electron-positron collision into $q\bar{q}$ give rise to two jets. Indeed, the development of two-jet-like structures in e^+e^- collisions as the total energy gradually increased with the use of more powerful colliders was an important piece of evidence that the quark picture of hadrons was more than just an intellectual device.

A number of qualitative predictions can be made on the basis of the string fragmentation picture. In the first place, we expect that, the higher the total energy of the produced quark-antiquark pair, the more particles will be produced. Secondly, we expect that, in general, the fastest particle inside a given jet is most likely to contain the 'original' (anti)quark. In the third place, it is easier for a $u\bar{u}$ or a $d\bar{d}$ pair to 'pop out of the vacuum' than for a $s\bar{s}$ pair[71], and therefore we expect more pions than kaons to be produced in jets. The charm quark is again much heavier, and D mesons are therefore quite likely to contain the 'original' charm quark rather than one coming from the string fragmentation. Finally, it is of course also possible, but less likely, for the string to snap into *two* $q\bar{q}$ pairs at the breaking point rather than just one such pair. In such a case one would get baryons from the string fragmentation, and this is what is observed: baryons are produced as well, but not a frequently as mesons.

[71]Because the strange quarks are heavier.

6.8 Excercises

Excercise 38 Computing the strength of the colour force
We can estimate the strength of the strong interactions as follows.

1. Find the rest mass of the nucleon (proton or neutron, approximately the same mass), and express their rest energy in GeV.

2. Convince yourself that this energy cannot be ascribed to the rest masses of the three quarks making up the nucleon.

3. Argue that the nucleon mass must be accounted for by the kinetic and potential energy of the quarks in their mutual potential well.

4. Assume that the binding force between the quarks is constant, in other words, the potential is linear in the separation. From this, and the fact that the diameter of a proton is approximately one Fermi (fm = 10^{-15} meter), estimate the slope of this potential.

5. Express the resulting force in *tons*.

Excercise 39 Constructing hadrons
Denote by q^a the quantum state of a quark with colour a(=r,b,g). Let us consider a change of basis in colour space, that is, we replace the three orthonormal colour basis states with three *other* orthonormal basis states. This operation is represented by a matrix $M^a{}_b$, where a and b are colour labels. In the new basis, the quark state is then written as

$$(q')^a = M^a{}_b \; q^b \quad (\text{summation convention!})$$

1. Argue that the matrix M must be *unitary*, that is, $M^\dagger = M^{-1}$.

2. Show that the transformation character for the *antiquarks* must be given by

$$\overline{q}'_a = \overline{q}_b \; (M^{-1})^b{}_a \quad .$$

3. Consider a quark and an antiquark in the combined quantum state

$$|\text{meson}\rangle = \sum_{a=r,g,b} q^a \; \overline{q}_a \quad .$$

4. Show that the colour basis change M does *not* change the state $|\text{meson}\rangle$, by applying M to the quark and the antiquark components.

5. Prove that the matrix M, being unitary, must have a determinant with absolute value *one*.

6. Consider three quarks, in the following quantum state

$$|\text{baryon}\rangle = \sum_{a,c,d=r,g,b} \epsilon_{acd}\; q^a\; q^c\; q^d \quad .$$

   Here ϵ is the Levi-Civita symbol. It is perfectly antisymmetric under the interchange of any of its three indices:

$$\epsilon_{rbg} = \epsilon_{bgr} = \epsilon_{grb} = -\epsilon_{gbr} = -\epsilon_{rgb} = -\epsilon_{brg} = 1 \quad .$$

   Apply the colour basis change M to the quatun state $|\text{baryon}\rangle$, and show that it only changes by a factor $\det(M)$.

7. Do the same for the *antibaryon*, made up from three antiquarks.

8. Prove that mesons are bosons, and baryons are fermions.

Excercise 40 Two- and three-quark model
Supposing only the up and down quark types to be available, write down all possible hadrons that can be formed using them. Do not bother with the angular momentum. Do the same, but now taking into account the strange quark as well.

Excercise 41 Drawing Feynman diagrams for QCD
Show that the process $q\overline{q} \to gg$ is (in lowest order!) described by 3 diagrams, by drawing them explicitly. Show that the process $gg \to gg$ is (in lowest order) described by 4 Feynman diagrams, again by drawing them explicitly. Show that (again, at lowest order), any process in whicih the initial state as well as the final state contain *only* gluons must be described by purely gluonic interactions, that is, *no* vertices of quark-quark-gluon type can occur. Compute the number of Feynman diagrams that decribe, to lowest order, the process $gg \to ggg$.

7 Electroweak physics

The third fundamental interaction to be discussed in these notes is the weak one. It shares many similarities with the other two, but it has its own peculiarities as well. As viewed nowadays, the weak interaction's coupling to the fermions is not governed by the electric or colour charge, but by the so-called *isospin*. Its nature is discussed below.

7.1 Weak doublets

The fermions in the SM table of section 5.3 are grouped into pairs of two quarks or two leptons each. In each pair, the electric charges of the fermions differ by exactly one unit of charge. This pattern is, in fact, determined by the weak-interaction properties of the fermions. As an analogy with the more familiar division of electron states in 'spin-up' and 'spin-down' states, we call these pairs the *isospin* 'up' and 'down' states. The up-type quarks (u,c,t) and the neutrinos have isospin +1/2, whereas the down-type quarks (d,s,b) and the charged leptons carry isospin -1/2. The pairs of fermions with differing isospin are called *weak doublets*[72]. In usual circumstances, the energy of a spin-up electron and a spin-down electron will differ only very little compared to the rest energy of the electron. It therefore makes sense to say that the spin-up and spin-down states refer to the same particle (see also excercise (42)). For isospin, this is no longer true. An electron and its neutrino differ a lot in mass, as do, say, the charm and the strange quarks[73]. They are therefore usually not considered to be the same particle. The particle 'type' is commonly called the particle's *flavour*, and the processes governed by the weak interactions are also described as *flavour dynamics*.

7.2 Beta decay and the W particle

The most familiar processes in which the weak interactions play a rôle are the radioactive processes known as beta decay (or inverse beta decay). Before the quark model of nucleons was establihed, beta decay was described as the decay of a neutron into a proton and leptons:

$$n \quad \to \quad p^+ \, e^- \, \bar{\nu}_e \ .$$

[72]There is a subtlety here, to be discussed below.

[73]The up and down quarks (or the proton and neutron) do not differ too dramatically in mass, and this was the idea behind isospin in the first place.

Nowadays we rather see this as the decay of a down quark into an up quark and leptons:

$$d \quad\to\quad u\, e^-\, \bar{\nu}_e \ .$$

Seen as written down, the weak interaction appears to be different from the electromagnetic or strong ones in that it involves a coupling between four fermions rather than between two fermions and a boson. This difference is, however, only an apparent one. What *really* happens is in fact a two-step process:

$$d \quad\to\quad u\, W^- \qquad \text{followed by} \qquad W^- \quad\to\quad e^-\, \bar{\nu}_e \ ,$$

where we have introduced a new particle, the W.

7.3 Weak bosons

In the previous paragraph we postulated the boson of the weak interactions, the W^- particle. We shall now examine the consequences of this proposal.

In the first place, the W particle must be 'heavy' so that it is not readily produced on its mass shell. This is borne out by the fact that the weak interactions are indeed weak: processes like beta decay have a very long lifetime (on the scale of subatoic physics) so the interactions responsible cannot be very drastic. Now, as it turns out, the isospin coupling constant is actually somewhat larger than the electromagnetic one: the reason for the weak interactions' weakness is in the propagator. Suppose the boson has momentum p^μ. Then the W propagator reads

$$\approx \ \frac{1}{p\cdot p - m_W^2 + i m_W \Gamma_W} \ \ll \ \frac{1}{p\cdot p} \quad \text{if } m_W^2 \gg p\cdot p$$

so the W propagator is very small compared to, say, the photon propagator, and paths from an initial to a final state that contain a W propagator are correspondingly suppressed. An important corollary follows: if the mass of the W is larger than the sum of the masses of two fermions that it couples to, it can also decay into them when it is produced on-shell. Therefore W particles are *unstable*.

In the second place, the W^- particle is electrically charged. It is, therefore, unavoidable that it has an antiparticle distinct from itself. Indeed, the W^+ particle exists, with the same mass and lifetime as the W^-[74].

[74]For well-known particles such as eletrons, there is general consensus that the e^- is the 'particle', and e^+ is the 'antiparticle'. For W's there is no such consensus. This is no reason for worry: it does not matter in the slightest whether the d quark becomes an u quark by emitting an W^- or by absorbing a W^+.

In the third place, the W's themselves carry isospin, since by being emitted or absorbed they change the isospin of the fermions. Therefore, they must have isospin interactions between themselves as well, in analogy with gluons. If three-boson interactions are to be possible at all, there must therefore exist yet another weak boson, with zero charge: the Z^0 boson. Being neutral, it cannot change the isospin of fermions. It is its own antiparticle. Naturally, one would expect the mass of the Z to be roughly as large as that of the W, and this is indeed the case. Likewise its lifetime is comparable to that of the W's.

Since the W's have electric charge, there is of course also interaction between W's and photons. One might think, therefore, that no Z boson is needed, and just the $W^\pm$ and the γ would describe the electromagnetic and weak interactions. This turns out to be not true (see below). The full bosonic sector of what is called the *electroweak SM* contains two charged and two neutral bosons: W^+, W^-, Z^0 and γ. What *is* true, however, is the fact that Z^0 and γ look alike in many aspects: if a Feynman diagram contains and internal γ line, we may also draw a Z line there. The converse is not exactly true, since neutrinos have interactions with the Z (as the do with the W's) but not with the photon. Nevertheless, the photon and Z 'mix' to some extent.

7.4 Left- and right-handed particles

The most peculiar property of the weak interactions is the following. For the moment, assume that all fermions are massless (and therefore move around with the speed of light). We can dististinguish between two spin states of these fermions: one state in which the spin points *along* the momentum ('right-handed' fermions), and one in whihch the spin points *against* the momentum ('left-handed' fermions). All other spin states are superpositions of these two. As excercise 46 shows, these notions are nicely Lorentz-invariant. Now comes the peculiar fact: *the W only couples to left-handed particles (and to right-handed antiparticles)*. This has some consequences.

In the first place, the W, being sensitive to the spin of the fermions, must have a spin of itself (just like, in fact, the gluons and the photon). The W is a spin-one boson. Likewise, then, the Z has spin one, just like the photon. Due to the mixing with the photon, however, the Z does not couple to left-handed particles only: after all, the photon couples to left- and right-handed particles equally. The Z therefore, couples to left- and right-handed particles with different strengths, depending on the amount to which these fermions couple to the photons, *i.e.* their electric charge. For instance, since neutrinos are neutral the Z couples *only* to left-handed neutrinos.

In the second place, the fermions do not, strictly speaking, form weak doublets. It is the *left-handed* pieces of the femions that form part of weak doublets, whereas the *right-handed* pieces do not feel any weak interactions at all: we say that they are 'neutral', or *weak singlets*.

In the third place, we know that the fermions are not strictly massless. A physical electron state therefore always contains both a left- and a right-handed piece. The above picture must, therefore, be somewhat modified.

7.5 Quark mixing / Lepton Mixing

For the moment, let us consider only two families of quarks. The *W* bosons couple isospin 1/2 and -1/2 states. This is *not* the same as saying that they couple up and down quarks. The coupling of a W to the combination (u, d) is *stronger* than to the combination (u, s), but the latter is certainly not negligible. We may formulate this a bit more quantum-mechanically as follows. If $|q_1 q_2\rangle$ is the qantum state of two free, on-shell quarks, the W do not couple to the pure states $|ud\rangle$ and $|cs\rangle$, but rather to 'rotated' combinations

$$\cos\alpha_c|ud\rangle + \sin\alpha_c|us\rangle \quad \text{and} \quad -\sin\alpha_c|ud\rangle + \cos\alpha_c|us\rangle \;\; . \tag{143}$$

The 'rotation' angle α_c is called the Cabibbo angle: its numerical value is about 13 degrees. If we disregard this effect, *all couplings of fermions to the W have equal strength*. It is the mixing that makes the strengths apparently different. If we include the third family, we of course get a three-by-three rotation, or *unitary* matrix, in which the W couples to linear combinations of $|ud\rangle$, $|cs\rangle$ and $|tb\rangle$. This 3×3 matrix is the Cabibbo-Kobayashi-Maskawa (CKM) matrix. The fact that this matrix becomes approximately diagonal if we collect the quarks in the groups (u,d), (c,s) and (t,b) is what makes us say that 'the up and down quark belong in one doublet'.

Up until recently, it was assumed that leptons do not suffer from mixing. This turns out to be too optimistic: for leptons there is a mixing matrix similar to that for quarks. This can be seen using solar neutrinos: neutrinos that are made in the sun's fusion processes are predominantly of type ν_e but with an admixture of the other types. Restricting ourselves to two families for simplicity, the neutrino state is made by interactions involving *W*'s:

$$|\text{made in Sun}\rangle = \eta_1|\nu_e\rangle + \eta_2|\nu_\mu\rangle \quad , \quad |\eta_1|^2 + |\eta_2|^2 = 1 \;\; . \tag{144}$$

As the neutrinos move towards the earth, the coefficients $\eta_{1,2}$ evolve into some $\zeta = 1, 2$, in a way determined by the propagators of ν_e and ν_μ. Upon arrival, they will be observed by, say, the inverse process, involving another W interaction. The neutrino state as it arrives reads, however:

$$|\text{seen on Earth}\rangle = \zeta_1 |\nu_e\rangle + \zeta_2 |\nu_\mu\rangle \quad , \quad |\zeta_1|^2 + |\zeta_2|^2 = 1 \quad , \tag{145}$$

and the probability of observing the process may therefore change appreciably. In the extreme case the neutrino may appear to have diasppeared competely [75]. The PMNS matrix which is the leptonic analogue of the CKM matrix is under a lot of sstudy at this moment: it appears to be even less clearly diagonal.

A final remark is in order here. The abovementioned *neutrino oscillations* can only occur if the evolution $\eta_1 \to \zeta_1$ is *different* from $\eta_2 \to \zeta_2$, otherwise the state would not change. Therefore, the propagators of the two neutrino states $|\nu_{e,\mu}\rangle$ must be different, in other words, the *masses* of the neutrinos must be different. But this implies that one neutrino species, at least, has nonzero mass! The observations on neutrino mixing indicate that the neutrino masses are quite small indeed. For some physicists this implies that some 'new physics' must underlie the neutrino sector of the SM.

[75] In fact, about 70% of the amount of 'original' neutrino made is seen by the reverse process on Earth. This used to be called the 'Solar neutrino problem'.

7.6 Excercises

Excercise 42 Extreme magnetic fields
Suppose an electron is put inside a uniform magnetic field of 30 Tesla. Due to its own magnetic moment is has different energy when its spin points parallel to the magnetic field than when it points antiparallel to the magnetic field. Compute the energy difference as a fraction of the electron rest energy.

Excercise 43 Beta decay of nucleons
Draw the Feynman diagram (picturing the quarks inside the nucleons) for the process of beta decay:

$$n \rightarrow p^+ e^- \bar{\nu}_e$$

Do the same for inverse beta decay:

$$e^- p^+ \rightarrow n \nu_e$$

Explain how both processes are, in principle, possible inside an atom. Why is the analogous process $n \rightarrow p^+ \mu^- \bar{\nu}_\mu$ not possible?

Excercise 44 Lifetimes and estimates
The Compton wavelength of a particle is delted to its mass m as

$$\lambda_C = \frac{2\pi\hbar}{mc} \quad . \tag{146}$$

Argue that the *smallest possible* lifetime of a particle of mass m is therefore of the order of the Compton lifetime:

$$\tau_C = \frac{\lambda_C}{c} = \frac{2\pi\hbar}{mc^2} \quad . \tag{147}$$

Compute the Compton lifetime for pions, with a rest energy of about 140 MeV. The dominant decay modes are:

$$\begin{aligned} \pi^0 &\rightarrow \gamma\,\gamma \qquad \text{in } 8 \times 10^{-17} \text{ seconds} \\ \pi^\pm &\rightarrow \text{charged lepton and neutrino} \qquad \text{in } 2 \times 10^{-8} \text{ seconds} \quad . \end{aligned} \tag{148}$$

Draw the Feynman diagrams (at the quark level) for these decays. Argue that the π^0 decay must be governed by the electromagnetic, and the $\pi^\pm$ decay by the weak, interactions.

Excercise 45 Boson self-interactions
Write down all admissible Feynman interaction vertices between three and four bosons of the electroweak SM. Take into accoutn the conditions imposed by charge conservation. Also assume that nature preserves isospin, that is, in a vetex the *total isospin* must be zero. Use this argument to show that a $Z^0\gamma\gamma$ vertex is not allowed.

Excercise 46 Handedness of particles
Consider a massive spinning particle, that is a given Lorentz frame has its spin pointing along its momentum. Now imagine that you make a Lorentz boost to a Lorentz frame that moves *faster* than the particle. Show that in this other frame the spin points against the momentum. Show that this is *not* possible if the particle is massless, and that therefore the handedness of massless particles is a Lorentz-invariant notion.

Excercise 47 Some weak processes
Write down the electroweak Feynman diagrams (lowest order) for the following processes:

$$\begin{aligned} e^+\,e^- &\to \nu_\mu\,\bar{\nu}_\mu \\ e^+\,e^- &\to \nu_e\,\bar{\nu}_e \\ e^+\,e^- &\to W^+\,W^- \\ u\,\bar{d} &\to W^+\,Z^0 \\ W^+\,W^- &\to Z^0\,Z^0 \\ W^+\,W^- &\to W^+\,W^- \end{aligned}$$

Show that the two Feynman diagrams describing the second process must have a relative minus sign.

Excercise 48 The GIM mechanism
The K^0 meson is the flavour combination $|d\bar{s}\rangle$. Consider the following process: the down and strange quark may exchange an up quark to produce a W^+W^- pair (off-shell). These can then produce an electron-positron pair. Write down the corresponding Feynman diagrams: these contain a closed loop involving the W's and the exchanged up quark. In spite of the fact that this diagram exists, the decay $K^0 \to e^+e^-$ is hardly ever observed, at a rate far below the calculated one. Glashow, Iliopoulos and Maiani (GIM) explained this by postulating (before it was discovered!) the charm quark, with Cabibbo-rotated couplings to the W as

described in the text. Now draw also the charm-exchange diagrams contributing to the K^0 decay, and show that they precisely cancel the up-quark mediated contribution, if the charm mass equals the up mass. Explain how the observed decay rate may be used to put upper limits on the charm quark mass[76].

[76] This is how it went historically: GIM proposed the charm quark, and put upper limits on its mass in 1970, before it was discovered in 1974.

8 The Higgs sector

So far, the fact that the weak bosons have mass and that physical fermions are never purely left-handed didn't bother us too much. We shall address these issues now.

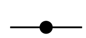
8.1 Mass viewed as interaction: Dyson summation

As excercise 46 shows, a difference between massive and massless spinning particles is that for massless particles the handedness is unchanged under Lorentz transformations, while for massive particles a Lorentz boost can be found that transforms a right-handed particle into a left-handed one, or vice versa. This suggests that mass might be viewed as an 'interaction' that transforms right-handed particles into left-handed ones. Since no other particles are involved, we are led to write the mass of a particle as a two-point vertex.

Let us now see how a massless particle becomes massive, by studying Feynman diagrams (forgetting about the $i\epsilon$ for simplicity). The propagator for a massless particle with momentum p is given by

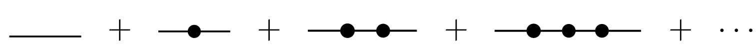
$$= \frac{i}{p \cdot p} \quad . \tag{149}$$

If, now, there is available a two-point vertex with Feynman rule



$$= iK \quad , \tag{150}$$

with some coupling constant K (to be determined), then any path (or part thereof) described by the above propagator is, in fact, described by an infinite set of paths, each with its own number of two-point vertices (but the same starting and end point):



$$\begin{aligned} &= \frac{i}{p \cdot p} + \frac{i}{p \cdot p}(iK)\frac{i}{p \cdot p} + \frac{i}{p \cdot p}(iK)\frac{i}{p \cdot p}(iK)\frac{i}{p \cdot p} \\ &\quad + \frac{i}{p \cdot p}(iK)\frac{i}{p \cdot p}(iK)\frac{i}{p \cdot p}(iK)\frac{i}{p \cdot p} + \cdots \\ &= \frac{i}{p \cdot p} \sum_{n=0}^{\infty} \left(\frac{-K}{p \cdot p}\right)^n = \frac{i}{p \cdot p}\,\frac{1}{1 + K/p \cdot p} = \frac{i}{p \cdot p + K} \quad . \end{aligned} \tag{151}$$

Therefore, by taking $K = -m^2$ we see that a massless particle's propagator turns into that of a massive one by taking into account all such two-point vertices: this is called *Dyson summation*.

8.2 Trouble with mass: the Higgs picture

In the theory of weak interactions, particle masses are problematic. This follows from the above-mentioned picture of mass-as-interaction. Since left-handed fermions have isospin one-half, and right-handed particles have isospin zero, the interaction term $-im^2$ of the above paragraph *cannot* be a simple number, but rather must have isospin one-half in order to have a isospin-zero vertex (necessary for isospin conservation). But mass, after all, is just a number! By simply putting masses into the SM, the theory becomes internally inconsistent[77].

The way out of this conundrum is the Higgs picture of mass. This idea says that the above two-point vertex is actually an interaction of the particle with some other field, the *Higgs field*. Let us suppose that, *for some reason*, this field has a nonzero value everywhere in spacetime, called a *nonzero vacuum expectation value*: all particles move through a 'sea' of Higgs field, as it were. The particles interact with this field; these interactions, however, do not involve a change of momentum of the particles for reasons explained below. They are, therefore, indistinguishable from two-point interactions, and Dyson summation works as before. The attractive point is that the Higgs field can be assigned isospin one-half without more ado, and the interaction vertices become isospin-conserving. The pattern of particle masses is therefore nothing but the pattern of particle couplings to the Higgs field.

An important consequence of the Higgs picture is in the weak boson sector. Since the Higgs field carries isospin, it couples to W's as well, so W's obtain masses. The interesting thing here is that the coupling of the W's to the Higgs field is not arbitrary; if we believe in the universality of the weak interactions, the W mass is therefore more or less fixed by its weak interactions. Some freedom is left by the mixing of Z's and photons, but the fact remains that, after all their couplings *to the fermions* have been determined, the masses of the W and the Z are fixed. Measuring the couplings and the masses therefore provides a nice check on the validity of the Higgs picture.

[77]Technically, it becomes *non-renormalizable*.

8.3 Properties of the Higgs field

From the Higgs picture of mass, we can immediately infer some properties that the Higgs field ought to have.

- The photon is massless. Therefore it does not couple to the Higgs field: the Higgs field is electrically neutral[78].

- The gluon is massless. Therefore it does not couple to the Higgs field: the Higgs field is white.

- The theory of special relativity (which agrees with all experiments in which gravitation is unimportant) holds that space does *not* contain a special, privileged direction. If the Higgs field had spin, its nonzero value would present physics with such a special direction, which would unavoidably have effect on particle physics. For example, one would then see that free electrons with spin in one direction would have different mass than electrons with spin in another direction. The Higgs field, therefore, is spinless.

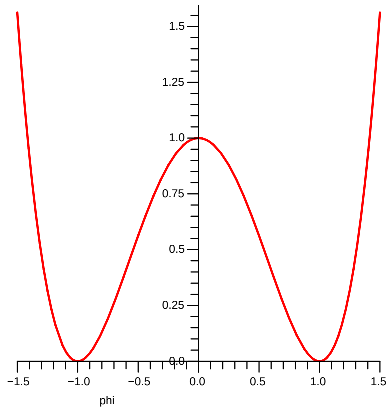
- In contrast to the last point, particles with different *isospin* do have different masses. The Higgs field therefore has nonzero isospin: in fact that is precisely what it was introduced for. However, special relativity has nothing to say about isospin, and a preferred *isospin* direction in the universe is perfectly acceptable.

- By the very idea of the Higgs picture, the Higgs field couples more strongly to particles, the heavier they are.

The big question remaining is: how does the Higgs field get its nonzero value? All other fields in particle physics, such as the electromagnetic fields containing photons, or the electron field containing electrons, and so on, are zero in the ground state of the universe, that is, the vacuum does *not* contain any nnzero electron field, nor any nonzero photon field. Remember that if the electric field $\vec{E}$ or the magnetic field $\vec{B}$ are nonzero, space contains an energy density proportional to $\vec{E}^2 + \vec{B}^2$. Therefore, switching off the electric and the magnetic field decreases the energy content, leading to a state of lower overall energy. The energy density (or

[78] A remark is in order here. The Higgs field is an isospin doublet: it must therefore contain *two* fields, differing by one unit of charge. In the standard version of the SM, we have one neutral component and one charged component. It is the *neutral* component that has the nonzero value which permeates all of spacetime. The charged part does not have a constant nonzero value.

potential) for the Higgs field must therefore be assumed to be such that the state of lowest energy is obtained for a nonzero field value.



An illustration of the potential of the Higgs field. The field value zero corresponds to an unstable equilibrium. The states of lowest energy, corresponding to the vacuum of the physical world, contain nonzero field values. This situation goes under the name of *spontaneous symmetry breaking*.

The Higgs field, and in particular its potential, have as yet not really been probed by experiment. With the commissioning of the LHC in 2007 (and ongoing experiments at Fermilab) this will holpefully change in the ear future.

8.4 The Higgs boson and degree-of-freedom counting

The masses of the particles are, in the Higgs picture, ascribed to a constant nonzero value of the Higgs field. But, if the Higgs field is to be an acceptable quantum field like the photon, lepton, quark and gluon fields, it must also allow for fluctuations, oscillations, and hence propagating waves: in short, it must also contain *particles*.

Before we continue, a word is in order on the spin states of spin-1 particles. In principle, quantum mechanics tellls us that a state of total spin S contains $2S + 1$ distinct orthonormal spin states: that is why electrons of spin-1/2 come in two spin states. *Massless* particles, however, contain only two spin states. The photon (and gluon) therefore come in only two spin states (called *polarization states*): the free photon always has transverse polarization[79]. The $W^{\pm}$ and Z^0, on the other hand, come in three polarization states. This apparent qualitative difference between the weak and the non-weak bosons is explained as follows.

The Higgs field contains a charged component, which (as we have seen) has zero vacuum expectation value. This component therefore describes particles of charge $-e$, and their antiparticles of charge e. In the SM, in which the 'primordial' W fields are massless like the photon and gluon and therefore have only two spin

[79] The polarization of light may be linear or circular, or elliptic: but in any case there are two orthonormal states.

states, the charged Higgs components play precisley the rôle of the ‘missing’ third polarization states of the W’s[80]. If the charged component of the Higgs field has two components itself, then so must the neutral component of the Higgs ffield have two components by itself. *One* of these components links up with the ‘primordial’ Z field so that the Z, too, becomes massive. The photon remains massless, and so does not team up with the remaining Higgs field component. There will, therefore, remain one propagating part of the Higgs field: this is the *Higgs boson*.

The properties of the, so far undiscovered[81] are, of course, those of the neutral part of the Higgs field itself: it is a *charge-zero*, *uncoloured*, *spinless* particle that couples to all particles the stronger the heavier they are. The values of the couplings to the particles of the SM are known, since they depend only on known couplings and the particle masses. The *mass* of the Higgs boson is, however, not really known although there are good arguments to guesstimate that it is larger than 115 GeV but smaller than, say, 250 GeV. The couplings of the Higgs to fermions are of the usual fermion-fermioni-boson type. The coupling of the Higgs to the bosons are of the familiar three- and four-boson types. Note that there are also self-interactions between three and four Higgs bosons, which are stronger the larger the Higgs mass itself is.

In a sense, finding the Higgs boson explicitly, and measuring its mass and couplings to the other particles, would be a capstone on the development of the SM. Note that finding the Higgs will not ‘explain the particle masses’, as it is often misstated in polpular reading. Rather, the existence of a neutral Higgs boson proves the existence of a neutral Higgs field, and makes the existence of the whole SM Higgs field more probable, therefore suggesting that the Higgs picture *might* be correct. The issue of spontaneous symmetry breaking is, however, *not* proven by the discovery of the Higgs: to get a handle on that, we will need to examine the Higgs potential.

[80]It is possible to prove that the two ‘original’ spin states and the third, ‘Higgs-induced’ spin state, all have precisely the same mass: a small but derivable miracle that we shall not prove here.

[81]That is, not seen directly in any collision experiment. It *has* been seen indirectly, in a sense, by the fact that all experimental precision results agree with a SM containing a Higgs, and *not* with a SM-like theory without the Higgs boson.

8.5 Excercises

Excercise 49 Dyson summation
Do the Dyson summation of Eq.(151) for the case where you start with a massive particle of mass m, and take K to be $-\mu^2$. Show that the resulting mass is given by $\sqrt{m^2+\mu^2}$. Now, instead of simple two-point vertices, consider the case of Feynman diagrams turning a particle into itself after some interactions. Give an example of such diagrams that turn an electron into itself in quantum electrodynamics. Show that the same Dyson summation can be performed. Prove the following: in any theory in which particles have interaction, their masses will be influenced by the interactions.

Excercise 50 The weight of wet light
A nice metaphor for the Higgs picture is provided by light and water. In air (as an excellent approximatin to vacuum) light travels at speed c. In water, light travels more slowly, as evidenced by the index of refraction of water. Now, perform the following steps.

1. The colour of light's photons is a measure of theor energy. Prove that the energy of the photons as they enter water does not change (Hint: consider what happens if light traverses a water layer).

2. Prove that photons in water become *massive*.

3. Assume that the index of reffraction of water is 4/3, independent of the colour of the light (this is true over a range of visible colours). Compute the mass of photons of energies between 2 and 3 eV when they are in water.

4. Compute the number K mentioned in the section discussing Dyson summation. Prove that this number depends on the photon momentum, and compte this dependence.

Excercise 51 Higgs couplings
Write down all Feynman vertices of couplings involving Higgs particles. Take into account spin, charge, and isospin conservation. Now return to excercise 47 and draw all extra Feynman diagrams for these processes thath become available with the advent of the Higgs boson in the theory.

Excercise 52 Boson polarizations
Spin-one particles are described not only by their momentum but, as stated in the

text, also by their spin state. More concretely, in addition to the momentum four-vector p^μ, a spin-1 boson also has a *polarization vector* s^μ, with the following properties:

$$p \cdot p = m^2c^2 \quad , \quad s \cdot s = -1 \quad , \quad p \cdot s = 0\,;. \tag{152}$$

Show that, if $m \neq 0$, there are precisely *three* possible solutions $s^\mu_{1,2,3}$ to the above equation that are orthonormal, in other words, $s_k \cdot s_l = 0$ if $k \neq l$ $(k, l = 1, 2, 3)$. Hint: go to the rest frame. Show that, if $m = 0$, there are only *two* such solutions.