Extra excercises

The first three lectures of this course form the basis of almost everything that follows. These excercises deal with the various subjects that were discussed so far.

Exercise 1 : A state
Consider a quantum system with an observable Q that can take on the values 1,2, and 3. The corresponding eigenstates are, of course, $|j\rangle$, $j = 1, 2, 3$. Now consider the state

$$|\psi\rangle = \frac{1}{2}|1\rangle + \frac{1}{3}|2\rangle + \frac{1}{6}|3\rangle$$

1. Compute the various probabilities for finding $Q = 1$, $Q = 2$, $Q = 3$ for this state.
2. Suppose that we find $Q = 2$. If we (immediately afterwards) repeat the measurement, what is the probability of finding $Q = 1$?
3. Give the (normalized) quantum state after the measurement.

Exercise 2 : Some operators
Consider a quantum system that can occupy one out of two states. These eigenstates we denote by $|a\rangle$ and $|b\rangle$. Consider the following operators :

$$\hat{A}_1 = |a\rangle\langle a| - |b\rangle\langle b| \quad , \quad \hat{A}_2 = |a\rangle\langle b| \quad , \quad \hat{A}_3 = |a\rangle\langle b| + |b\rangle\langle a| \quad ,$$
$$\hat{A}_4 = |a\rangle\langle b| - |b\rangle\langle a| \quad , \quad \hat{A}_5 = |a\rangle\langle a| + |b\rangle\langle b| \quad , \quad \hat{A}_6 = 0$$

1. Give the marix form of these operators.
2. Determine which of these operators can correspond to an observable, and, for those, compute the spectrum.

Exercise 3 : Another operator
Consider a quantum system that can be observed to be in either of three eigenstates $|1\rangle$, $|2\rangle$, or $|3\rangle$. Now, an operator is given as follows :

$$\hat{Z} = |1\rangle\langle 1| + |2\rangle\langle 2| + \sqrt{6}\,|2\rangle\langle 3| + \sqrt{6}\,|3\rangle\langle 2|$$

1. Show that this operator may correspond to an observable Z.
2. Compute the spectrum of Z.
3. Compute the eigenstates of Z.

Exercise 4 : Heisenberg strikes again
Consider a two-state quantum system. We define the following operators, in matrix notation :

$$\hat{M}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \hat{M}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Let the system be in a normalized state, in matrix form given by

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

1. Given that $[\hat{M}_1, \hat{M}_2] = i\hat{M}_3$, compute $\hat{M}_3$ and show that it is, in principle, an observable.
2. Compute $\langle M_1 \rangle$, $\langle M_2 \rangle$, $\langle {M_1}^2 \rangle$, $\langle {M_2}^2 \rangle$, $\sigma(M_1)$ and $\sigma(M_2)$.
3. Show that it is possible to have $\sigma(M_1) = 0$, and find values for α and β for which this happens.
4. Show that in the above case, still the Heisenberg uncertainty relation holds.

Exercise 5 : Matrix operations are not measurements
Consider a nontrivial quantum system in a general state $|\psi\rangle$. There is, for this system, an observable P with its corresponding operator $\hat{P}$. Define

$$|\chi\rangle \equiv \hat{P}\ |\psi\rangle$$

Show that $|\chi\rangle$ is ***not*** the result of measuring P on the system in state $|\psi\rangle$, except for one case : and identify the exceptional case.