The Physics of Spacetime

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Chapter 0

Prolegomena

0.1 Metaphysical manifesto

Physicists, mostly, are realistic reductionist Copernicist mathematicists.

Realism is the prejudice that there is a universe out there, distinct from, but in communication with the physicist[1], and that understanding of this universe ought to be obtained by careful observation[2].

Reductionism is the prejudice that in order to understand complicated things it helps to understand its simpler parts[3].

Copernicism is the prejudice that the universe doesn't care about the physicist's tastes and preferences[4]: the universe will keep going its own way as physicists fall for fads and fashions. In short, *you* are neither the center, nor at the center, of the universe.

Mathematicism is the prejudice that we can describe and, with luck, predict phenomena using the rules of the mathematical game[5]; and that the universe

[1]Denying this is solipsism: the solipsist cannot be proven wrong by logical argument, but is, perhaps for that very reason, not a nice person to be with.

[2]Denying this disqualifies one from doing physics (but not from doing mathematics); in extreme forms it can lead to revelationism and religious fundamentalism.

[3]Do not be misled by fashionable slogans like 'holism', 'complexity', or 'self-organization'. In the language we are using here, holims just states that the simplest part of a thing may happen to be the thing itself; complexity just states that simple things obeying simple rules may actually result in something pretty complicated-looking; and self-organization just states that the complicated-looking result may actually be pretty robust. The Pseudo-Scientific New Ager will sit back and be impressed by the complicated-looking result; the scientist will be impressed by the simplicity of the parts and rules. The *science* of complexity is not the antithesis of reductionism, but an extreme case of it.

[4]Do not be misled by fashionable slogans like 'observer effect','collapse of the wavefunction', or 'the measurement problem'. Observer effect just states that the physicist, after all, is unavoidably part of the universe, and a separation between self and the universe is an idealization; collapse of the wavefunction just states that as the status of the self changes because of information gained from the universe, it should not come as a surprise that the status of the universe also changes; the measurement problem just states that at present it is not yet completely clear how these status changes are to be explained reductionistically.

[5]After centuries of brainwashing by physics' successes, it is extremely hard for us even to

deigns to conform to the rules of human mathematics[6].

All these prejudices may be wrong[7], but I will stick to them until they are *clearly* wrong.

0.2 The classical arena

Physics is, to a large extent, the science of change and movement. The classical picture of physics is that of an independently existing universe filled with *events*, such as: a raindrop hits a puddle over *here* and *now*, two galaxies collide *over there* and *then*, a flash of light goes off *there*, and its light moves to *that* place, being absorbed by *that* object at *such* and *such* a moment. The events themselves take place, whether observed or not, but any one *can*, in principle, be observed and described, without being disturbed by the act of observation[8].

Motion is the phenomenon in which an identifiable object is at different places at different times (these positions may conceivably be labelled by events taking place there). In order to describe things like motion mathematically, we need a systematic way of keeping track of events. This is a *coordinate system.* The idea behind this is that the universe consists of a set of locations where an event can, or does, take place, and one ought to be able to tell someone else how to find a particular location[9]. A coordinate system, together with a set of numerical coordinates, provides a description of the location of the event in question. In the reductionist spirit, a *small* set of instructions is best: therefore, although this is an idealization, a single set of coordinate values is often deemed to describe the location of an event. In our actual world, there are at least three spatial coordinates, and one temporal coordinate; but it is easy to conceive of a more general set of coordinates. In addition, a phenomenon like a moving object may require more numbers for an adequate description. For instance, the *mass* of an object may play a rôle.

We are led, then, to think of the totality of locations as a *space-time*, with a number of a-priori properties:

1. Each *point* (*i.e.* location) in spacetime has some coordinate values, and these coordinate values can, in principle, be determined unambiguously[10];

conceive of any other way of doing it.

[6]An undeserved but very fortunate condescension! This behaviour cannot be *proven*, but so far we have been succesful. On the other hand, it may mean that our mathematics has hit upon some fundamental *actual* properties of the world — in that case, an alien thechnology-based civilization on a distant planet would probably do mathematics the same way.

[7]An open mind is a good thing, but a prejudice such as 'do not jump out of this high window, because gravity *always* works' sees you safely through the day.

[8]'Classical', mind you.

[9]It may not be *physically* feasible to go to that location, for instance if it is in the past; but at least it ought to be possible *in principle.*

[10]This is again something of a prejudice, and has to be accompanied by some operational procedure describing how coordinates are to be assigned. In general relativity, the coordinates are, indeed, unambiguous in any chosen coordinate frame. For now, we shall simply assume this property.

2. Each allowed set of coordinate values corresponds to an actual location in spacetime[11];

3. Each neighbourhood of a spacetime point contains other spacetime points[12]

Again, these are just prejudices, but for now they seem to work. The paradigmatic object in classical physics is a *point particle*: a system whose essential instantaneous characteristics can be embodied by a single set of position coordinates, in addition to a short list of properties such as mass and charge[13].

0.3 Physical laws and coordinate transforms

A useful[14] physical law is most often formulated in a form like

$$A \;=\; B \;.$$

Here A and B are usually sets of numbers relating to properties of objects. For instance, A may refer to a point particle's acceleration, and B to the gravitational influence of another object: in that case, we are talking about Newton's third law. There are two issues here. The primary one is that of *truth*: the law may be true or not. This is something that ultimately depends from observation. By continuously discarding physical laws that disagree with observation, we hope to arrive at a set of laws that defy all attempts at falsification, and are therefore deemed to be valid pysical laws[15]. The secondary issue is that of *consistency*: in order to have any chance at all of being a true physical law, the law ought to be Copernican enough to allow us to change our preferences and coordinate systems and still be sensible in the new coordinate system[16]. That is, if under a change of coordinate system, the object A goes over into an object A', and B goes over into B', we should have

$$A' \;=\; B' \;;$$

if this is not the case, the physical law $A = B$ cannot possibly be valid in both coordinate systems, and is therefore contemptible to a universe that doesn't care

[11] Again, a prejudice that has worked so far. We have never researched *all* points indicated by allowed coordinate values, even in our direct neighbourhood. A spacetime with 'holes', that is, regions with valid coordinate values that are 'not there', may exist, but its properties would presumably be quite exotic.

[12] A topological statement! We do not believe that there are 'isolated' points of spacetime; but it is eminently possible that spacetime consists of disconnected regions. In that case, whatever we have to say holds for one such disconnected piece. Of course, it is hard to see how disconnected pieces could communicate in a physical way, so we might as well forget about the pieces that don't contain *us*.

[13] For a high-minded cosmologist, a galaxy may be a point particle; on the other end of the spectrum, a point particle may be a quark, inside a nucleon, inside an atom, inside a body, on a planet, inside a solar system, inside such a galaxy.

[14] In the sense that it allows predictions.

[15] Note the provisional wording! At *any* given moment, *any* law of physics may be invalidated by experiment; but nowadays you'll have to be pretty smart to come up with an observation that falsifies, say, momentum conservation.

[16] Note carefully: consistency and thruth are quite different properties!

about coordinate systems anyway. In order to have any idea of the structure of physical laws, it is therefore useful to examine the effect of changes of coordinate system on all kinds of quantities; and to construct a repertory of objects with fairly simple, and understood, transformation properties such that they may serve as the ingredients of physical laws that, even if untested, are at least internally consistent. This will lead us to consider things like vectors and tensors.

0.4 The rationale of relativity

The idea of General Relativity can be captured in a few statements. In the first place, physical objects move about in the spacetime arena. Their behaviour is therefore influenced by the properties of spacetime. Secondly, such properties of spacetime as are relevant ought to be discernible by observations done *inside* it, *i.e.* they should be *intrinsic* properties[17] Thirdly, there is no reason why, as objects are influenced by the properties of spacetime, these properties should not be influenced by the objects. In Newtonian mechanics, the spacetime properties are aloof; but in Einsteinian mechanics, they indeed vary with the rest of the physics. Fourthly, the most useful formulation of physical laws appears to be in terms of mathematical objects that transform in the 'relatively simple' ways of covariant and contravariant vectors and tensors[18].

[17]Nobody has ever stepped outside spacetime and looked at it 'from the outside'. At any rate, if they did, they didn't leave any farewell note, nor did they ever come back.

[18]In principle, nothing forbids physical laws that are formulated using mathematical objects A and B that transform in ways radically different from vectors and tensors. On the other hand, no such formulations are known to me.

Chapter 1

Materia Mathematica

1.1 Preliminaries: metric spaces

1.1.1 Distance and metric

We consider a D-dimensional space. In this space, each point is determined by D real parameters, the coordinates:

$$x^\mu \quad , \quad \mu = 1, 2, \ldots, D \ .$$

The coordinates are considered to be nothing more than a recipe that tells one how to arrive at a particular given point. The point is supposed to have its own identity; the coordinate system, combined with the point's coordinate values, are just the information on how to get there. We assume the coordinated space to be smooth, in the sense that two points with infinitesimally differing coordinates are also physically close[1]. Consider two nearby points, A with coordinates x^μ, and B with coordinates $x^\mu + dx^\mu$. The *physical* distance ds between A and B is then given by the following expression[2]

$$(ds)^2 = \sum_{\mu=1}^{D} \sum_{\nu=1}^{D} g_{\mu\nu}\ dx^\mu\ dx^\nu \quad , \tag{1.1}$$

Adopting the Einstein summation convention, in which repeated (once upper, once lower) indices are to be summed over, we have

$$(ds)^2 = g_{\mu\nu}\ dx^\mu\ dx^\nu \quad . \tag{1.2}$$

The object $g_{\mu\nu}$ is, without loss of generality, taken to be symmetric in its indices:

$$g_{\mu\nu} = g_{\nu\mu} \quad , \tag{1.3}$$

[1] There is a philosophical, or rather metaphysical, issue here: in what sense are two points close together, unless we are told so by the coordinates? Only physical, rather than mathematical considerations can decide this.

[2] This notation *suggests*, at least, that $(ds)^2$ is a positive number: but in fact we shall allow both positive, null, and negative values.

and is called the *metric tensor*. Note that the measure $(ds)^2$ of the distance between two nearby points is assumed to be an *intrinsic* property of the space, and hence independent of the particular coordinate system we employ; on the other hand, $g_{\mu\nu}$ depends on the coordinate system, and may also depend on the coordinate values x^μ themselves: $g_{\mu\nu} = g_{\mu\nu}(x)$. We shall also assume that $g_{\mu\nu}$ has an inverse, $g^{\mu\nu}$:

$$g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu = \begin{cases} 1 & , \quad \mu = \nu \\ 0 & , \quad \mu \neq \nu \end{cases} \quad . \tag{1.4}$$

The object δ^μ_ν is called the *Kronecker scalar* ; it is a simple set of constants. Eq.(1.4), together with the symmetry of the metric, implies that a coordinate transform can be found that diagonalizes the metric tensor: however, that transform may differ from point to point.

1.1.2 A note on co-factors

Let us, for the moment, consider $g_{\mu\nu}$ as a matrix. Then, $g^{\mu\nu}$ must be its inverse. Let g denote the determinant of the matrix $g_{\mu\nu}$: this determinant is nonzero by assumption. By the rules of matrix inversion, therefore, $gg^{\mu\nu}$ must be the co-factor of the element $g_{\mu\nu}$ in that matrix. We shall have occasion to use the derivative of the determinant with respect to the x^μ. Such a derivative is obtained by taking the derivative of each element, and multiplying it with the corresponding co-factor. This means that

$$\frac{\partial}{\partial x^\mu} g = g \, g^{\alpha\beta} \, \frac{\partial}{\partial x^\mu} g_{\alpha\beta} \quad . \tag{1.5}$$

This identity can, in any given case, be checked by explicit computation.

Excercise 1 *Check Eq.(1.5) for the cases $D = 2$ and $D = 3$ by explicit computation. Show that it does not depend on the metric being symmetrical.*

1.1.3 Embedded spaces

Sometimes a metric space can be viewed as a part ('hypersurface') of a higher-dimensional space; an example is the two-dimensional surface of a sphere in three dimensions. The space is then said to be **embedded** in the larger one. Let us denote the coordinates X in the higher-dimensional 'embedding' space by Roman indices:

$$X^m \quad , \quad m = 1, 2, \ldots, N \quad , \quad N \geq D \quad .$$

The positions of the points in the embedded space are then given by N functions of the coordinates x^μ of the embedded space:

$$X^m = F^m(x^1, x^2, \ldots, x^D) \quad , \quad m = 1, 2, \ldots, N \quad . \tag{1.6}$$

Let the metric tensor of the embedding space be h_{mn}. The most visually appealing case is, of course, that in which the embedding space is the Euclidean space with Cartesian coordinates, in which case $h_{mn} = 1$ if $m = n$, otherwise zero; but in pinciple it could be any metric[3]. The distance between nearby points *in the embedding space* is then given by

$$(ds)^2 = h_{mn}\, dX^m\, dX^n = h_{mn}\, \frac{\partial F^m}{\partial x^\mu}\, \frac{\partial F^n}{\partial x^\nu}\, dx^\mu\, dx^\nu \quad . \tag{1.7}$$

It is natural to choose the distance definition in the embedded space to coincide with that of the embedding space. This means that the embedding space's metric *induces* a metric on the embedded space:

$$g_{\mu\nu} \equiv h_{mn}\, \frac{\partial F^m}{\partial x^\mu}\, \frac{\partial F^n}{\partial x^\nu} \quad . \tag{1.8}$$

Knowing the embedding space's metric and the functions F^m that define the hypersurface, we can then compute the metric of the embedded space; note that it only depends on the x^μ, as it should.

A last remark: embedding the space in a (possibly larger) one is to be considered more or less as a visual aid[4]. In practice we shall only be interested in objects that *can* be formulated without any reference to embedding. As a visual aid, the most attractive embedding space is of course Euclidean, with Cartesian coordinates.

Excercise 2 *Show the following: if a given space can be embedded in a Euclidean-Cartesian one, then all the diagonal elements of the metric must be non-negative (cf. Eq.(1.8)).*

1.1.4 Raising and lowering indices

We shall adopt the following convention: to every object with an upper index, A^μ, say, we associate a similar one with a lower index:

$$A_\mu = g_{\mu\nu}\, A^\nu \quad . \tag{1.9}$$

Conversely, the definition of $g^{\mu\nu}$ allows us to write

$$A^\mu = g^{\mu\nu}\, A_\nu \quad . \tag{1.10}$$

A similar convention holds for objects with several indices; for instance,

$$A^{\mu\nu}{}_{\alpha}{}^{\beta} = g^{\mu\rho} g^{\nu\sigma} A_{\rho\sigma\alpha}{}^{\beta} = g_{\alpha\kappa} g^{\beta\delta} A^{\mu\nu\kappa}{}_{\delta} \quad , \tag{1.11}$$

[3]Consider embedding the embedding space in a yet larger embedding space, and the *next* step . . . the mind boggles.

[4]The question of whether a given metric space *can* be viewed as an embedded one is not completely answered. *Compact* metric spaces can in principle be embedded in a sufficiently high-dimensional Euclidean space. This was first proven by Nash, who subsequently suffered a mental breakdown for about twenty years, and was afterwards awarded the Nobel prize in economics for something completely different.

and so on. In particular, we have

$$dx_\mu = g_{\mu\nu}\, dx^\nu \quad , \tag{1.12}$$

so that

$$(ds)^2 = dx_\mu\, dx^\mu = dx^\nu\, dx_\nu \quad . \tag{1.13}$$

1.1.5 Examples

Euclidean space

The simplest case is that of a D-dimensional Euclidean space, $\mathcal{E}^D$, with standard Cartesian coordinate axes. In that case the metric is simply

$$g_{\mu\nu} = \delta_{\mu\nu} \equiv \begin{cases} 1 & \text{if } \mu = \nu \\ 0 & \text{if } \mu \neq \nu \end{cases} \quad , \quad \mu, \nu = 1, 2, \ldots, D \quad . \tag{1.14}$$

With this metric, raising or lowering indices has no numerical effect, and the value of $g^{\mu\nu}$ is the same as that of $g_{\mu\nu}$. The space is considered to be *flat*, in the following sense: take two arbitrary points, with coordinates X^μ and Y^μ. Then, $X^\mu + z\, Y^\mu$ also refers to a point in the space, for any real z value[5].

Excercise 3 *We may also take a rectilinear but not orthonormal coordinate system. In that case the Cartesian coordinates X are linear functions of the non-Cartesian rectilinear ones, x:*

$$X^\mu = H^\mu{}_\nu\, x^\nu \quad , \tag{1.15}$$

with H some nonsingular matrix with constant elements. Show that the induced metric is then given by

$$g_{\mu\nu} = \sum_{\alpha=1}^{D} H^\alpha{}_\mu\, H^\alpha{}_\nu \quad . \tag{1.16}$$

Now, $g^{\mu\nu}$ and $g_{\mu\nu}$are no longer numerically equal. This metric, being nonsingular and symmetric, can of course be diagonalized by the opposite transform. Therefore, if a space admits of a coordinate system in which the metric is constant, that space is flat.

A less trivial case is that of spherical coordinates. In that case, we have

$$\begin{aligned}
X^1 &= x^1\, \sin(x^2)\, \sin(x^3) \cdots \sin(x^{D-2})\, \sin(x^{D-1})\, \sin(x^D) \quad , \\
X^2 &= x^1\, \sin(x^2)\, \sin(x^3) \cdots \sin(x^{D-2})\, \sin(x^{D-1})\, \cos(x^D) \quad , \\
X^3 &= x^1\, \sin(x^2)\, \sin(x^3) \cdots \sin(x^{D-2})\, \cos(x^{D-1}) \quad , \\
X^4 &= x^1\, \sin(x^2)\, \sin(x^3) \cdots \cos(x^{D-2}) \quad , \\
&\vdots \\
X^{D-1} &= x^1\, \sin(x^2)\, cos(x^3) \quad , \\
X^D &= x^1\, \cos(x^2) \quad .
\end{aligned} \tag{1.17}$$

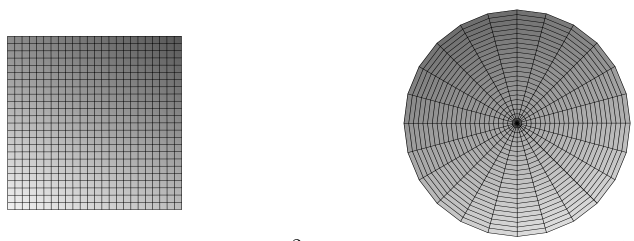
Here $x^1 \geq 0$, $0 \leq x^j \leq \pi$ $(2 \leq j \leq D-1)$, and $0 \leq x^D < 2\pi$.

[5] In words: by travelling straight on, one never leaves the space.

Excercise 4 *Show that the induced metric is diagonal, with*

$$\begin{aligned} g_{11} &= 1 \ , \\ g_{22} &= (x^1)^2 \ , \\ g_{33} &= (x^1)^2 \ \sin(x^2)^2 \ , \\ g_{44} &= (x^1)^2 \ \sin(x^2)^2 \ \sin(x^3)^2 \ , \\ &\vdots \\ g_{DD} &= (x^1)^2 \ \sin(x^2)^2 \ \sin(x^3)^2 \cdots \sin(x^{D-1})^2 \ . \end{aligned} \tag{1.18}$$

The space, being still $\mathcal{E}^D$, is still flat, but this is by no means obvious from the metric! In fact, g can actually vanish (for instance, for $x^1 = 0$) which leads to a word of caution: even if the metric becomes singular for some coordinate values, this does *not* automatically mean that the space has some special property at that point.



Parts of the Euclidean plane $\mathcal{E}^2$, in Cartesian and in polar (spherical) coordinates, with lines of constant x^1 or x^2.

Minkowski space

Special relativity operates in four-dimensional Minkowski space, which is non-Euclidean. It *is* flat, however: in Cartesian coordinates, the metric is diagonal and given by[6]

$$g_{jj} = -1 \ \ (j = 1, 2, 3) \quad , \quad g_{44} = +1 \ . \tag{1.19}$$

Excercise 5 *Show that Minkowski space cannot be embedded in any Euclidean space: this may make visualizations harder.*

Spherical surface

A simple example of a D-dimensional spherical surface, $\mathcal{S}^D$, is given by considering the sphere of radius R to be embedded in $\mathcal{E}^{D+1}$, and parametrize the spherical surface with polar coordinates:

$$\begin{aligned} X^1 &= R \ \sin(x^1) \ \sin(x^2) \cdots \sin(x^{D-2}) \ \sin(x^{D-1}) \ \sin(x^D) \ , \\ X^2 &= R \ \sin(x^1) \ \sin(x^2) \cdots \sin(x^{D-2}) \ \sin(x^{D-1}) \ \cos(x^D) \ , \end{aligned}$$

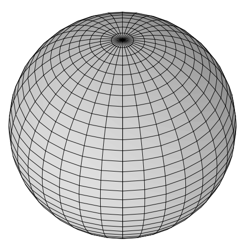
[6] In physical applications, is conventional to label the 4th index by '0' rather than by '4': of course this does not affect any result.

$$
\begin{aligned}
X^3 &= R\ \sin(x^1)\ \sin(x^2)\cdots\sin(x^{D-2})\ \cos(x^{D-1})\ \ , \\
X^4 &= R\ \sin(x^1)\ \sin(x^2)\cdots\cos(x^{D-2})\ \ , \\
&\vdots \\
X^D &= R\ \sin(x^1)\ cos(x^2)\ \ , \\
X^{D+1} &= R\ \cos(x^1)\ \ . \qquad (1.20)
\end{aligned}
$$

Excercise 6 *Show that the induced metric is diagonal, with*

$$
\begin{aligned}
g_{11} &= R^2\ \ , \\
g_{22} &= R^2\ \sin(x^1)^2\ \ , \\
g_{33} &= R^2\ \sin(x^1)^2\ \sin(x^2)^2\ \ , \\
&\vdots \\
g_{DD} &= R^2\ \sin(x^1)^2\ \sin(x^2)^2\cdots\sin(x^D)^2\ \ . \qquad (1.21)
\end{aligned}
$$

Although this metric looks suspiciously like that of Eq.(1.18), the sphere is *not* flat.



The sphere $\mathcal{S}^2$ in polar coordinates, with lines of constant x^1 or x^2.

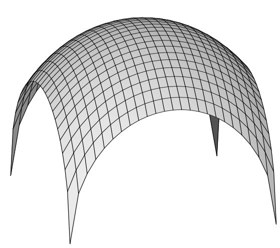
There are, of course, other ways to embed the sphere $\mathcal{S}^D$ in $\mathcal{E}^{D+1}$. We may take, for instance,

$$X^\mu = x^\mu\ \ (\mu = 1,\ldots,D)\quad ,\quad X^{D+1} = \left(R^2 - \sum_{j=1}^{D}(x^j)^2\right)^{1/2}\quad , \qquad (1.22)$$

which describes the upper half of the sphere.

Excercise 7 *Show that the metric tensor for this coordinate system has the form*

$$g_{\mu\nu} = \delta_{\mu\nu} + \frac{x^\mu\ x^\nu}{(X^{D+1})^2}\ \ . \qquad (1.23)$$



Part of the sphere $\mathcal{S}^2$ in the coordinate representation (1.22), with lines of constant x^1 or x^2.

An 'antispherical' surface

This is a D-dimensional space with a diagonal metric

$$g_{11} = \frac{R^2}{(x^1)^2} \quad , \quad g_{jj} = -\frac{R^2}{(x^1)^2} \ , \ j = 2, \ldots, D \ . \tag{1.24}$$

Excercise 8 *Show that, like Minkowski space, this surface, $\mathcal{AS}^D$, does not allow of an embedding in Euclidean space.*

The name 'antispherical' will become clear later.

A cyclinder

A two-dimensional cylindrical tube of radius R can be embedded in $\mathcal{E}^3$ as follows:

$$X^1 = R\ \sin(x^1) \ , \ X^2 = R\ \cos(x^1) \ , \ X^3 = x^2 \ . \tag{1.25}$$

Excercise 9 *Show that the induced metric is that of $\mathcal{E}^2$, so that, in the sense discussed here, this space is flat.*

Any distinction between the cylinder and Euclidean space is only apparent from considerations of a more global character (as we shall see).

A torus

A two-dimensional toroidal surface embedded in $\mathcal{E}^3$ can be written as

$$\begin{aligned} X^1 &= (R + a\cos(x^1))\ \sin(x^2) \ , \\ X^2 &= (R + a\cos(x^1))\ \cos(x^2) \ , \\ X^3 &= a\sin(x^1) \ . \end{aligned} \tag{1.26}$$

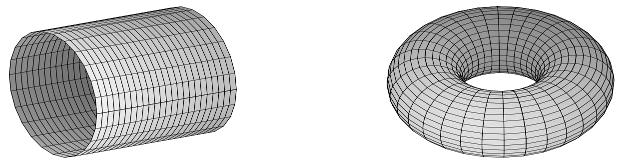
This describes a torus lying parallel to the plane $X^3 = 0$, with inner radius $R - a$ and outer radius $R + a$.

Excercise 10 *Show that the induced metric is diagonal, with*

$$g_{11} = a^2 \ , \ g_{22} = (R + a\cos(x^1))^2 \ . \tag{1.27}$$

It should be noted that the torus can be introduced in another way: one may simply take the plane $\mathcal{E}^2$, and *identify* points (x^1, x^2) and $(x^1 + k_1, x^2 + k_2)$ for any two integers $k_{1,2}$. These *periodic boundary conditions* also give a toroidal topology, but the metric (after all, a locally defined object) is still[7] that of $\mathcal{E}^2$. Some care is usually required in distinguishing these two ways of introducing a toroidal structure.

[7]Things are not so trivial as they may seem, since the point (0.0001,0.5) is now considered to be equally far from (0.0003,0.5) and from (0.9999,0.5).



Cylinder and torus, with lines of constant x^1 or constant x^2 (in the representation discussed in the text).

Excercise 11 *Consider the following situation: an intelligent ant does geometry on a flat surface of temperature T_0. It uses a Cartesian coordinate system, using a very small ruler to determine distances. Now the surface temperature changes in an uneven manner, so that there are 'hot' regions, with temperature $T > T_0$, and 'cold' regions, with $T < T_0$. The ant is not aware of this since it is wearing thick shoes: the ruler, however, contracts or expands according to the surface's temperature. Show that the ant will observe that the metric has changed. Show that in 'hot' regions, the metric tensor becones* smaller*, and in 'cold' regions it becomes* larger*. Show that the change in the metric tensor is given by a factor* $\exp(\epsilon(T-T_0))^{-2}$*, where ϵ is the linear coefficient of expansion of the ant's ruler. You may assume that the surface itself has zero zoefficient of expansion, and that the hot and cold regions are large compared to the length of the ruler. Find a distribution of temperature that will fool the ant into thinking that it moves about on the surface of (part of) a sphere, and one that makes the ant believe it is on a torus.*

1.2 Tensors

1.2.1 Coordinate transformations

In the introduction we have argued that it is important to examine the effects of *coordinate transformations*. Consider such a transformation, in which the original coordinates x^μ are expressed in a set of new coordinates x'^μ:

$$x^\mu = x^\mu(x') \ . \tag{1.28}$$

We shall assume that we can also invert this, *i.e.* x'^μ can also be written as a function of the x^μ.

Excercise 12 *Show that he matrices of the partial derivatives must be each other's inverse:*

$$\frac{\partial x^\mu(x')}{\partial x'^\nu}\,\frac{\partial x'^\nu(x)}{\partial x^\alpha} = \delta^\mu{}_\alpha \quad , \quad \frac{\partial x'^\mu(x)}{\partial x^\nu}\,\frac{\partial x^\nu(x')}{\partial x'^\alpha} = \delta^\mu{}_\alpha \ . \tag{1.29}$$

For the infinitesimal steps dx^μ we then have

$$dx^\mu = \frac{\partial x^\mu(x')}{\partial x'^\nu}\, dx'^\nu \ , \tag{1.30}$$

or

$$dx'^{\mu} = \frac{\partial x'^{\mu}(x)}{\partial x^{\nu}}\, dx^{\nu} \quad . \tag{1.31}$$

1.2.2 Contravariant vectors

We use Eq.(1.31) to arrive at the following definition. Consider a function $A^{\mu}(x)$, and the transformation (1.28). Let us denote the object A^{μ}, *expressed in terms of the x' rather than of the x*, by $A'^{\mu}(x')$. Note that, in A', the index $^{\mu}$ refers to the μ-th component in the new system, rather than the μ-th component in the old one. We indicate this by a prime. Now, *if* it so happens that

$$A'^{\mu}(x') = \frac{\partial x'^{\mu}(x)}{\partial x^{\nu}}\, A^{\nu}(x) \quad , \tag{1.32}$$

then A is called a **contravariant vector**. The object dx^{μ} is a contravariant vector. The components of A are, in general, mixed by the coordinate transform. Note that the mere presence of an upper index does *not* guarantee contravariance: in principle, the contravariant property should be checked by applying an explicit coordinate transformation.

Excercise 13 *Show that any linear combination of contravariant vectors is also a contravariant vector.*

1.2.3 Scalars

An object that does not change under coordinate tansformations:

$$S(x) = S(x(x')) = S'(x') \quad , \tag{1.33}$$

is called a **scalar**. An example is the metric, $(ds)^2$. The value of a scalar may change, of course, from point to point: but its value *at a given point* is independent of the choice of the coordinate system at that point.

1.2.4 Covariant vectors

Consider the scalar distance function, $(ds)^2$. We must have

$$(ds)^2 = dx_{\mu}\, dx^{\mu} = dx'_{\nu}\, dx'^{\nu} \quad , \tag{1.34}$$

which implies that we need to have

$$dx'_{\nu} = \frac{\partial x^{\mu}(x')}{\partial x'^{\nu}}\, dx_{\mu} \quad . \tag{1.35}$$

We use this also as a definition: *every object A_{μ} that, under the coordinate transformation (1.28), obeys*

$$A'_{\mu}(x') = \frac{\partial x^{\nu}}{\partial x'^{\mu}}\, A_{\nu}(x) \tag{1.36}$$

is defined to be a **covariant vector**. The object dx_{μ} is a covariant vector.

Excercise 14 *Show that if A^μ is a contravariant vector, then A_μ is automatically a covariant vector, and vice versa. Show that any linear combination of covariant vectors is a covariant vector.*

Excercise 15 *Show that if A^μ is contravariant, then the object $A^\mu + A_\mu$ is not a vector unless $A^\mu = 0$.*

1.2.5 Higher-rank tensors

Consider two contravariant vectors, A^μ and B^μ. We can construct their outer product as a two-index object:

$$T^{\mu\nu} = A^\mu \; B^\nu \quad . \tag{1.37}$$

By construction, its transformation character is given by

$$T^{\mu\nu}(x) = \frac{\partial x^\mu(x')}{\partial x'^\alpha} \; \frac{\partial x^\nu(x')}{\partial x'^\beta} \; T'^{\alpha\beta}(x') \quad , \tag{1.38}$$

which defines a *contravariant tensor of rank two*. For covariant tensors of rank two, and mixed co- and contravariant tensors of rank two, the definition is straightforward:

$$T_{\mu\nu}(x) = \frac{\partial x'^\alpha}{\partial x^\mu} \; \frac{\partial x'^\beta}{\partial x^\nu} \; T'_{\alpha\beta}(x') \quad , \tag{1.39}$$

and

$$T^\mu{}_\nu(x) = \frac{\partial x^\mu}{\partial x'^\alpha} \; \frac{\partial x'^\beta}{\partial x^\nu} \; T'^\alpha{}_\beta(x') \quad . \tag{1.40}$$

Higher-rank tensors are defined in the same way. Two final notes are in order here. In the first place, since a linear combination of any two tensors of the same co/contravariance and the same rank is again a tensor of the same co/contravariance and the same rank, the combination (1.37) is general enough to derive all necessary properties. In the second place, as we have already indicated before, the mere presence of indices does not guarantee co- or contravariance.

Excercise 16 *Prove the following trivial but useful facts: if a co/contravariant object is zero in one coordinate system, it is also zero in any other coordinate system. As a corollary, two tensorial quantities with the same transformation character, when equal in any one coordinate system, are automatically equal in any other system. Conversely, if any object vanishes is one system but not in another one, it is not a tensor.*

1.2.6 Covariance of the metric

The metric tensor has its own transformation character: we must have

$$(ds)^2 = g_{\mu\nu}(x) \; dx^\mu \; dx^\nu = g'_{\alpha\beta}(x') \; dx'^\alpha \; dx'^\beta \quad , \tag{1.41}$$

Excercise 17 *Show from this that the metric transforms as*

$$g'_{\alpha\beta}(x') = \frac{\partial x^\mu(x')}{\partial x'^\alpha}\, \frac{\partial x^\nu(x')}{\partial x'^\beta}\, g_{\mu\nu}\, (x(x'))\ \ , \tag{1.42}$$

and is therefore a covariant tensor of rank 2.

Excercise 18 *Show that $g^{\mu\nu}$ is a contravariant tensor.*

Excercise 19 *Show that δ^μ_ν is a tensor. Show that its form is invariant under coordinate transformations. Show that the individual components of δ^μ_ν are scalars.*

On the other hand, the constant object $\delta_{\mu\nu}$ used in section 1.1.5 is *not* a tensor: it coincides with the metric tensor in one coordinate system, but not in another.

Excercise 20 *Show that if A^μ is a contravariant vector, and B_μ is a covariant one, then $A^\mu B_\mu$ is a scalar.*

1.2.7 Examples

Cartesian and polar coordinates in two dimensions

One of the simplest nontrivial coordinate transforms is that between Cartesian and polar coordinates, especially for $\mathcal{E}^2$. Let us again denote the Cartesian coordinates by capitals, and the polar ones by lower-case symbols: we then have

$$X^1 = x^1\ \sin(x^2)\quad ,\quad X^2 = x^1\ \cos(x^2)\ \ . \tag{1.43}$$

Excercise 21 *Show tha the converse relations are*

$$x^1 = \left((X^1)^2 + (X^2)^2\right)^{1/2}\quad ,\quad x^2 = arctan\left(\frac{X^1}{X^2}\right)\ \ . \tag{1.44}$$

Excercise 22 *Show that the components of the transformation matrices are*

$$\begin{aligned} \frac{\partial X^1}{\partial x^1} &= \sin(x^2)\ \ ,\ \ & \frac{\partial X^1}{\partial x^2} &= x^1\ \cos(x^2)\ \ ,\\ \frac{\partial X^2}{\partial x^1} &= \cos(x^2)\ \ ,\ \ & \frac{\partial X^2}{\partial x^2} &= -x^1\ \cos(x^2)\ \ , \end{aligned} \tag{1.45}$$

and

$$\begin{aligned} \frac{\partial x^1}{\partial X^1} &= \frac{X^1}{R}\ \ ,\ \ & \frac{\partial x^1}{\partial X^2} &= \frac{X^2}{R}\ \ ,\\ \frac{\partial x^2}{\partial X^1} &= \frac{X^2}{R^2}\ \ ,\ \ & \frac{\partial x^2}{\partial X^2} &= -\frac{X^1}{R^2}\ \ , \end{aligned} \tag{1.46}$$

where $R = ((X^1)^2 + (X^2)^2)^{1/2}$.

Excercise 23 *Check explicitly that*

$$\frac{\partial X^\mu}{\partial x^\alpha}\,\frac{\partial x^\alpha}{\partial X^\nu} = \delta^\mu{}_\nu \quad , \quad \frac{\partial x^\mu}{\partial X^\alpha}\,\frac{\partial X^\alpha}{\partial x^\nu} = \delta^\mu{}_\nu \ . \tag{1.47}$$

Let us denote a contravariant vector, expressed in the polar coordinates, by A_p, and the corresponding vector in Cartesian coordinates by A_c. We then have

$$A_c^\mu = \frac{\partial X^\mu}{\partial x^\nu}\, A_p^\nu \ , \tag{1.48}$$

that is,

$$\begin{aligned} A_c^1 &= A_p^1 \, \sin(x^2) + A_p^2 \, x^1 \, \cos(x^2) \ , \\ A_c^2 &= A_p^1 \, \cos(x^2) - A_p^2 \, x^1 \, \sin(x^2) \ ; \end{aligned} \tag{1.49}$$

and conversely,

$$\begin{aligned} A_p^1 &= \frac{1}{R}\,\left(A_c^1 X^1 + A_c^2 X^2\right) \ , \\ A_p^2 &= \frac{1}{R^2}\,\left(A_c^1 X^2 - A_c^1 X^2\right) \ . \end{aligned} \tag{1.50}$$

Excercise 24 *The* D*-dimensional Levi-Civita symbol* $\epsilon_{\mu_1\mu_2\cdots\mu_D}$ *is defined to be totally antisymmetric in all its indices, and is therefore completely determined by its component* $\epsilon_{12\cdots D\equiv 1}$*. For* $D = 2$*, we therefore have* $\epsilon_{11} = \epsilon_{22} = 0$, $\epsilon_{12} = -\epsilon_{21} = 1$*. Show that this symbol is* not *a covariant tensor.*

1.3 Geodesics and covariant derivatives

1.3.1 Christoffel symbols

It turns out to be extremely useful to define an object that depends on the point-to-point variation of the metric. This is called the *Christoffel symbol*. It is conventional to denote, by a comma'ed index, a partial derivative: for any object $A(x)$, we write

$$A(x)_{,\mu} = \frac{\partial}{\partial x^\mu}\, A(x) \ . \tag{1.51}$$

The Christoffel symbol of the first kind is defined by

$$\Gamma_{\mu\nu\alpha} = \frac{1}{2}\,(\, g_{\mu\nu,\alpha} + g_{\mu\alpha,\nu} - g_{\nu\alpha,\mu}\,) \ . \tag{1.52}$$

This object is symmetric in its last two indices. In addition, we have the Christoffel symbols of the second kind:

$$\Gamma^\mu{}_{\nu\alpha} = g^{\mu\beta}\,\Gamma_{\beta\nu\alpha} \ . \tag{1.53}$$

The reason for the introduction of these objects appears below.

Excercise 25 *Show that Christoffel symbols are* not *tensors, by examining the metric of* $\mathcal{E}^2$ *in Cartesian and polar coordinates.*

1.3.2 Geodesics

In Euclidean geometry, the concept of a straight line is fundamental. In a space with more general metric, 'straightness' of a curve is a more problematic concept. We shall adhere to the Euclidean notion that the straight line forms the shortest curve between two given points. In a general space, we define the *geodesic* to be that curve between two given points that has the shortest (or, at least, extremal) length: any slight first-order deviation from the geodesic path results in a curve that is, to first order in the change of path, of equal length. Clearly, a variational excercise is in order here. Let us consider a general curve $x^\mu(s)$. Here, s is the *arc length*, the distance along the path, which we use here as the parameter that changes continuously as one moves along the curve. The starting point of the curve is at x_0^μ, the final point is at x_1^μ. The total length of the curve is, of course,

$$L = \int ds \quad , \tag{1.54}$$

and depends on the choice of path. Let us consider an infinitesimal variation of the path into a neighbouring one:

$$x^\mu(s) \quad \to \quad x^\mu(s) + \delta x^\mu(s) \quad . \tag{1.55}$$

We consider only variations in which the endpoints are kept fixed:

$$\delta x_0^\mu = 0 \quad , \quad \delta x_1^\mu = 0 \quad . \tag{1.56}$$

The changed path will, in general, have a different length: $L \to L + \delta L$. Geodesics are those paths for which, to first order, $\delta L = 0$. We now work out the condition for a geodesic. First of all, we have, for the infinitesimal contravariant vector:

$$dx^\mu \quad \to \quad dx^\mu + \delta(dx^\mu) \quad , \quad \text{with} \;\; \delta(dx^\mu) = d(\delta x^\mu) \quad . \tag{1.57}$$

The variation of $(ds)^2$ is then

$$\begin{aligned} 2(ds)\,\delta(ds) &= (\delta g_{\mu\nu})\, dx^\mu\, dx^\nu + 2\, g_{\mu\nu}\, dx^\mu\, d(\delta x^\nu) \\ &= g_{\mu\nu,\rho}\, dx^\mu\, dx^\nu\, \delta x^\rho + 2\, g_{\mu\nu}\, dx^\mu\, d(\delta x^\nu) \quad . \end{aligned} \tag{1.58}$$

Excercise 26 *Show that upon division by $2(ds)$, this implies*

$$\delta(ds) = \left(\frac{1}{2} g_{\mu\nu,\alpha}\, \dot{x}^\mu\, \dot{x}^\nu\, \delta x^\alpha + g_{\mu\alpha}\, \dot{x}^\mu\, \frac{\delta(dx^\alpha)}{ds}\right) ds \quad . \tag{1.59}$$

Here, and in the following, a dot will indicate derivation with respect to s

. The change in the total path length is

$$\begin{aligned} \delta L &= \int \delta(ds) \\ &= \int \left(\frac{1}{2} g_{\mu\nu,\alpha}\, \dot{x}^\mu\, \dot{x}^\nu\, \delta x^\alpha + g_{\mu\alpha}\, \dot{x}^\mu\, \frac{\delta(dx^\alpha)}{ds}\right) ds \\ &= \int \left(\frac{1}{2} g_{\mu\nu,\alpha}\, \dot{x}^\mu\, \dot{x}^\nu - \frac{d}{ds}(g_{\mu\alpha}\, \dot{x}^\mu)\right) \delta x^\alpha \quad , \end{aligned} \tag{1.60}$$

where we have used partial integration on the second term: there are no boundary contributions because of the fixity of the endpoints, Eq.(1.56). The fact that the geodesic must have extremal length under an arbitrary first-order change in the path thus implies

$$\frac{1}{2} g_{\mu\nu,\alpha}\ \dot{x}^\mu\ \dot{x}^\nu - \frac{d}{ds}(g_{\mu\alpha}\ \dot{x}^\mu) = 0 \ . \tag{1.61}$$

Excercise 27 *Show that*

$$\frac{d}{ds}(g_{\mu\alpha}\ \dot{x}^\mu) = g_{\mu\alpha}\ \ddot{x}^\mu + \frac{1}{2}\left(g_{\mu\alpha,\beta} + g_{\beta\alpha,\mu}\right)\ \dot{x}^\mu\ \dot{x}^\beta \ . \tag{1.62}$$

The requirement for the curve x^μ to be a geodesic can therefore be written as

$$\ddot{x}_\mu + \Gamma_{\mu\alpha\beta}\ \dot{x}^\alpha\ \dot{x}^\beta = 0 \ , \tag{1.63}$$

or, in a more 'standard' way,

$$\ddot{x}^\mu + \Gamma^\mu{}_{\alpha\beta}\ \dot{x}^\alpha\ \dot{x}^\beta = 0 \ . \tag{1.64}$$

This is a set of D differential equations, the simultaneous solution of which determines the geodesic.

1.3.3 The kinematic condition

A final result follows directly from the definition of the invariant $(ds)^2$. Let us assume that $(ds)^2$ is positive, as expected for, for instance, the coordinates of a point particle with nonzero rest mass. From the fact that, for a curve $x^\mu(s)$, the derivative is $dx^\mu(s)/ds$, we have immediately what we may call the *kinematic condition*:

$$\dot{x}^\mu(s)\ \dot{x}_\mu(s) = 1 \ . \tag{1.65}$$

This equation holds for *any* curve parametrized by s, not just for geodesics. However, it is related to the set of D geodesic equations, as can be seen from the following reasoning. Let us consider the fact that along a curve parametrized by s, the metric will also vary as s varies since it depends, in general, on x. Thus,

$$\begin{aligned}
\frac{d}{ds}\ (\dot{x}_\mu \dot{x}^\mu) &= \frac{d}{ds}\ \dot{x}^\mu\ \dot{x}^\nu\ g_{\mu\nu} = 2\ \ddot{x}^\mu\ \dot{x}^\nu\ g_{\mu\nu} + \dot{x}^\mu\ \dot{x}^\nu\ \frac{d}{ds} g_{\mu\nu} \\
&= -2\ \Gamma^\mu{}_{\alpha\beta}\ \dot{x}^\alpha\ \dot{x}^\beta\ \dot{x}^\nu\ g_{\mu\nu} + \dot{x}^\mu\ \dot{x}^\nu\ \dot{x}^\alpha\ g_{\mu\nu,\alpha} \\
&= \left(-2\ \Gamma_{\mu\nu\alpha} + g_{\mu\nu,\alpha}\right)\ \dot{x}^\mu\ \dot{x}^\nu\ \dot{x}^\alpha \\
&= \left(-g_{\mu\alpha,\nu} + g_{\nu\alpha,\mu}\right)\ \dot{x}^\mu\ \dot{x}^\nu\ \dot{x}^\alpha = 0 \ .
\end{aligned} \tag{1.66}$$

The *constancy* of $\dot{x}_\mu \dot{x}^\mu$, therefore, follows from the geodesic equations; the fact that it equals unity does not follow, since the geodesic equations are homogenous (of degree -2) in ds. Nevertheless, Eq.(1.65) tells us that of the D geodesic equations, only $D - 1$ are actually independent.

It should be noted that there exist metrics for which $(ds)^2$ may be negative. We might then replace ds by $\sqrt{-(ds)^2}$ to parametrize the curves, which would result in $\dot{x}_\mu \dot{x}^\mu = -1$, with the same consequence for the interdependence of the geodesic equations. Finally, in the case that $(ds)^2 = 0$, the arc length is of course useless to parametrize a curve, and one should look for another parametrization. TThis would, for instance, be the case of the path traced by a light signal, for which $(ds) = 0$ along the path: in that case, a possible parametrization of the path is given by observing the light signal's path from *another* coordinate system, in which the subsequent positions of the light signal *can* be times unambiguously. The notion of a geodesic as an extremal curve, however, remains also in this case.

Excercise 28 *Find the kinematic condition on $\mathcal{E}^n$, on the n-dimensional spherical surface, and on the two-dimnsional torus.*

1.3.4 Examples

Geodesics in the Euclidean space $\mathcal{E}^2$

Excercise 29 *Consider $\mathcal{E}^2$ with Cartesian coordinates. Show that all Christoffel symbols vanish, and that the geodesic equations are*

$$\ddot{x}^1 = \ddot{x}^2 = 0 \ . \tag{1.67}$$

At this point we may make a useful observation: for physicists it is handy to imagine a point particle of unit mass, with coordinates x^μ, and to let s play the rôle of an evolution parameter, or simply of *time*, or rather of *proper time*. The geodesic equations then take the form of equations of motion, and are, like Newton's mechanics, of second order. Eq.(1.67) is then seen to embody nothing else than Newton's first law: the geodesics describe uniform motion along straight lines, with unit velocity. Such trajectories are described by 3 parameters, for instance the position at $s = 0$ and the directional coefficient of the motion.

We now examine the same $\mathcal{E}^2$, in polar coordinates. The following excercises lead us through this treatment. For ease of visualization, we shall denote x^1 by r, and x^2 by φ.

Excercise 30 *Show that the metric is in this case given by*

$$g_{rr} = 1 \ , \ g_{\varphi\varphi} = r^2 \ , \ g_{r\varphi} = g_{\varphi r} = 0 \ , \tag{1.68}$$

Excercise 31 *Show that the only nonvanishing Christoffel symbols are*

$$\Gamma_{r\varphi\varphi} = -r \ , \ \Gamma_{\varphi r\varphi} = \Gamma_{\varphi\varphi r} = r \ , \ \Gamma^r{}_{\varphi\varphi} = -r \ , \ \Gamma^\varphi{}_{r\varphi} = \Gamma^\varphi{}_{\varphi r} = \frac{1}{r} \ . \tag{1.69}$$

Excercise 32 *Show that the kinematic and geodesic equations are given by*

$$\dot{r}^2 + r^2\dot{\varphi}^2 = 1 \ , \tag{1.70}$$

$$\ddot{r} - r\dot{\varphi}^2 = 0 \ , \tag{1.71}$$

$$\ddot{\varphi} + \frac{2}{r}\dot{r}\dot{\varphi} = 0 \ . \tag{1.72}$$

Excercise 33 *Show that Eq.(1.72) implies that*

$$\frac{d}{ds}\left(r^2\,\dot{\varphi}\right) = 0 \;\; , \tag{1.73}$$

so that $L \equiv r^2\dot{\varphi}$ *is conserved.*

Actually, this is nothing but conservation of angular momentum.

Excercise 34 *Show that Eq.(1.70) implies*

$$\dot{r}^2 + \frac{L^2}{r^2} = 1 \;\; . \tag{1.74}$$

Excercise 35 *Show that this leads to*

$$\frac{1}{2}\frac{d}{ds}r^2 = \sqrt{r^2 - L^2} \;\; . \tag{1.75}$$

Excercise 36 *Show that this can be integrated, over* r *rather than over* s*, to give*

$$s = \int \frac{d(r^2)}{2\sqrt{r^2 - L^2}} = \sqrt{r^2 - L^2} + s_0 \;\; , \tag{1.76}$$

and that the general solution reads

$$r(s) = \sqrt{(s - s_0)^2 + L^2} \quad , \quad \varphi(s) = \arctan\left(\frac{s - s_0}{L}\right) + a \;\; . \tag{1.77}$$

The geodesic again depends on 3 parameters (L, s_0 and a), but in terms of polar coordinates it does not look straight at all, of course[8]. Nevertheless, it describes the same class of curves in $\mathcal{E}^2$. Note, also, that we never used Eq.(1.71), although the solution (1.77) satisfies it: the three equations are not independent, and that is why it is easier to use Eq.(1.70), which is of first order, rather than Eq.(1.71) which is of second order in the derivatives.

Geodesics on the two-sphere $\mathcal{S}^2$

Let us again consider the two-sphere $\mathcal{S}^2$ with radius R. We denote x^1 by ϑ and x^2 by φ, so that

$$X^1 = R\sin(\vartheta)\sin(\varphi) \;\; , \;\; X^2 = R\sin(\vartheta)\cos(\varphi) \;\; , \;\; X^3 = R\cos(\vartheta) \;\; . \tag{1.78}$$

The metric is of course

$$g_{\vartheta\vartheta} = R^2 \;\; , \;\; g_{\varphi\varphi} = R^2\;\sin(\vartheta)^2 \;\; , \;\; g_{\vartheta\varphi} = g_{\varphi\vartheta} = 0 \;\; , \tag{1.79}$$

[8]For $L \to 0$, the solution describes a straight line with directional coefficient a, crossing the origin at $s = s_0$.

Excercise 37 *Show that the nonvanishing Christoffel symbols of the second kind are*

$$\Gamma^{\vartheta}{}_{\varphi\varphi} = -\cos(\vartheta)\ \sin(\vartheta) \ \ , \ \ \Gamma^{\varphi}{}_{\vartheta\varphi} = \Gamma^{\varphi}{}_{\varphi\vartheta} = \frac{\cos(\vartheta)}{\sin(\vartheta)} \ \ . \tag{1.80}$$

Excercise 38 *Show that the kinematic and geodesic equations are*

$$\begin{aligned} R^2\ \dot\vartheta^2 + R^2\ \sin(\vartheta)^2\ \dot\varphi^2 &= 1 \ , \\ \ddot\vartheta - \sin(\vartheta)\ \cos(\vartheta)\ \dot\varphi^2 &= 0 \ , \\ \ddot\varphi + 2\cos(\vartheta)\ \dot\vartheta\ \dot\varphi/\sin(\vartheta) &= 0 \ . \end{aligned} \tag{1.81}$$

The simplest insight in the solutions is obtained by working in the embedded space.

Excercise 39 *Verify that,* if *the geodesic conditions (1.81) hold,*

$$\frac{d^2}{ds^2} X^m(s) = -X^m(s) \ \ , \ \ m = 1,2,3 \ . \tag{1.82}$$

Viewing the geodesic as the trajectory in time of a moving particle, we see that the particle's acceleration is constant, and always pointing to the origin. Therefore, the *effective* constraint forces that keep the particle on the spherical surface are central. The motion is therefore confined to a plane (in $\mathcal{E}^3$!) through the origin. All the geodesics therefore lie in intersections of the spherical surface and planes through the origin, *i.e.* they are parts of great circles. The particle travels along one of these with constant (unit) velocity.

Geodesics on the cylinder: winding number

We consider the cylinder with unit radius, embedded in $\mathcal{E}^3$ according to Eq.(1.25). Since the metric in terms of these coordinates is that of $\mathcal{E}^2$, so are the geodesics.

Excercise 40 *Show that between the points* $\vec{x}_0 = (x_0^1, x_0^2)$ *and* $\vec{x}_1 = (x_1^1, x_1^2)$ *there is a geodesic of the form*

$$\vec{x}_{(0)}(s) = \vec{x}_0 + s\ \vec{v}_0 \ \ \ , \ \ \ \vec{v}_0 = (\vec{x}_1 - \vec{x}_0)\,/|\vec{x}_1 - \vec{x}_0| \ \ . \tag{1.83}$$

The geodesic is defined such that it crosses the point $\vec{x}_0$ at $s = 0$, and is traversed with unit velocity. There is an observation to be made here, however. Since coordinates (x^1, x^2) and $(x^1 + 2\pi, x^2)$ refer to *the same point*, there is actually an infinite set of geodesics[9] between the points referred to by the coordinates $\vec{x}_0$ and $\vec{x}_1$.

Excercise 41 *Show that these alternative geodesics are given by*

$$\begin{aligned} &\vec{x}_{(k)}(s) = \vec{x}_0 + s\ \vec{v}_k \ \ \ , \\ &\vec{v}_k = \vec{w}_k/|\vec{w}_k| \ \ \ , \ \ \ \vec{w}_k = (x_1^1 + 2k\pi - x_0^1\ ,\ x_1^2 - x_0^2) \ \ , \end{aligned} \tag{1.84}$$

for arbitrary integers k.

[9] We disregard the possibility of shifting both x_0^1 and x_1^1 by the same multiple of 2π: that would give paths that are truly indistinguishable from one another.

These geodesics are all extremal against small variations of the path. They differ in their *winding number*, that is the number of times the path winds around the cylinder. The winding number is a *topological* property: two paths with different winding number cannot be transformed continuously from one to the other without cutting. Since all geodesics are traversed by the point particle with unit velocity, the particle will, in general, not arrive at $\vec{x}_1$ at the same time s over all geodesics.

Geodesics on the antisphere $\mathcal{AS}^2$

We consider the antisphere $\mathcal{AS}^2$. The metric, as we have seen, is

$$g_{11} = -g_{22} = (x^1)^{-2} \quad , \quad g_{12} = g_{21} = 0 \ , \tag{1.85}$$

Excercise 42 *Show that the nonvanishing Christoffel symbols are*

$$\begin{aligned} \Gamma_{111} = \Gamma_{122} = -\Gamma_{212} = -\Gamma_{221} = -(x^1)^{-3} \quad , \\ \Gamma^1{}_{11} = \Gamma^1{}_{22} = \Gamma^2{}_{12} = \Gamma^2{}_{21} = -(x^1)^{-1} \ , \end{aligned} \tag{1.86}$$

and that the kinematic and geodesic equations read

$$\frac{1}{(x^1)^2}\left((\dot{x}^1)^2 - (\dot{x}^2)^2\right) = 1 \ , \tag{1.87}$$

$$\ddot{x}^1 - \frac{1}{x^1}\left((\dot{x}^1)^2 + (\dot{x}^2)^2\right) = 0 \ , \tag{1.88}$$

$$\ddot{x}^2 - \frac{2}{x^1}\,\dot{x}^1\,\dot{x}^2 = 0 \ . \tag{1.89}$$

We see that Eq.(1.89) implies that $\dot{x}^2/(x^1)^2$ is conserved, or

$$\dot{x}^2 = K\,(x^1)^2 \ , \tag{1.90}$$

which, inserted in Eq.(1.87), implies that

$$\dot{x}^1 = x^1\,\sqrt{1+K^2(x^1)^2} \ . \tag{1.91}$$

Excercise 43 *Show that the solution for $x^1(s)$ is, in this case,*

$$x^1(s) = \frac{1}{K\sinh(s-s_0)} \ . \tag{1.92}$$

The shape of the geodesics can be found most easily from

$$\frac{dx^2}{dx^1} = \frac{\dot{x}^2}{\dot{x}^1} = \frac{K\,x^1}{\sqrt{1+K^2(x^1)^2}} = \frac{1}{K}\,\frac{d}{dx^1}\sqrt{1+K^2(x^1)^2} \ . \tag{1.93}$$

Excercise 44 *Show that x^1 and x^2 now have the relation*

$$(x^2-a)^2 - (x^1)^2 = 1/K^2 \ . \tag{1.94}$$

In these coordinates, the geodesics are hyperbolae, traversed with unit velocity.

Geodesics on a 'rotating disk'

We consider a three-dimensional space. For ease of visualization, we rename the three coordinates as

$$x^1 = r \quad , \quad x^2 = \varphi \quad , \quad x^3 = t \ . \tag{1.95}$$

We use the following embedding in $\mathcal{E}^3$:

$$X^1 = r\sin(\varphi + \omega t) \quad , \quad X^2 = r\cos(\varphi + \omega t) \quad , \quad X^3 = t \ . \tag{1.96}$$

Here, ω is a constant. The simplest picture here is that of a plane with polar coordinates, which rotates with constant angular velocity ω as the time, t, progresses.

Excercise 45 *Show that the induced covariant metric tensor has the following nonzero elements:*

$$g_{rr} = 1 \ , \ g_{\varphi\varphi} = r^2 \ , \ g_{tt} = 1 + \omega^2 r^2 \ , \ g_{t\varphi} = g_{\varphi t} = \omega r \ , \tag{1.97}$$

and that the covariant metric tensor has nonzero elements

$$g^{rr} = g^{tt} = 1 \ , \ g^{\varphi\varphi} = \omega^2 + \frac{1}{r^2} \ , \ g^{-\varphi t} = g^{t\varphi} = -\omega \ . \tag{1.98}$$

Excercise 46 *Show that the nonvanishing Christoffel symbols of the second knd are*

$$\begin{aligned} &\Gamma^r{}_{\varphi\varphi} = -r \ , \ \Gamma^r{}_{\varphi t} = \Gamma^r{}_{t\varphi} = -\omega r \ , \ \Gamma^r{}_{tt} = -\omega^2 r \ , \\ &\Gamma^\varphi{}_{r\varphi} = \Gamma^\varphi{}_{\varphi r} = \frac{1}{r} \ , \ \Gamma^\varphi{}_{rt} = \Gamma^\varphi{}_{tr} = \frac{\omega}{r} \ . \end{aligned} \tag{1.99}$$

Excercise 47 *Show that the four equations describing the geodesics are, then*

$$\dot{r}^2 + \left(r\dot{\varphi} + \omega\dot{t}\right)^2 + (\dot{t})^2 = 1 \ , \tag{1.100}$$

$$\ddot{r} - r\dot{\varphi}^2 - 2\omega r\dot{\varphi}\dot{t} - r\omega^2\dot{t}^2 = 0 \ , \tag{1.101}$$

$$\ddot{\varphi} + \frac{2}{r}\dot{r}\dot{\varphi} + \frac{2\omega}{r}\dot{r}\dot{t} = 0 \ , \tag{1.102}$$

$$\ddot{t} = 0 \ . \tag{1.103}$$

The 'time' t is a linear function of s (*cf.* Eq.(1.103)), and we may write

$$t = s\,c + t_0 \ , \ |c| \le 1 \ , \tag{1.104}$$

since $|dt/ds|$ has to be smaller than 1 from Eq.(1.100).

Excercise 48 *Show that this defintion of t modifies Eq.(1.102) into*

$$\ddot{\varphi} + \frac{2}{r}\dot{\varphi}\dot{r} + \frac{2\omega c}{r}\dot{r} = 0 \ , \tag{1.105}$$

and that this implies that

$$\dot{\varphi} = \frac{K}{r^2} - \omega c \ , \tag{1.106}$$

where K is a constant.

Excercise 49 *Show that this last result transforms Eq.(1.100) into*

$$\frac{dr}{ds} = \frac{1}{r}\sqrt{\beta r^2 - K^2} \quad , \quad \beta = 1 - c^2 \ . \tag{1.107}$$

Excercise 50 *show that, by straightforward integration of ds over sr we then find, for r:*

$$r = \sqrt{\beta(s - s_0)^2 + \frac{K^2}{\beta}} \ , \tag{1.108}$$

with s_0 a constant, and moreover

$$\varphi = \arctan\left(\frac{\beta}{K}(s - s_0)\right) - \omega c s + \varphi_0 \ . \tag{1.109}$$

We see that the geodesic approaches closest to the origin for $s = s_0$. For large $|s - s_0|$, r and φ are approximately linear in s. We see that, indeed, the 'time' t and the 'time' (or, more appropriately, the *eigentime*) s are equivalent since one is transformed into the other by a simple change of units and displacement of the origin. Let us therefore arrange things such that $t_0 = -cs_0$, so that $t = 0$ corresponds to closest approach to the origin. Moreover, let us define constants u and v such that

$$\beta = c^2 v^2 \quad , \quad K = \frac{c\, v^2}{u} \ ; \tag{1.110}$$

then, we can express r and φ as functions of the time t:

$$r(t) = \sqrt{v^2 t^2 + \frac{v^2}{u^2}} \quad , \quad \varphi(t) = \arctan(ut) - \omega t + \varphi_0 \ . \tag{1.111}$$

The interpretation is as follows: v is the asymptotic radial velocity, and v/u is the closest approach to the origin: at that moment $t = 0$, the angular velocity is $u - \omega$.

1.4 Covariant derivatives

1.4.1 Covariant derivatives of scalars

In physical laws, (partial) derivatives of objects occur all over the place. It is therefore sensible to investigate the behaviour of such derivatives under coordinate transformations.

Consider a scalar function $S(x)$ of x. A coordinate transformation to x' results in a function S' of x' by

$$S(x) = S(x(x')) = S'(x') \ . \tag{1.112}$$

For the partial derivative to x^μ we may therefore write

$$S_{,\mu} = \frac{\partial}{\partial x^\mu} S(x) = \frac{\partial x'^\alpha}{\partial x^\mu}\, \frac{\partial}{\partial x'^\alpha} S'(x') = \frac{\partial x'^\alpha}{\partial x^\mu}\, S'_{,\alpha} \ . \tag{1.113}$$

Excercise 51 *Show that this implies that, for scalar $S(x)$, the partial derivative is automatically a covariant vector.*

1.4.2 Non-covariant derivatives of covariant vectors

We now try to repeat the treatment of the previous section for a covariant vector A_μ. In the first place, since A is assumed to be covariant, we must have

$$A_\mu(x) = A'_\alpha(x')\,\frac{\partial x'^\alpha}{\partial x^\mu}\quad . \tag{1.114}$$

The partial derivative now has the result

$$\begin{aligned} A_{\mu,\nu}(x) &= \frac{\partial}{\partial x^\nu}\left[A_\mu(x)\right] = \frac{\partial}{\partial x^\nu}\left[A'_\alpha(x')\,\frac{\partial x'^\alpha}{\partial x^\mu}\right] \\ &= \frac{\partial x'^\beta}{\partial x^\nu}\,\frac{\partial x'^\alpha}{\partial x^\mu}\,A'_{\alpha,\beta}(x') + \frac{\partial^2 x'^\alpha}{\partial x^\mu\,\partial x^\nu}\,A'_\alpha(x')\quad . \end{aligned} \tag{1.115}$$

Excercise 52 *Show that the above result implies that $A_{\mu,\nu}$ is* not *covariant.*

The same holds for derivatives of covariant tensors.

Excercise 53 *Show that*

$$\begin{aligned} g_{\mu\nu,\alpha} &= \frac{\partial^2 x'^\rho}{\partial x^\mu\,\partial x^\alpha}\,\frac{\partial x^\sigma}{\partial x^\nu}\,g'_{\rho\sigma} + \frac{\partial x'^\rho}{\partial x^\mu}\,\frac{\partial^2 x^\sigma}{\partial x^\nu\,\partial x^\alpha}\,g'_{\rho\sigma} \\ &\quad + \frac{\partial x'^\rho}{\partial x^\mu}\,\frac{\partial x'^\sigma}{\partial x^\nu}\,\frac{\partial x'^\beta}{\partial x^\alpha}\,g'_{\rho\sigma,\beta}\quad , \end{aligned} \tag{1.116}$$

and that the last term alone would give covariance.

Excercise 54 *Show that, for the Christoffel symbol, we have*

$$\Gamma_{\alpha\mu\nu} = \frac{\partial x'^\beta}{\partial x^\alpha}\,\frac{\partial^2 x'^\lambda}{\partial x^\mu\,\partial x^\nu}\,g'_{\beta\lambda} + \frac{\partial x'^\beta}{\partial x^\alpha}\,\frac{\partial x'^\rho}{\partial x^\mu}\,\frac{\partial x'^\sigma}{\partial x^\nu}\,\Gamma'_{\beta\rho\sigma}\quad . \tag{1.117}$$

As we have already anticipated, the Christoffel symbol is not a tensor.

1.4.3 Covariant derivatives of covariant vectors

Using the Christoffel symbol, we can construct a derivative-like procedure that *does* behave in a covariant manner. We define, by a semicolon, the *covariant derivative of a covariant vector* as

$$A_{\mu;\nu} \equiv A_{\mu,\nu} - \Gamma^\alpha{}_{\mu\nu}\,A_\alpha = A_{\mu,\nu} - A^\alpha\,\Gamma_{\alpha\mu\nu}\quad . \tag{1.118}$$

Under a coordinate transform, we find

$$\begin{aligned} A_{\mu;\nu} &= \frac{\partial x'^\beta}{\partial x^\nu}\,\frac{\partial x'^\alpha}{\partial x^\mu}\,A'_{\alpha,\beta}(x') + \frac{\partial^2 x'^\alpha}{\partial x^\mu\,\partial x^\nu}\,A'_\alpha(x') \\ &- \left(A'^\gamma\,\frac{\partial x^\alpha}{\partial x'^\gamma}\right)\left(\frac{\partial x'^\beta}{\partial x^\alpha}\,\frac{\partial^2 x'^\lambda}{\partial x^\mu\,\partial x^\nu}\,g'_{\beta\lambda} + \frac{\partial x'^\beta}{\partial x^\alpha}\,\frac{\partial x'^\rho}{\partial x^\mu}\,\frac{\partial x'^\sigma}{\partial x^\nu}\,\Gamma'_{\beta\rho\sigma}\right) \\ &= \frac{\partial x'^\rho}{\partial x^\mu}\,\frac{\partial x'^\sigma}{\partial x^\nu}\left(A'_{\mu,\nu} - A'^\alpha\,\Gamma'_{\alpha\rho\sigma}\right)\quad , \end{aligned} \tag{1.119}$$

which shows that this object is, indeed, a covariant rank-two tensor.

1.4.4 Covariant derivatives of covariant tensors

Let us again consider a covariant rank-two tensor built up from two covariant vectors:

$$T_{\mu\nu} = A_\mu B_\nu \quad , \tag{1.120}$$

Covariant differentiation is now of course defined such that it satisfies Leibniz' rule:

$$\begin{aligned} T_{\mu\nu;\alpha} &= A_{\mu;\alpha} B_\nu + A_\mu B_{\nu;\alpha} \\ &= A_{\mu,\alpha} B_\nu - \Gamma^\beta{}_{\alpha\mu} A_\beta B_\nu + A_\mu B_{\nu,\alpha} - A_\mu \Gamma^\beta{}_{\alpha\nu} B_\beta \end{aligned} \tag{1.121}$$

or, in more general terms,

$$T_{\mu\nu;\alpha} = T_{\mu\nu,\alpha} - \Gamma^\beta{}_{\alpha\mu} T_{\beta\nu} - \Gamma^\beta{}_{\alpha\nu} T_{\mu\beta} \quad . \tag{1.122}$$

The generalization to higher-rank covariant tensors is of course straightforward: each index gets its own Christoffel-symbol compensation term. For a scalar, therefore, we can write

$$S_{;\alpha} = S_{,\alpha} \quad . \tag{1.123}$$

1.4.5 Covariant derivatives of contravariant vectors

Consider a scalar product of a co- and a contravariant vector:

$$S = A^\mu \; B_\mu \quad . \tag{1.124}$$

Its covariant derivative is equal to its normal derivative. Using Leibniz' rule, we can therefore write

$$S_{,\alpha} = A^\mu{}_{,\alpha} \; B_\mu + A^\mu \; B_{\mu,\alpha} \quad , \tag{1.125}$$

$$\begin{aligned} S_{;\alpha} &= A^\mu{}_{;\alpha} \; B_\mu + A^\mu \; B_{\mu;\alpha} \\ &= A^\mu{}_{;\alpha} \; B_\mu + A^\mu \; B_{\mu,\alpha} - A^\mu \; \Gamma^\beta{}_{\mu\alpha} \; B_\beta \quad . \end{aligned} \tag{1.126}$$

Excercise 55 *Show that the correct definition of the covariant derivative of the contravariant vector A^μ reads*

$$A^\mu{}_{;\alpha} = A^\mu{}_{,\alpha} + \Gamma^\mu{}_{\nu\alpha} \; A^\nu \quad . \tag{1.127}$$

Note the different sign in front of the Christoffel-symbol term! Again, the extension to higher-rank tensors, as well as to tensors with mixed upper and lower indices, is straightforward.

1.4.6 Covariant derivatives of the metric

A case of special interest is the covariant derivative of the metric tensor itself.

Excercise 56 *Show that, by the definition of covariant differentiation, we have*

$$g_{\mu\nu;\alpha} = 0 \ . \tag{1.128}$$

The metric is constant under *covariant* differentiation. Also, we would of course like to have δ^μ_ν constant under covariant differentiation.

Excercise 57 *Show that, indeed,*

$$\delta^\mu_{\nu;\alpha} = 0 \ . \tag{1.129}$$

The Kronecker tensor is constant under both normal and covariant derivation.

Excercise 58 *Show that from Leibniz' rule we can immediately infer that*

$$g^{\mu\nu}{}_{;\alpha} = 0 \ . \tag{1.130}$$

1.4.7 A note on curls

For the covariant derivative of a covariant vector,

$$A_{\mu;\nu} = A_{\mu,\nu} - \Gamma^\alpha{}_{\mu\nu}\, A_\alpha \ , \tag{1.131}$$

the last term is symmetric in $\mu \leftrightarrow \nu$.

Excercise 59 *Show that the covariant curl is equal to the standard curl:*

$$A_{\mu;\nu} - A_{\nu;\mu} = A_{\mu,\nu} - A_{\nu,\mu} \ . \tag{1.132}$$

This is a cheap and efficient way of building a rank-two covariant tensor from a covariant vector field[10].

Excercise 60 *Show that that the same trick wouldn't work for a* contravariant *vector, since the object*

$$A^\mu{}_{;\nu} - A^\nu{}_{;\mu}$$

does not have the transformation properties of any tensor[11].

1.4.8 Geodesics revisited

Consider a geodesic $x^\mu(s)$. The 'velocity' is defined as

$$v^\mu(x) = \frac{d}{ds} x^\mu(s) \ . \tag{1.133}$$

We can write the geodesic equation as

$$\frac{d}{ds} v^\mu + \Gamma^\mu{}_{\alpha\beta}\, v^\alpha\, v^\beta = v^\mu_{,\beta} \frac{d}{ds} x^\beta + \Gamma^\mu{}_{\alpha\beta}\, v^\alpha\, v^\beta = \left(v^\mu_{,\beta} + \Gamma^\mu{}_{\alpha\beta}\, v^\alpha \right) v^\beta \ . \tag{1.134}$$

Excercise 61 *Show that the geodesic equation can be written as follows:*

$$v^\mu_{;\beta}\, v^\beta = 0 \ . \tag{1.135}$$

[10]It is not for nothing that this is the form of the field strength in electromagnetism!

[11]It *does* have transformation properties, of course; they are just usueless for building any physical theory.

1.4.9 Examples

A non-conservative vector field in $\mathcal{E}^2$

We consider a vector field on $\mathcal{E}^2$, and use polar coordinates. As usual, we write $x^1 = r$, $x^2 = \varphi$, and the embedding into Cartesian coordinates X^1, X^2 is $X^1 = r\sin(\varphi)$, $X^2 = r\cos(\varphi)$. For any contravariant vector A, the relation between its polar and Cartesian coordinates is then

$$\begin{aligned} A^1 &= A^r\ \sin(\varphi) + A^\varphi\ r\ \cos(\varphi)\ \ , \\ A^2 &= A^r\ \cos(\varphi) - A^\varphi\ r\ \sin(\varphi)\ \ . \end{aligned} \tag{1.136}$$

The metric and Christoffel symbols are given in Eqs.(1.68,1.69).

Excercise 62 *Show that the covariant derivatives are given by*

$$\begin{aligned} A^r{}_{;r} &= \frac{\partial}{\partial r} A^r \quad , \quad A^r{}_{;\varphi} = \frac{\partial}{\partial \varphi} A^r - r\ A^\varphi \quad , \\ A^\varphi{}_{;r} &= \frac{\partial}{\partial r} A^\varphi + \frac{1}{r}\ A^\varphi \quad , \quad A^\varphi{}_{;\varphi} = \frac{\partial}{\partial \varphi} A^\varphi + \frac{1}{r}\ A^r \quad . \end{aligned} \tag{1.137}$$

Note that the covariant derivatives may be nonzero even for constant vectors.

As an example, we consider the vector field with polar-coordinate form[12]

$$\vec{A} = (A^r, A^\varphi) \quad , \quad A^r = f(r)\ ,\ A^\varphi = g(\varphi)\ \ . \tag{1.138}$$

Excercise 63 *Show that the covariant derivatives are*

$$\begin{aligned} & A^r{}_{;r} = f'(r) \quad , \quad A^r{}_{;\varphi} = -r g(\varphi) \quad , \\ & A^\varphi{}_{;r} = \frac{1}{r} g(\varphi) \quad , \quad A^\varphi{}_{;\varphi} = g'(\varphi) + \frac{1}{r} f(r) \quad . \end{aligned} \tag{1.139}$$

Excercise 64 *Show that the covariant* divergence *is given by*

$$A^\mu{}_{;\mu} = A^r{}_{;r} + A^\varphi{}_{;\varphi} = f'(r) + \frac{1}{r} f(r) + g'(\varphi)\ \ . \tag{1.140}$$

To check whether this is indeed a scalar quantity, we compute the vector $\vec{A}$ in Cartesian coordinate form ('c' for 'Cartesian'):

$$\begin{aligned} \vec{A^c} &= (A^{c1}, A^{c2})\ \ , \\ A^{c1} &= f(R)\ \sin(\Phi) + g(\Phi)\ R\ \cos(\Phi)\ \ , \\ A^{c2} &= f(R)\ \cos(\Phi) - g(\Phi)\ R\ \sin(\Phi)\ \ , \\ R &= \left((X^1)^2 + (X^2)^2\right)^{1/2} \quad , \quad \Phi = \arctan\left(X^1/X^2\right)\ \ . \end{aligned} \tag{1.141}$$

Note that the *numerical values* of R and Φ coincide with those of r and φ, respectively, but that they are nontrivial functions of the Cartesian coordinates.

[12] This is *not* the most general case. Also, the field is single-valued provided $g(\varphi+2\pi) = g(\varphi)$, but we shall not assume this in what follows.

Since the metric is trivial, the covariant derivative is for these coordinates equal to the standard partial derivative. It is now straightforward to check that

$$A^{c\nu}{}_{;\nu} = \frac{\partial}{\partial X^1} A^{c1} + \frac{\partial}{\partial X^2} A^{c2} = f'(R) + \frac{1}{R} f(R) + g'(\Phi) \quad , \tag{1.142}$$

so that the divergence has the same numerical value, *i.e.* it is a scalar.

Since the metric has vanishing covariant derivative, we may simply use

$$A_{\mu;\nu} = (g_{\mu\alpha}\, A^\alpha)_{;\nu} = g_{\mu\alpha}\, A^\alpha{}_{;\nu} \quad , \tag{1.143}$$

so that in this case

$$\begin{aligned} A_{r;r} &= f'(r) \quad , \quad A_{r;\varphi} = -r\, g(\varphi) \quad , \\ A_{\varphi;r} &= r\, g(\varphi) \quad , \quad A_{\varphi;\varphi} = r^2\, g'(\varphi) + r\, f(r) \quad . \end{aligned} \tag{1.144}$$

The (covariant) curl is, in polar coordinates,

$$M_{\mu\nu} = A_{\mu;\nu} - A_{\nu;\mu} \quad , \tag{1.145}$$

so that

$$M_{rr} = M_{\varphi\varphi} = 0 \quad , \quad M_{r\varphi} = -M_{\varphi r} = -2r\, g(\varphi) \quad . \tag{1.146}$$

The vector field is not conservative unless $g(\varphi) = 0$. In the Cartesian coordinate system, on the other hand, we have simply $A^c_1 = A^{c1}$, $A^c_2 = A^{c2}$ and therefore

$$M^c_{\mu\nu} = A^c{}_{\mu;\nu} - A^c{}_{\nu;\mu} \quad , \tag{1.147}$$

leading to

$$M^c_{11} = M^c_{22} = 0 \quad , \quad M^c_{12} = M^c_{21} = 2\, g(\Phi) \quad . \tag{1.148}$$

These two curls are not equal, since the curl is not a scalar; rather, we must use the transformation character of a covariant rank-two tensor:

$$\begin{aligned} M^c_{12} &= \frac{\partial x^\mu}{\partial X^1} \frac{\partial x^\nu}{\partial X^2}\, M_{\mu\nu} \\ &= \frac{\partial r}{\partial X^1} \frac{\partial \varphi}{\partial X^2}\, M_{r\varphi} + \frac{\partial \varphi}{\partial X^1} \frac{\partial r}{\partial X^2}\, M_{\varphi r} \quad = \quad -\frac{1}{r}\, M_{r\varphi} \quad . \end{aligned} \tag{1.149}$$

This checks the transformation character of the covariant curl.

1.5 Parallel displacement

1.5.1 Parallel displacement in an embedding space

The covariant derivative may be attractive as a tensor quantity, but it is not very easily visualized. To improve on this situation, we proceed as follows. We assume that our D-dimensional space can be embedded in a N-dimensional space that admits of a rectilinear coordinate system. Denoting the coordinates in the embedding space by X, and labelling the coordinates by Roman letters, we have, as before,

$$X^m = F^m(x) \ \ (m = 1, 2, \ldots, N) \quad , \quad g_{\mu\nu} = h_{mn} \frac{\partial F^m(x)}{\partial x^\mu} \frac{\partial F^n(x)}{\partial x^\nu} \quad , \tag{1.150}$$

where h_{mn} is the (constant!) metric of the embedding space.

Now, consider a contravariant vector A^μ at the point x. In the embedding space, it is denoted by N components:

$$A^m = \frac{\partial F^m(x)}{\partial x^\mu} A^\mu = F^m_{,\mu}(x) \ A^\mu \quad . \tag{1.151}$$

At the point x, the vector A^m lies, of course, in the D-dimensional plane tangential to the surface at that point. Let us now move the vector A^n from x to a nearby point $x + dx$, with coordinates $x^\mu + dx^\mu$, *without changing its direction.* At the new point, the vector A^m does, in general, *not* lie in the tangential plane, because the embedded space is in general curved in the embedding one. It has a component B^m tangential to the surface, and a component K^m normal to the surface, at the new point $x + dx$. Let us decide to drop K^m, and keep only B^m. Since it lies in the tangential plane at $x + dx$, it has also an expression in the D-dimensional indices, B^μ, and we have

$$A^m = B^m + K^m \quad , \quad B^m = B^\mu \ F^m_{,\mu}(x+dx) \quad , \quad K^m \ F_{m,\mu}(x+dx) = 0 \ \ . \tag{1.152}$$

Therefore

$$\begin{aligned} A^m \ F_{m,\nu}(x + dx) &= B^\mu \ F^m_{,\mu}(x + dx) \ F_{m,\nu}(x + dx) \\ &= B^\mu \ g_{\mu\nu}(x + dx) \\ &= B_\nu \ \ , \end{aligned} \tag{1.153}$$

and we find

$$\begin{aligned} B_\nu &= A^m \ F_{m,\nu}(x + dx) \\ &= A^\mu \ F^m_{,\mu}(x) \ F_{m,\nu}(x + dx) \\ &\approx A^\mu \ \left(F^m_{,\mu}(x) \ F_{m,\nu}(x) + F^m_{,\mu}(x) \ F_{m,\nu\alpha}(x) \ dx^\alpha\right) \quad . \end{aligned} \tag{1.154}$$

Now, $F^m_{,\mu}(x) \ F_{m,\nu}(x)$ is precisely the metric $g_{\mu\nu}(x)$.

Excercise 65 *Verify that the Christoffel symbol can be computed as follows:*

$$\Gamma_{\mu\nu\alpha} = \frac{\partial F^m}{\partial x^\mu} \frac{\partial^2 F_m}{\partial x^\nu \ \partial x^\alpha} \quad . \tag{1.155}$$

Therefore, for covariant vectors we have the following rule for parallel displacment from x to $x + dx$:

$$A_\mu \quad \to \quad A_\mu + A^\rho \ \Gamma_{\rho\mu\nu} \ dx^\nu \quad . \tag{1.156}$$

1.5.2 Rules for parallel displacement of vectors and tensors

We have seen that, under parallel displacement from x to $x + dx$, a covariant vector changes as

$$A_\mu \quad \rightarrow \quad A_\mu + A_\rho \, \Gamma^\rho{}_{\mu\nu} \, dx^\nu \quad . \tag{1.157}$$

A scalar must, of course, be unaffected by parallel displacement. Therefore, if A^μ is contravariant and B_μ is covariant, we must have, under parallel displacement,

$$\begin{aligned} A^\mu \, B_\mu \quad &\rightarrow \quad (A^\mu + dA^\mu) \; (B_\mu + B^\rho \, \Gamma_{\rho\mu\nu} \, dx^\nu) \\ &\approx \quad A^\mu \, B_\mu + dA^\mu \, B_\mu + A^\mu \, \Gamma_{\rho\mu\nu} \, dx^\nu \, B^\rho \\ &= \quad A^\mu \, B_\mu + dA^\mu \, B_\mu + A^\alpha \, \Gamma^\mu{}_{\alpha\nu} \, dx^\nu \, B_\mu \quad , \end{aligned} \tag{1.158}$$

Excercise 66 *Show that this leads to the rule for contravariant vectors:*

$$A^\mu \quad \rightarrow \quad A^\mu - A^\alpha \, \Gamma^\mu{}_{\alpha\nu} \, dx^\nu \quad . \tag{1.159}$$

Parallel displacement for higher-rank tensors follows, of course in the standard way, with one Christoffel term for each index: for instance, for rank-two tensors we have

$$\begin{aligned} T_{\mu\nu} \quad &\rightarrow \quad T_{\mu\nu} + T_{\alpha\nu} \, \Gamma^\alpha{}_{\mu\rho} \, dx^\rho + T_{\mu\alpha} \, \Gamma^\alpha{}_{\nu\rho} \, dx^\rho \quad , \\ T^{\mu\nu} \quad &\rightarrow \quad T^{\mu\nu} - T^{\alpha\nu} \, \Gamma^\mu{}_{\alpha\rho} \, dx^\rho - T^{\mu\alpha} \, \Gamma^\nu{}_{\alpha\rho} \, dx^\rho \quad , \\ T^\mu{}_\nu \quad &\rightarrow \quad T^\mu{}_\nu - T^\mu{}_\alpha \, \Gamma^\alpha{}_{\nu\rho} \, dx^\rho + T^\alpha{}_\nu \, \Gamma^\mu{}_{\alpha\rho} \, dx^\rho \quad . \end{aligned} \tag{1.160}$$

Excercise 67 *Show that we find for the covariant metric tensor the parallel-displacement rule*

$$g_{\mu\nu} \quad \rightarrow \quad g_{\mu\nu} + g_{\mu\nu,\rho} \, dx^\rho \quad ; \tag{1.161}$$

Excercise 68 *Show that for the contravariant metric tensor we find the parallel-displacement rule*

$$g^{\mu\nu} \quad \rightarrow \quad g^{\mu\nu} + g^{\mu\nu}{}_{,\rho} \, dx^\rho \quad , \tag{1.162}$$

where use may be made of

$$\begin{aligned} g^{\mu\alpha} \, g^{\beta\nu} \, g_{\alpha\beta,\rho} \quad &= \quad g^{\mu\alpha} \, \left(\delta^\nu{}_{\alpha,\rho} - g_{\alpha\beta} \, g^{\beta\nu}{}_{,\rho}\right) = - \, g^{\mu\alpha} \, g_{\alpha\beta} \, g^{\beta\nu}{}_{,\rho} \\ &= \quad - \, \delta^\mu{}_\beta \, g^{\beta\nu}{}_{,\rho} = - \, g^{\mu\nu}{}_{,\rho} \quad . \end{aligned} \tag{1.163}$$

Excercise 69 *Show that for the Kronecker tensor we find the parallel-displacement rule*

$$\delta^\mu{}_\nu \quad \rightarrow \quad \delta^\mu{}_\nu \quad . \tag{1.164}$$

In short, under parallel displacement we find explicitly that

$$g_{\mu\nu}(x) \;\rightarrow\; g_{\mu\nu}(x + dx) \;\;,\;\; g^{\mu\nu}(x) \;\rightarrow\; g^{\mu\nu}(x + dx) \;\;,\;\; \delta^\mu{}_\nu \;\rightarrow\; \delta^\mu{}_\nu \;\; . \tag{1.165}$$

1.5.3 Covariant differentiation from parallel displacement

The notion of parallel displacement serves to define a concept of two vectors at nearby points being parallel: they are *defined* to be parallel if, after parallel displacement to the same point, they are parallel at that point. Now consider a vector *field* $A_\mu(x)$, where A^μ may change from point to point. The standard derivative $A_{\mu,\alpha}(x)$ considers evaluating A_μ at $x+dx$, and subtracting from it A_μ at x. Here, we subtract two vectors that are located at different points, and hence lying in two different tangential planes. It is reasonable to replace this by considering, rather, the difference between $A_\mu(x+dx)$ and *the vector* $A_\mu(x)$, *shifted by parallel replacement to the point* $x+dx$. This is

$$\begin{aligned} & A_\mu(x+dx) - \big(A_\mu(x) + A_\alpha(x)\, \Gamma^\alpha{}_{\mu\nu}\, dx^\nu\big) \\ & \quad = \big(A_{\mu,\nu}(x) - A_\alpha(x)\, \Gamma^\alpha{}_{\mu\nu}\big)\ dx^\nu \quad , \end{aligned} \tag{1.166}$$

and we recognize here the rule for covariant differentiation.

The results (1.165) immediately show that the metric tensors and the Kronecker tensor are constant under covariant differentiation.

A final remark: the *visualization* of parallel displacement is easiest if we use an embedding space. The *rule* for parallel displacement, Eq.(1.156), is however expressed purely in terms of the Christoffel symbols, and hence independent of the existence of an embedding space. It is a notion that can be formulated entirely in terms of the D-dimensional space alone.

1.5.4 Examples

Parallel displacement in $\mathcal{E}^2$

In the Euclidean space $\mathcal{E}^2$ with Cartesian coordinates, all Christoffel symbols vanish, and parallel displacement leaves a constant vector unchanged, except for the position of the point where it is defined. To see how it works in another coordinate system, we shall use polar coordinates. For ease of visualization,we shall denote x^1 by r and x^2 by φ. The Cartesian coordinates of a point are then given, as usual by

$$X^1 = F^1(r,\varphi) = r\ \sin(\varphi) \quad , \quad X^2 = F^2(r,\varphi) = r\ \cos(\varphi) \ . \tag{1.167}$$

The Cartesian coordinates of a vector are therefore related to its polar coordinates, according to Eq.(1.150), by

$$\begin{aligned} A^1 &= A^r\ \sin(\varphi)\ +\ A^\varphi\ r\ \cos(\varphi)\ , \\ A^2 &= A^r\ \cos(\varphi)\ -\ A^\varphi\ r\ \sin(\varphi)\ . \end{aligned} \tag{1.168}$$

We now take a vector at point $x=(r,\varphi)$ and perform parallel displacement to point $x+dx=(r+dr,\varphi+d\varphi)$. The metric,

$$g_{rr} = 1 \ , \ g_{\varphi\varphi} = r^2 \ , \ g_{r\varphi} = g_{\varphi r} = 0 \ , \tag{1.169}$$

leads to the nonvanishing Christoffel symbols

$$\Gamma^r{}_{\varphi\varphi} = -r \ \ , \ \ \Gamma^\varphi{}_{r\varphi} = \Gamma^\varphi{}_{\varphi r} = 1/r \ \ . \tag{1.170}$$

Under parallel displacement of the vector (A^r, A^φ) from x to $x + dx$, its components are therefore changed (to first order!) as

$$\begin{aligned} A^r &\rightarrow A^r + dA^r \quad , \quad dA^r = r\, A^\varphi\, d\varphi \ \ , \\ A^\varphi &\rightarrow A^\varphi + dA^\varphi \quad , \quad dA^\varphi = -\frac{1}{r}\,(A^r\, d\varphi + A^\varphi\, dr) \ \ . \end{aligned} \tag{1.171}$$

We now compute the *Cartesian* components of the parallel-displaced vector, keeping in mind that its position has been moved from x to $x + dx$:

$$\begin{aligned} A^1 &\rightarrow (A^r + dA^r)\sin(\varphi + d\varphi) \\ &\quad + (A^\varphi + dA^\varphi)(r + dr)\cos(\varphi + d\varphi) \ \ , \\ A^2 &\rightarrow (A^r + dA^r)\cos(\varphi + d\varphi) \\ &\quad - (A^\varphi + dA^\varphi)(r + dr)\sin(\varphi + d\varphi) \ \ . \end{aligned} \tag{1.172}$$

To first order, all infintesimals cancel, and we find that the vector (A^1, A^2) remains unchanged, as it should in this case.

Parallel displacement on $\mathcal{S}^2$ and Foucault's pendulum

Next, we consider the two-sphere $\mathcal{S}^2$ with unit radius. Denoting x^1 by ϑ and x^2 by φ, we then have

$$X^1 = \sin(\vartheta)\ sin(\varphi) \ \ , \ \ X^2 = \sin(\vartheta)\ \cos(\varphi) \ \ , \ \ X^3 = \cos(\vartheta) \tag{1.173}$$

as a possible embedding. The relation between the Cartesian and the polar components of a contravariant vector A are, therefore

$$\begin{aligned} A^1 &= A^\vartheta\cos(\vartheta)\sin(\varphi) + A^\varphi\sin(\vartheta)\cos(\varphi) \ \ , \\ A^2 &= A^\vartheta\cos(\vartheta)\cos(\varphi) - A^\varphi\sin(\vartheta)\sin(\varphi) \ \ , \\ A^3 &= -A^\vartheta\sin(\vartheta) \ \ . \end{aligned} \tag{1.174}$$

The metric, as we have seen, is

$$g_{\vartheta\vartheta} = 1 \ \ , \ \ g_{\varphi\varphi} = \sin(\vartheta)^2 \ \ , \ \ g_{\vartheta\varphi} = g_{\varphi\vartheta} = 0 \ \ , \tag{1.175}$$

so that the nonvanishing Christoffel symbols are

$$\Gamma^\vartheta{}_{\varphi\varphi} = -\sin(\vartheta)\cos(\vartheta) \quad , \quad \Gamma^\varphi{}_{\vartheta\varphi} = \Gamma^\varphi{}_{\varphi\vartheta} = \frac{\cos(\vartheta)}{\sin(\vartheta)} \ \ . \tag{1.176}$$

The effect of a parallel displacement of the vector (A^ϑ, A^φ) from $x = (\vartheta, \varphi)$ to $x + dx = (\vartheta + d\vartheta, \varphi + d\varphi)$ is therefore

$$\begin{aligned} dA^\vartheta &= \sin(\vartheta)\ \cos(\vartheta)\ A^\varphi\ d\varphi \ \ , \\ dA^\varphi &= -\frac{\cos(\vartheta)}{\sin(\vartheta)}\ \left(A^\vartheta\ d\varphi + A^\varphi\ d\vartheta\right) \ \ . \end{aligned} \tag{1.177}$$

There are some simple cases of interest here. In the first place, the displacement may take place with constant φ. A *finite* displacement (that is, over a finite rather than an infinitesimal trajectory) with constant φ follows a geodesic, one of the 'meridians'. Putting $d\varphi = 0$ in Eq.(1.177) we then find

$$A^\vartheta = \text{constant} \quad , \quad A^\varphi \ \sin(\vartheta) = \text{constant} \ . \tag{1.178}$$

The other case is that of constant ϑ: in general this displacement, when pursued over a finite distance, only follows a geodesic for $\vartheta = \pi/2$, when the curve follows the 'equator'. For fixed ϑ, we have

$$\frac{d}{d\varphi}A^\vartheta = \sin(\vartheta)\ \cos(\vartheta)\ A^\varphi \quad , \quad \frac{d}{d\varphi}A^\varphi = -\frac{\cos(\vartheta)}{\sin(\vartheta)}\ A^\vartheta \ . \tag{1.179}$$

This leads us to an equation for A^ϑ as a function of φ with constant ϑ:

$$\left(\frac{d^2}{(d\varphi)^2} + \cos(\vartheta)^2\right)\ A^\vartheta = 0 \ . \tag{1.180}$$

The general solution is, of course,

$$\begin{aligned} A^\vartheta &= K_1\cos(\varphi\cos(\vartheta)) + K_2\sin(\varphi\cos(\vartheta)) \ , \\ A^\varphi &= -K_1\frac{\sin(\varphi\cos(\vartheta))}{\sin(\vartheta)} + K_2\frac{\cos(\varphi\cos(\vartheta))}{\sin(\vartheta)} \ . \end{aligned} \tag{1.181}$$

with boundary conditions provided by the values of A^ϑ and A^φ. To gain some insight in the behaviour of the vector A it is easiest to return to the embedding formulation: we have (remember, for constant ϑ!)

$$\begin{aligned} A^1(\varphi) &= K_1\left(\cos(\vartheta)\sin(\varphi)\sin(\varphi\cos(\vartheta)) + \cos(\varphi)\cos(\varphi\cos(\vartheta))\right) \\ &+ K_2\left(\cos(\vartheta)\sin(\varphi)\cos(\varphi\cos(\vartheta)) - \cos(\varphi)\sin(\varphi\cos(\vartheta))\right) \ , \\ A^2(\varphi) &= K_1\left(\cos(\vartheta)\cos(\varphi)\sin(\varphi\cos(\vartheta)) - \sin(\varphi)\cos(\varphi\cos(\vartheta))\right) \\ &+ K_2\left(\cos(\vartheta)\cos(\varphi)\cos(\varphi\cos(\vartheta)) + \sin(\varphi)\sin(\varphi\cos(\vartheta))\right) \ , \\ A^3(\varphi) &= -K_1\sin(\vartheta)\sin(\varphi\cos(\vartheta)) - K_2\sin(\vartheta)\cos(\varphi\cos(\vartheta)) \ . \end{aligned} \tag{1.182}$$

It is easily checked that, for all values of φ, the length of the vector is conserved:

$$\vec{A}(\varphi)^2 = K_1^2 + K_2^2 \ . \tag{1.183}$$

If we increase φ by 2π, the vectors $A(\varphi)$ and $A(\varphi+2\pi)$ lie in the same plane, namely the plane in $\mathcal{E}^3$ tangential to the sphere at the same points. However, the vector is rotated over an angle $2\pi\cos(\vartheta)$:

$$\vec{A}(\varphi)\cdot\vec{A}(\varphi+2\pi) = \left(K_1^2 + K_2^2\right)\cos(2\pi\cos(\vartheta)) \ . \tag{1.184}$$

This lies at the basis of Foucault's pendulum. Nijmegen is a point moving, with constant ϑ of about 40 degrees, over the surface of an ideal 'stationary' (that is non-rotating) earth with a velocity of 360 degrees longitude in 24 hours. A parallel-displaced vector fixed in Nijmegen turns by about 11 degrees per hour[13].

[13] At the poles, it would be 15 degrees per hour; at the equator, zero.

1.6 Curvature

1.6.1 Noncommuting differentiation of covariant vectors: the Riemann tensor

Covariant differentiation of covariant objects yields again covariant objects. However, whereas the noncovariant, standard partial differentiations may be taken in any order as usual, repeated covariant differentiations do not automatically commute, as we shall see.

For a scalar we have

$$S_{;\mu;\nu} = (S_{;\mu})_{;\nu} = (S_{,\mu})_{;\nu} = S_{,\mu\nu} - \Gamma^{\alpha}{}_{\mu\nu} S_{,\alpha} \tag{1.185}$$

Since this is symmetric in μ and ν, the order of covariant differentiation does not matter here. For covariant vectors, things are different:

$$\begin{aligned} A_{\mu;\alpha;\beta} &= (A_{\mu;\alpha})_{,\beta} - \Gamma^{\lambda}{}_{\mu\beta}\,(A_{\beta;\alpha}) - \Gamma^{\lambda}{}_{\alpha\beta}\,(A_{\mu\lambda}) \\ &= A_{\mu,\alpha\beta} - \Gamma^{\rho}{}_{\mu\alpha,\beta} A_{\rho} - \Gamma^{\rho}{}_{\mu\alpha} A_{\rho,\beta} - \Gamma^{\lambda}{}_{\mu\beta} A_{\lambda,\alpha} + \Gamma^{\lambda}{}_{\mu\beta}\Gamma^{\rho}{}_{\lambda\alpha} A_{\rho} \\ &\quad -\Gamma^{\lambda}{}_{\alpha\beta} A_{\mu,\lambda} + \Gamma^{\lambda}{}_{\alpha\beta}\Gamma^{\rho}{}_{\mu\lambda} A_{\rho} \;\; . \end{aligned} \tag{1.186}$$

This is not symmetric in α and β. This is best expressed by considering the commutator of the two orderings. This commutator does *not* contain any derivatives of A, and can be written as

$$A_{\mu;\alpha;\beta} - A_{\mu;\beta;\alpha} = R^{\rho}{}_{\mu\alpha\beta} A_{\rho} \;\; , \tag{1.187}$$

where

$$R^{\rho}{}_{\mu\alpha\beta} = \Gamma^{\rho}{}_{\lambda\alpha}\Gamma^{\lambda}{}_{\mu\beta} - \Gamma^{\rho}{}_{\lambda\beta}\Gamma^{\lambda}{}_{\mu\alpha} - \Gamma^{\rho}{}_{\mu\alpha,\beta} + \Gamma^{\rho}{}_{\mu\beta,\alpha} \;\; . \tag{1.188}$$

This is called the *Riemann-Christoffel tensor*, or the *curvature tensor*. It is, by construction, a tensor with one contravariant index and three covariant indices. It does not depend on A, but only on the metric, via the Christoffel symbols. A fully covariant form of the Riemann tensor is, of course,

$$R_{\mu\nu\alpha\beta} \equiv g_{\mu\rho}\, R^{\rho}{}_{\nu\alpha\beta} \;\; . \tag{1.189}$$

Excercise 70 *Show that we can write*

$$\begin{aligned} R_{\mu\nu\alpha\beta} &= \frac{1}{2}\left(g_{\mu\beta,\nu\alpha} - g_{\mu\alpha,\nu\beta} - g_{\nu\beta,\mu\alpha} + g_{\nu\alpha,\mu\beta}\right) \\ &\quad +\Gamma_{\lambda\mu\beta}\,\Gamma^{\lambda}{}_{\nu\alpha} - \Gamma_{\lambda\mu\alpha}\,\Gamma^{\lambda}{}_{\nu\beta} \;\; . \end{aligned} \tag{1.190}$$

Excercise 71 *Show that the above form implies the following symmetry properties:*

$$\begin{aligned} &R_{\mu\nu\alpha\beta} = -R_{\mu\nu\beta\alpha} = -R_{\nu\mu\alpha\beta} = R_{\alpha\beta\mu\nu} \;\; , \\ &R_{\mu\nu\alpha\beta} + R_{\mu\alpha\beta\nu} + R_{\mu\beta\nu\alpha} = 0 \;\; . \end{aligned} \tag{1.191}$$

Excercise 72 *Show that these symmetry properties reduce the number of independent components of the Riemann tensor, as follows. Show that there are no components with three or four equal indices; that there are* $D\begin{pmatrix} D \\ 2 \end{pmatrix}$ *independent components with two equal indices; and that there are* $2\begin{pmatrix} D \\ 4 \end{pmatrix}$ *independent components with all indices different; only for this last case is the cyclic-sum property relevant. Show that the number of independent componentns is therefore reduced from* D^4 *to* $D^2(D^2-1)/12$.

1.6.2 Noncommuting differentiation of contravariant vectors

From the fact that the metric is constant under covariant differentiation we can construct how a contravariant vector behaves.

Excercise 73 *Show that*

$$\begin{aligned} (A^\mu)_{;\alpha;\beta} - (A^\mu)_{;\beta;\alpha} &= (g^{\mu\nu}\ A_\nu)_{;\alpha;\beta} - (g^{\mu\nu}\ A_\nu)_{;\beta;\alpha} \\ &= -A^\nu\ R^\mu{}_{\nu\alpha\beta}\ \ . \end{aligned} \tag{1.192}$$

1.6.3 Noncommuting differentiation of tensors

Define a rank-two covariant tensor as

$$T_{\mu\nu} = A_\mu\ B_\nu\ \ . \tag{1.193}$$

From Leibniz' rule we derive

$$\begin{aligned} T_{\mu\nu;\alpha;\beta} &= (A_{\mu;\alpha}\ B_\nu + A_\mu\ B_{\nu;\alpha})_{;\beta} \\ &= A_{\mu;\alpha;\beta}\ B_\nu + A_\mu\ B_{\nu;\alpha;\beta} + A_{\mu;\alpha}\ B_{\nu;\beta} + A_{\mu;\beta}\ B_{\nu;\alpha}\ \ , \end{aligned} \tag{1.194}$$

Excercise 74 *Show that*

$$\begin{aligned} T_{\mu\nu;\alpha;\beta} - T_{\mu\nu;\beta;\alpha} &= A_\lambda\ R^\lambda{}_{\mu\alpha\beta}\ B_\nu + A_\mu\ B_\lambda\ R^\lambda{}_{\nu\alpha\beta} \\ &= T_{\lambda\nu}\ R^\lambda{}_{\mu\alpha\beta} + T_{\mu\lambda}\ R^\lambda{}_{\nu\alpha\beta}\ \ . \end{aligned} \tag{1.195}$$

The derivative-commutator leads to one Riemann term per index. The extension to contravariant indices, and to higher-rank tensors, is straightforward.

Excercise 75 *Given that* $A^{\alpha\beta\gamma}{}_{\mu\nu\rho}$ *is a tensor, show its behaviour under the commutator of covariant differentiation to* x^κ *and* x^λ.

1.6.4 Why 'curvature'?

The name 'curvature tensor' needs some justification. Consider a space that is flat, by which we mean that it admits of a rectilinear coordinate system (for instance, a Cartesian one). If we adopt that coordinate system for our metric tensor, the metric tensor will be constant; the Christoffel symbols will all vanish; and the Riemann tensor will also vanish identically. If we change to another coordinate system, the Christoffel symbols may become nonzero (after all, they do not form a tensor), but the Riemann tensor *does* transform covariantly, *i.e.* it is also zero for the new coordinate system.

It can be proven that the converse also holds: if the Riemann tensor vanishes identically in some extended region of coordinate space, that region admits of a rectilinear coordinate system, with a (not necessarily diagonal, but) constant metric tensor. Therefore, that region is flat.

We may conclude that a nonvanishing Riemann tensor tells us that the space is *not* flat, but curved.

1.6.5 The Bianchi identities for the Riemann tensor

An additional built-in property of the Riemann tensor can be found by some sneaky algebra, as follows. Consider a rank-two covariant tensor, which is, itself, the covariant derivative of a covariant vector. We then have, from Eq.(1.195),

$$A_{\mu;\nu;\alpha;\beta} - A_{\mu;\nu;\beta;\alpha} = A_{\rho;\nu} R^{\rho}{}_{\mu\alpha\beta} + A_{\mu;\rho} R^{\rho}{}_{\nu\alpha\beta} \quad . \tag{1.196}$$

This is where the sneaky part starts. We perform cyclic permutations of the indices ν, α and β, and arrange the resulting three equations:

$$\begin{aligned} A_{\mu;\nu;\alpha;\beta} - A_{\mu;\nu;\beta;\alpha} &= A_{\rho;\nu} R^{\rho}{}_{\mu\alpha\beta} + A_{\mu;\rho} R^{\rho}{}_{\nu\alpha\beta} \\ A_{\mu;\alpha;\beta;\nu} - A_{\mu;\alpha;\nu;\beta} &= A_{\rho;\alpha} R^{\rho}{}_{\mu\beta\nu} + A_{\mu;\rho} R^{\rho}{}_{\alpha\beta\nu} \\ A_{\mu;\beta;\nu;\alpha} - A_{\mu;\beta;\alpha;\nu} &= A_{\rho;\beta} R^{\rho}{}_{\mu\nu\alpha} + A_{\mu;\rho} R^{\rho}{}_{\beta\nu\alpha} \quad . \end{aligned} \tag{1.197}$$

We now sum them. The left-hand side is *anti*symmetric in α and ν. For instance, we encounter the combination

$$\begin{aligned} A_{\mu;\nu;\alpha;\beta} - A_{\mu;\alpha;\nu;\beta} &= \left(A_{\mu;\nu;\alpha} - A_{\mu;\alpha;\nu}\right)_{;\beta} \\ &= \left(A_{\rho}\ R^{\rho}{}_{\mu\nu\alpha}\right)_{;\beta} \\ &= A_{\rho;\beta}\ R^{\rho}{}_{\mu\nu\alpha} + A_{\rho}\ R^{\rho}{}_{\mu\nu\alpha;\beta} \quad . \end{aligned} \tag{1.198}$$

So, the left-hand side becomes

$$\begin{aligned} \text{LHS} \; = \; & A_{\rho;\beta}\ R^{\rho}{}_{\mu\nu\alpha} + A_{\rho}\ R^{\rho}{}_{\mu\nu\alpha;\beta} \\ + \; & A_{\rho;\nu}\ R^{\rho}{}_{\mu\alpha\beta} + A_{\rho}\ R^{\rho}{}_{\mu\alpha\beta;\nu} \\ + \; & A_{\rho;\alpha}\ R^{\rho}{}_{\mu\beta\nu} + A_{\rho}\ R^{\rho}{}_{\mu\beta\nu;\alpha} \quad . \end{aligned} \tag{1.199}$$

On the right-hand side, the three terms with $A_{\rho;\mu}$ cancel because of the properties (1.191), so we find

$$\text{RHS} = A_{\rho;\beta}\ R^{\rho}{}_{\mu\nu\alpha} + A_{\rho;\nu}\ R^{\rho}{}_{\mu\alpha\beta} + A_{\rho;\alpha}\ R^{\rho}{}_{\mu\beta\nu} \quad . \tag{1.200}$$

We therefore have

$$A_\rho \; R^\rho{}_{\mu\nu\alpha;\beta} + A_\rho \; R^\rho{}_{\mu\alpha\beta;\nu} + A_\rho \; R^\rho{}_{\mu\beta\nu;\alpha} = 0 \;\; ; \tag{1.201}$$

and, since this must hold for any covariant vector A, we have

$$R^\rho{}_{\mu\nu\alpha;\beta} + R^\rho{}_{\mu\alpha\beta;\nu} + R^\rho{}_{\mu\beta\nu;\alpha} = 0 \;\; . \tag{1.202}$$

These are the so-called *Bianchi identities*. Note that they are not purely algebraic, but rather *differential* identities.

1.6.6 The Ricci tensor

A more compact object can be constructed by contracting two indices in the Riemann tensor $R_{\mu\nu\alpha\beta}$ by multiplying it with a metric tensor. If we choose either $g^{\mu\nu}$ or $g^{\alpha\beta}$ we get of course zero because of the antisymmetry (1.191); any of the other four choices will change the result by at most a sign. We have therefore, essentially uniquely,

$$R_{\nu\beta} \equiv R_{\mu\nu\alpha\beta} \; g^{\mu\alpha} \;\; . \tag{1.203}$$

$R_{\nu\beta}$ is called the *Ricci tensor*; it is, by construction, a symmetric covariant tensor of rank two.

Excercise 76 *Show that, in terms of the Christoffel symbols, we can write the Ricci tensor as*

$$R_{\mu\nu} = -\Gamma^\alpha{}_{\mu\alpha,\nu} + \Gamma^\alpha{}_{\mu\nu,\alpha} + \Gamma^\alpha{}_{\mu\nu}\Gamma^\beta{}_{\alpha\beta} - \Gamma^\alpha{}_{\mu\beta}\Gamma^\beta{}_{\nu\alpha} \;\; . \tag{1.204}$$

In this form, the symmetry is evident except for the first term; but that is also symmetric, as we now prove.

Excercise 77 *Show that we can write*

$$\Gamma^\alpha{}_{\mu\alpha} = g^{\alpha\beta}\Gamma_{\beta\mu\alpha} = \frac{1}{2} g^{\beta\alpha} \; g_{\beta\alpha,\mu} \;\; , \tag{1.205}$$

since the first and third term in the brackets cancel upon contraction. Now, use the result of section 1.1.2 to obtain

$$\Gamma^\alpha{}_{\mu\alpha,\nu} = \frac{\partial}{\partial x^\nu}\left(\frac{1}{2g} g_{,\mu}\right) = \frac{1}{2} log(g)_{,\mu\nu} \tag{1.206}$$

which proves the symmetry.

An observation is in order here. In one-half of the literature the definitions of Riemann and/or Ricci tensor differ by an overall sign! This can lead to confusion.

1.6.7 The curvature scalar

We may condense the Ricci tensor even more, by yet another contraction:

$$R \equiv R_{\alpha\nu}\ g^{\alpha\nu} \quad . \tag{1.207}$$

The scalar number R is called the *Gauss curvature*, or simply the *curvature*. It is, by contruction, a bona fide scalar, but it can of course vary from point to point. In the literature, there is no sign confusion in the relation between Ricci tensor and Gauss curvature. The definitions used here are such that the D-dimensional spherical surface has a constant positive curvature equal to $D(D-1)$.

1.6.8 The Einstein tensor

The Bianchi identity (1.202) can of course also be translated into a statement about the Ricci tensor. We start with a contracted form of Eq.(1.202):

$$R^{\alpha}{}_{\mu\nu\rho;\alpha} + R^{\alpha}{}_{\mu\rho\alpha;\nu} + R^{\alpha}{}_{\mu\alpha\nu;\rho} = 0 \quad . \tag{1.208}$$

From the fact that the metric is constant under covariant differentiation we can rewrite this as

$$\left(g^{\mu\nu}\ R^{\alpha}{}_{\mu\nu\rho}\right)_{;\alpha} + \left(g^{\mu\nu}\ R^{\alpha}{}_{\mu\rho\alpha}\right)_{;\nu} + \left(g^{\mu\nu}\ R^{\alpha}{}_{\mu\alpha\nu}\right)_{;\rho} = 0 \quad . \tag{1.209}$$

By rewriting the three terms as follows:

$$\begin{aligned} g^{\mu\nu}\ R^{\alpha}{}_{\mu\nu\rho} &= g^{\beta\alpha}\ g^{\mu\nu}\ R_{\beta\mu\nu\rho} = -g^{\beta\alpha}\ g^{\mu\nu}\ R_{\mu\beta\nu\rho} = -g^{\beta\alpha}\ R_{\beta\rho} = -R^{\alpha}{}_{\rho} \quad , \\ g^{\mu\nu}\ R^{\alpha}{}_{\mu\rho\alpha} &= -g^{\mu\nu}\ R^{\alpha}{}_{\mu\alpha\rho} = -g^{\mu\nu}\ R_{\mu\rho} = -R^{\nu}{}_{\rho} \quad , \\ g^{\mu\nu}\ R^{\alpha}{}_{\mu\alpha\nu} &= g^{\mu\nu}\ R_{\mu\nu} = R \quad , \end{aligned} \tag{1.210}$$

we arrive at the following result:

$$2R^{\alpha}{}_{\mu;\alpha} - R_{;\alpha} = 0 \quad . \tag{1.211}$$

This is the Bianch identity for the Ricci tensor. We can write it also as

$$G^{\alpha}{}_{\nu;\alpha} = 0 \quad , \tag{1.212}$$

where the symmetric object

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \tag{1.213}$$

is called the *Einstein tensor*. It plays an important rôle in the general theory of relativity.

Excercise 78 *Show that the Einstein tensor can be written as*

$$G^{\mu\nu} = \frac{1}{2}\left(g^{\mu\alpha} g^{\nu\beta} + g^{\mu\beta} g^{\nu\alpha} - g^{\mu\nu} g^{\alpha\beta}\right) R_{\alpha\beta} \tag{1.214}$$

Excercise 79 *Show that the relation between Ricci and Einstein tensor can be inverted:*

$$R_{\mu\nu} = G_{\mu\nu} - \frac{1}{D-2} g_{\mu\nu}\; G^{\alpha}{}_{\alpha} \quad . \tag{1.215}$$

Excercise 80 *Show that for $D = 1$, the Riemann tensor vanishes identically.*

Excercise 81 *Show that for $D = 2$, we have*

$$g^{11} = \frac{g_{22}}{g} \quad , \quad g^{22} = \frac{g_{11}}{g} \quad , \quad g^{12} = -\frac{g_{12}}{g} \quad , \quad g = g_{11}\; g_{22} - (g_{12})^2 \quad . \tag{1.216}$$

Show that the Riemann tensor has only one independent term: assume it to be R_{1212}. Show that the Ricci tensor is therefore given by

$$R_{\mu\nu} = \frac{1}{g}\; g_{\mu\nu}\; R_{1212} \quad , \tag{1.217}$$

and the curvature by

$$R = g^{\mu\nu}\; R_{\mu\nu} = \frac{2}{g}\; R_{1212} \quad , \tag{1.218}$$

As a result, show that for $D = 2$ the Einstein tensor always vanishes.

1.6.9 Examples

The unit hypersphere $\mathcal{S}^D$

For the unit D-sphere $\mathcal{S}^D$, with the polar-coordinate metric as given by Eq.(1.21), explicit computation shows that

$$R_{\mu\nu} = -(D-1) g_{\mu\nu} \quad , \tag{1.219}$$

and hence

$$R = D(D-1) \quad , \quad G_{\mu\nu} = -\frac{1}{2}(D-1)(D-2) g_{\mu\nu} \quad . \tag{1.220}$$

the D-sphere has constant positive curvature. For a sphere with radius R, all these curvature measures acquire a factor $1/R^2$.

The 'antisphere' $\mathcal{AS}^D$

The antispherical surface $\mathcal{AS}^D$ described by the metric (1.24) turns out to have curvature properties 'opposite' to that of the sphere $\mathcal{S}^D$:

$$R_{\mu\nu} = -(D-1) g_{\mu\nu} \quad , \quad R = -D(D-1) \quad , \quad G_{\mu\nu} = \frac{1}{2}(D-1)(D-2) g_{\mu\nu} \quad . \tag{1.221}$$

The antisphere has constant *negative* curvature, hence the name.

In fact, some computer algebra[14] shows that the $D \times D$-diagonal metric given by

$$g_{\mu\nu} = \begin{cases} 0 & \text{if } \mu \neq \nu \\ \sigma_\mu \, (x^1)^p & \text{if } \mu = \nu \end{cases} \quad , \tag{1.222}$$

with σ_μ and p constants, also has a diagonal Ricci tensor, with diagonal elements

$$R_{11} = \frac{(D-1)p}{2(x^1)^2} \quad , \quad R_{nn} = \frac{\sigma_n}{\sigma_1} \frac{p(2-(D-2)p)}{4(x^1)^2} \quad . \tag{1.223}$$

The Gauss curvature is then

$$R = (x^1)^{-p-2} \, \frac{1}{4\sigma_1^2} \, p(D-1)(4-(D-2)p) \quad . \tag{1.224}$$

For $p = -2$, therefore, all these metrics have constant Gauss curvature equal to $-D(D-1)/\sigma_1$. Neither the sphere nor the antisphere is uniquely given by the requirement of constant curvature.

The torus

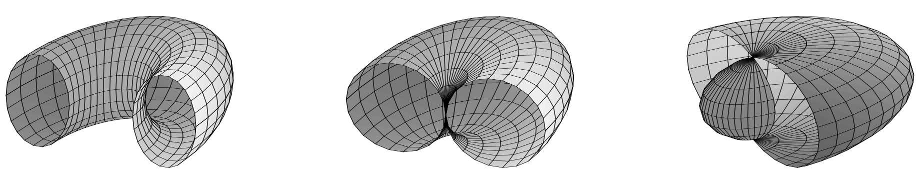
The torus described in section 1.1.5 has diagonal metric

$$g_{11} = a^2 \quad , \quad g_{22} = (R + a\cos(x^1))^2 \quad . \tag{1.225}$$

The Gauss curvature of this surface is

$$R = \frac{2\cos(x^1)}{a(R + a\cos(x^1))} \quad . \tag{1.226}$$

This curvature is finite as long as $R > a$; for $a > R$ there are singularities in the Gauss curvature; for $R = a$ we can indeed see the center of the torus, where the 'hole' has shrunk to a point, as a point with infinite curvature. For $R < a$, the values $cos(x^1) = -R/a$ correspond to points where the torus intersects itself: the question of whether the curvature is really infinite there depends on which parts of the toroidal surface are taken to be the continuation of a nonsingular part. For $R = 0$, however, we are back at the spherical surface with radius a, and the curvature is again constant. Note, however, the following: for the standard spherical surface, x^1 runs from 0 to π, whereas for the torus we have $0 \leq x^1 \leq 2\pi$. As $R \to 0$, therefore, the surface is covered twice.



Here we depict half of the torus (namely, with $0 \leq x^1 < 2\pi$, but with $0 \leq x^2 \leq \pi$) with $a = 1$ and $R = 2$, $R = 1$ and $R = 0.5$, respectively.

[14] Maple code `Riemann`

A surface with negative curvature

The antisphere $\mathcal{AS}$ cannot be viewed as embedded in a Euclidean space, which makes visualization difficult. A a second-best case, we may consider the two-dimensional surface defined by the embedding in $\mathcal{E}^3$:

$$X^1 = x^1 \quad , \quad X^2 = x^2 \quad , \quad X^3 = \frac{1}{2}\left((x^1)^2 - (x^2)^2\right) \quad . \tag{1.227}$$

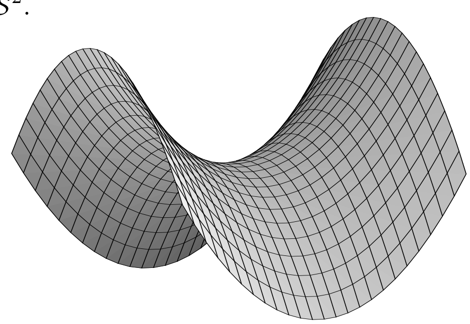
The induced metric is, then

$$g_{11} = 1 + (x^1)^2 \quad , \quad g_{22} = 1 + (x^2)^2 \quad , \quad g_{12} = g_{21} = -x^1 x^2 \quad , \tag{1.228}$$

and the Gauss curvature turns out to be

$$R = -\frac{2}{1 + (x^1)^2 + (x^2)^2} \quad . \tag{1.229}$$

Close to the point $(0,0)$, therefore, the surface looks like that of the antisphere $\mathcal{AS}^2$.



Part of the surface defined by Eq.(1.227), with lines of constant x^1 or x^2. The antisphere $\mathcal{S}^2$ looks like the neighbourhood of the point $(0,0)$, but then *everywhere*.

1.7 Integration and conservation

1.7.1 The integration element

We shall have occasion to perform integrals over (part of) the space. Therefore we must investigate how the integration elements transform. We have

$$d^D x' \equiv dx'^1 \; dx'^2 \cdots dx'^D = dx^1 \; dx^2 \cdots dx^D \; J \equiv d^D x \; J \quad , \tag{1.230}$$

where

$$J \equiv \frac{\partial(x'^1, x'^2, \ldots, x'^D)}{\partial(x^1, x^2, \ldots, x^D)} \tag{1.231}$$

is the Jacobian determinant of the transformation. From the fact that the metric is a covariant rank-two tensor, we immediately derive that

$$g \equiv \det(g_{\mu\nu}(x)) = J^2 \det(g'_{\mu\nu}(x')) \equiv J^2 \; g' \quad . \tag{1.232}$$

We may therefore write

$$\sqrt{|g|} = J \sqrt{|g'|} \quad . \tag{1.233}$$

Here, the absolute-value form ensures that the square root is real; since by assumption g is nowhere zero, it is quite harmless.

1.7.2 Scalar integrals

Let $S(x)$ be a scalar, so that $S'(x') = S(x)$. Let Z be some integration region given in terms of the x coordinates, and let Z' be the same region, now expressed in terms of the x' coordinates. We then have

$$\int\limits_Z S(x)\ \sqrt{|g|}\ d^D x = \int\limits_Z S(x')\ \sqrt{|g'|}\ J\ d^D x = \int\limits_{Z'} S(x')\ \sqrt{|g'|}\ d^D x' \quad , \tag{1.234}$$

so that this integral is invariant under coordinate transformations.

1.7.3 Tensorial densities

Let $A^{\mu\nu\cdots}(x)$ be a tensor. Then, the infinitesimal element

$$A^{\mu\nu\cdots}(x)\ \sqrt{|g|}\ d^D x$$

transforms in precisely the same way as $A^{\mu\nu\cdots}(x)$ itself, and is called a *tensor density*. The integral

$$\int\limits_Z A^{\mu\nu\cdots}(x)\ \sqrt{|g|}\ d^D x$$

will, however, not transform in any simple manner since the transformation behaviour may change from point to point under nontrivial coordinate transformations. It is only a tensor if Z is infinitesimally small.

1.7.4 Conserved vector currents

We shall employ the following identity, from Eq.(1.5)

$$\left(\sqrt{|g|}\right)_{,\mu} = \frac{1}{2}\ \sqrt{|g|}\ g^{\alpha\beta}\ g_{\alpha\beta,\mu} = \sqrt{|g|}\ \Gamma^{\alpha}{}_{\alpha\mu} \quad . \tag{1.235}$$

Now, consider a vector field $A^{\mu}(x)$. The 'covariant divergence', $A^{\mu}{}_{;\mu}$ is given by

$$A^{\mu}{}_{;\mu} = A^{\mu}{}_{,\mu} + \Gamma^{\mu}{}_{\mu\alpha}\ A^{\alpha} = A^{\mu}{}_{,\mu} + A^{\alpha}\ \frac{1}{\sqrt{|g|}} \left(\sqrt{|g|}\right)_{,\alpha} \quad , \tag{1.236}$$

so that we may write

$$A^{\mu}{}_{;\mu}\ \sqrt{|g|} = \left(\ A^{\mu}\ \sqrt{|g|}\right)_{,\mu} \quad . \tag{1.237}$$

Since the covariant divergence is a scalar, we see that the integral

$$\int A^{\mu}{}_{;\mu}\ \sqrt{|g|}\ d^D x = \int \left(\ A^{\mu}\ \sqrt{|g|}\right)_{,\mu}\ d^D x \tag{1.238}$$

is also a scalar invariant.

Let us call one of the coordinates x^μ (x^1, say),the 'time', and the other ones the 'space' ccordinates. We call $A^1\sqrt{|g|}$ the 'density' of a fluid, and $A^k\sqrt{|g|}$ ($k = 2, \ldots, D$) the 'current' of that fluid[15]. If the integration region is now taken at fixed 'time' (that is, x^1 is fixed at some value t by taking the integration region to contain a factor $\delta(x^1 - t)$, we have, by Gauss' theorem:

$$\begin{aligned} \frac{d}{dt}\left(\int A^1\ \sqrt{|g|}\ d^{D-1}x\right) &= -\int\left(A^k\ \sqrt{|g|}\right)_{,k}\ d^{D-1}x \\ &= \text{surface integral over the boundary} \end{aligned} \tag{1.239}$$

If no current crosses the boundary, the total amount of fluid is constant in 'time'. In this terminology, therefore, we can say that

$$A^\mu{}_{;\mu} = 0 \ \Leftrightarrow\ A^\mu \text{ is a conserved current} \ . \tag{1.240}$$

1.7.5 A tensorial conservation law

Consider an *antisymmetric* rank-two tensor:

$$F^{\mu\nu} = -F^{\nu\mu} \ . \tag{1.241}$$

For its covariant divergence we find

$$\begin{aligned} F^{\mu\nu}{}_{;\nu} &= F^{\mu\nu}{}_{,\nu} + F^{\mu\alpha}\ \Gamma^\nu{}_{\alpha\nu} + F^{\alpha\nu}\ \Gamma^\mu{}_{\alpha\nu} \\ &= F^{\mu\nu}{}_{,\nu} + F^{\mu\alpha}\ \Gamma^\nu{}_{\alpha\nu} \\ &= F^{\mu\nu}{}_{,\nu} + F^{\mu\alpha}\ \frac{1}{\sqrt{|g|}}\ \left(\sqrt{|g|}\right)_{,\alpha} \ , \end{aligned} \tag{1.242}$$

where in the second line we have used the fact that F is antisymmetric while Γ is symmetric in $\alpha \leftrightarrow \nu$, and in the second line we have again used Eq.(1.235). We may write this result as

$$\sqrt{|g|}\ F^{\mu\nu}{}_{;\nu} = \left(\sqrt{|g|}\ F^{\mu\nu}\right)_{,\nu} \ . \tag{1.243}$$

By exactly the reasoning applied to the case of conserved vectors, we may therefore conclude that

$$F^{\mu\nu}{}_{;\nu} = 0 \ \Leftrightarrow\ F^{\mu\nu} \text{ is a conserved tensor} \ . \tag{1.244}$$

Note that this will only work for antisymmetric tensors, since otherwise the third term in the second line of Eq.(1.242) would not vanish.

[15]Later on, we shall do precisely this.

1.7.6 Stokes' theorem

Consider a bounded two-dimensional surface S in our metric space: we shall call the closed curve determining its boundary C. We define an infintesimal surface integration element by taking two infintesimal vectors $d\xi^\mu$ and $d\eta^\mu$, and writing

$$d\Sigma^{\mu\nu} = d\xi^\mu \; d\eta^\nu - d\xi^\nu \; d\eta^\mu \quad . \tag{1.245}$$

for instance let the surface be given by some curve C in the x^1, x^2 plane, and $x^3, \ldots, x^D$ be fixed. Then, with $d\xi^\mu = dx^1 \; \delta^\mu{}_1$ and $d\eta^\mu = dx^2 \; \delta^\mu{}_2$, and

$$d\Sigma^{12} = dx^1 \; dx^2 \quad , \quad d\Sigma^{21} = - \; dx^2 \; dx^1 \quad .$$

For a covariant vector field A_μ we may then use Stokes' theorem:

$$\begin{aligned} &\int_S \left(A_{1;2} - A_{2;1}\right) \; dx^1 \; dx^2 = \\ &= \int_S \left(A_{1,2} - A_{2,1}\right) \; dx^1 \; dx^2 = \int_C \left(A_1 \; dx^1 + A_2 \; dx^2\right) \quad . \end{aligned} \tag{1.246}$$

Turning this back to the fully covariant form, we can formulate Stokes' theorem as

$$\int_S \left(A_{\mu;\nu} - A_{\nu;\mu}\right) \; d\Sigma^{\mu\nu} = \int_C A_\mu \; dx^\mu \quad . \tag{1.247}$$

Chapter 2

Materia Physica

2.1 Special relativity

2.1.1 Four-dimensional spacetime

It is an empirical fact that the speed of light,

$$c \approx 3\ 10^8\ \frac{\text{meter}}{\text{second}}$$

appears to be a fundamental constant of nature, *i.e.* it has the same value for all observers. Einstein raised this observation, which can of course only be made in a finite number of experiments, with a finite experimental accuracy, to the status of a fundamental postulate. The availability of c as a natural constant allows its use in natural laws whenever we please, in an unambiguous way[1]. Most physical *events*, that is, identifiable places where some identifiable happenings take place at identifiable moments, are readily described by three space coordinates and one time coordinate. Time is measured in seconds, and space distances in meters. These are *very different things*! After all, one moves about much more readily in space than in time. However, multiplying any time interval by c we construct a space interval, namely the distance travelled by light in that time interval[2].

Imagine an *inertial observer*: by this we mean an observer not under the influence of any external force (except possibly gravity). The observer comes equipped with three Cartesian coordinate axes, one time axis, and an origin: this set of equipment is called a *frame*. Using his/her frame the observer can measure the Cartesian position and time position of a physical event. We assume that this is possible in *some* neighbourhood of the origin. The situation of an inertial observer is called that of 'free fall': since special relativity does not acknowledge the effects of gravity, such an observer is supposed to be *'moving*

[1]Much like the fact that nobody will protest if a law of nature contains the constant π: being a constant, it has the same unambiguous meaning for everyone.

[2]For instance, the lightyear.

uniformly, along a straight line, with constant velocity'. Note the conundrum: such motion must be ascertainedby another interial observer, and who is to say that *that* observer is, in fact, inertial? Yet another observer. We are therefore forced to imagine a whole set of observers, each of them inertial with respect to all the others. Special relativity supposes that if observer A is inertial with respect to observer B, and B is inertial with respect to observer C, then A is also inertial with respect to observer C.

Let the coordinate values of a given event as measured by a given inertial observer be (x^1, x^2, x^3) and t, respectively. Using c, we can turn this into a four-dimensional set of coordinates, all with the dimension of length:

$$x^\mu = (x^0, x^1, x^2, x^3) \equiv (ct, x^1, x^2, x^3) \quad . \tag{2.1}$$

Note that, as is usual, we have assigned the index '0' rather than '4' to the time coordinate[3]. Physics is therefore viewed as taking place in a four-dimensional *spacetime*.

2.1.2 The Minkowski metric

The postulate of the theory of special relativity is that the metric of spacetime is given by

$$(ds)^2 = (dx^0)^2 - |d\vec{x}|^2 \equiv (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \quad , \tag{2.2}$$

so that the metric tensor is diagonal, with

$$g_{00} = 1 \quad , \quad g_{kk} = -1 \quad , \quad k = 1, 2, 3 \; . \tag{2.3}$$

Numerically, the elements of the contravariant and the covariant metric tensors *happen* to coincide. Since this metric is flat, the coordinate axes of the inertial observers are actually useful over arbitrarily long intervals. It also implies that the coordinates of an event with respect to the origin, x^μ, are themselves contravariant vectors, and the invariannt distance from an event to the origin is

$$s^2 = (x^0)^2 - |\vec{x}|^2 \tag{2.4}$$

2.1.3 Comoving frame and proper time

Since four-dimensional spacetime is the 'arena' of physics, the behaviour of a point particle is a line in this spacetime, the *worldline*. Since the 'time', x^0 is one of the coordinates, it is convenient to choose something else than x^0 as the parameter governing the 'motion' of the particle along its worldline. For this, one takes the arc length, s, since ds is nicely scalar and therefore all inertial observers agree on its value. Note that this will work properly only for wordlines along which $(ds)^2 > 0$ everywhere.

[3] In code `Riemann`, it is easier to keep '4' assigned to the time index. Of course this convention does not influence the physics at all.

Now consider an inertial observer who (possibly only for a small period) happens to move precisely along with the particle: this is the *comoving observer*, carrying along the *comoving frame*. For this observer, the particle is at standstill: during an infintesimal time interval dt the particle therefore moves a distance $|d\vec{x}| = 0$, and $(ds)^2 = c^2(dt)^2$. We see that $\tau \equiv s/c$ is *the time as measured by comoving obsevers*. Therefore τ is called the *proper time* of the particle.

An operational note is in order here. If external forces act on the particle, this is signalled by the fact that the particle does not move uniformly, and will therefore be 'at rest' only for an infinitesimal time with respect to a given comoving intertial observer. What we have to envisage, therefore, is an infinite host of intertial observers, moving criss-cross with any velocity and in any direction. During an infinitesimal interval $d\tau$ the particle will be comoving with one observer of this host, and the next instant it will be comoving with another one. Each observer monitors his/her own piece of proper time, $d\tau$. The picture here is therefore that of a relay race in which each comoving observer tells the value of τ to the next one, who then adds to this running total his own piece $d\tau$.

2.1.4 The speed of light

Consider a particle moving with the speed of light with respect to (inertial) observer A. For this observer, the distance travelled, $|d\vec{x}|_A$, and the time this takes, dt_A, are related by

$$\frac{|d\vec{x}|_A}{dt_A} = c \ , \tag{2.5}$$

for this is what it means to move with the speed of light. Therefore, the arc length as measured by A vanishes: $(ds)^2_A = 0$. But then another interial observer, B, must also arrive at $(ds)^2_B = 0$, and therefore

$$\frac{|d\vec{x}|_B}{dt_B} = c \ , \tag{2.6}$$

The particle also moves at light speed for observer B. The constancy of the speed of light is therefore consistent with the Minkowski metric.

2.1.5 Lorentz transformations: the generators

Consider two inertial observers, A and B, moving with uniform velocity with respect to one another. Let the origin of their frames coincide (that is, their space origins coincide, $\vec{x}_A = \vec{x}_B = 0$, at time $t_A = t_B = 0$). We shall now work out the relation between the coordinates of one given event as measured by A and those measured by B, called x^μ_A and x^μ_B, respectively.

Since the notion 'event 1 is twice as far from the origin as event 2' must be unambiguous for inertial observers, the coordinate transformation must be linear, so that we may write

$$x^\mu_A = \Lambda^\mu{}_\alpha \, x^\alpha_B \ . \tag{2.7}$$

Because the two origins coincide and the distance must be invariant, Eq.(2.4), we must have

$$g_{\mu\nu}\, x_A^\mu\, x_A^\nu = g_{\mu\nu}\, \Lambda^\mu{}_\alpha\, \Lambda^\nu{}_\beta\, x_B^\alpha\, x_B^\beta = g_{\alpha\beta}\, x_B^\alpha\, x_B^\beta \quad ; \tag{2.8}$$

and since this holds for any vector x_B the transformation matrix must satisfy

$$g_{\mu\nu}\, \Lambda^\mu{}_\alpha\, \Lambda^\nu{}_\beta = g_{\alpha\beta} \quad . \tag{2.9}$$

If the frames of A and B coincide exactly, then A and B are actually the same observer, and we must have $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu$. Let us suppose that the frames of A and B differ only very slightly, so that the transformation Λ is almost the identity:

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \epsilon\, T^\mu{}_\nu \quad , \tag{2.10}$$

where ϵ is infinitesimal. q.(2.9) then becomes

$$\begin{aligned} g_{\alpha\beta} &= g_{\mu\nu}\, (\delta^\mu{}_\alpha + \epsilon\, T^\mu{}_\alpha)\, (\delta^\nu{}_\beta + \epsilon\, T^\nu{}_\beta) \\ &= g_{\mu\nu}\, \delta^\mu{}_\alpha\, \delta^\nu{}_\beta + g_{\mu\nu}\, \delta^\mu{}_\alpha\, T^\nu{}_\beta + g_{\mu\nu}\, T^\mu{}_\alpha\, \delta^\nu{}_\beta + \mathcal{O}(\epsilon^2) \\ &= g_{\alpha\beta} + \epsilon\, \left(T_{\alpha\beta} + T_{\beta\alpha}\right) + \mathcal{O}(\epsilon^2) \quad , \end{aligned} \tag{2.11}$$

so that the requirement is that T be antisymmetric:

$$T_{\alpha\beta} = -T_{\beta\alpha} \quad . \tag{2.12}$$

There are, therefore, six linearly independent solutions. We may *label* these by antisymmetric labels (indicated by brackets):

$$\left(T^{(\mu\nu)}\right)^\alpha{}_\beta = g^{\mu\alpha}\, \delta^\nu{}_\beta - g^{\nu\alpha}\, \delta^\mu{}_\beta \quad . \tag{2.13}$$

Note that $T^{(\mu\nu)} = -T^{(\nu\mu)}$. These are called the *generators* of the Lorentz transformations. A non-infinitesimal Lorentz transform depends on antisymmetric parameters $\omega_{\mu\nu} = -\omega_{\nu\mu}$:

$$\Lambda^\mu{}_\nu = \left(\exp\left(\frac{1}{2}\omega_{\alpha\beta} T^{(\alpha\beta)}\right)\right)^\mu{}_\nu \quad . \tag{2.14}$$

2.1.6 Lorentz transformations: rotations

Consider the generator $T^{(12)}$. Using the convention that upper indices label *rows*, and lower indices label *columns*[4], we find the matrix form

$$T^{(12)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \text{-1} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad . \tag{2.15}$$

[4]This is consistent with writing x^μ as a column vector, and x_μ as a row vector.

This means that

$$\left(T^{(12)}\right)^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \tag{2.16}$$

so that, for integer $m \geq 0$,

$$\begin{aligned} \left(T^{(12)}\right)^{4m+1} &= T^{(12)} \quad , \quad \left(T^{(12)}\right)^{4m+2} = \left(T^{(12)}\right)^2 \quad , \\ \left(T^{(12)}\right)^{4m+3} &= -T^{(12)} \quad , \quad \left(T^{(12)}\right)^{4m+4} = -\left(T^{(12)}\right)^2 \quad . \end{aligned} \tag{2.17}$$

The finite Lorentz transform of this type reads, therefore,

$$\begin{aligned} \Lambda^\mu{}_\nu &= \left(\exp\left(\omega_{12}T^{(12)}\right)\right)^\mu{}_\nu \\ &= \delta^\mu{}_\nu + \sum_{m\geq 1} \frac{1}{m!}(\omega_{12})^m \left(\left(T^{(12)}\right)^m\right)^\mu{}_\nu \\ &= \delta^\mu{}_\nu + \sin(\omega_{12}) T^{(12)\mu}{}_\nu + (\cos(\omega_{12}) - 1)\left(\left(T^{(12)}\right)^2\right)^\mu{}_\nu \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\omega_{12}) & -\sin(\omega_{12}) & 0 \\ 0 & \sin(\omega_{12}) & \cos(\omega_{12}) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad . \end{aligned} \tag{2.18}$$

We recognize this as a rotation around the 3-axis over an angle ω_{12}; similarly, the generators $T^{(13)}$ and $T^{(23)}$ lead to rotations around the 2-axis and the 1-axis, respectively. These transforms leave the times x^0 unchanged, and therefore relate frames that are at rest with respect to each other.

2.1.7 Lorentz transformations: boosts

A treatment similar to that of the rotations is possible for the other generators: we have

$$T^{(01)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \left(T^{(01)}\right)^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \tag{2.19}$$

from which we can easily infer that

$$\left(\exp\left(\omega_{01}T^{(01)}\right)\right)^\mu{}_\nu = \begin{pmatrix} \cosh(\omega_{12}) & \sinh(\omega_{12}) & 0 & 0 \\ \sinh(\omega_{12}) & \cosh(\omega_{12}) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad . \tag{2.20}$$

Because of the mixing between space and time coordinates, this Lorentz transform relates two observer frames that are not at relative rest: instead, we have

$$\begin{pmatrix} ct_A \\ x_A^1 \end{pmatrix} = \begin{pmatrix} \cosh(\omega) & \sinh(\omega) \\ \sinh(\omega) & \cosh(\omega) \end{pmatrix} \begin{pmatrix} ct_B \\ x_B^1 \end{pmatrix} \quad , \quad x_A^{2,3} = x_B^{2,3} \quad , \tag{2.21}$$

with $\omega = \omega_{12}$. A particle fixed at the origin of B's frame has $\vec{x}_B = 0$, and has therefore the following coordinates in A's frame:

$$t_A = \cosh(\omega)\ t_B \quad , \quad x_A^1 = \sinh(\omega)\ t_B \quad , \quad x_A^2 = x_A^3 = 0 \ . \tag{2.22}$$

From A's point of view, therefore, the particle is moving along the 1-axis with velocity v, where

$$v = \frac{x_A^1}{t_A} = c\frac{\sinh(\omega)}{\cosh(\omega)} \ . \tag{2.23}$$

The parameter ω, called the *rapidity*, is related to the velocity by

$$\omega = \frac{1}{2}\log\left(\frac{1+v/c}{1-v/c}\right) \ . \tag{2.24}$$

In terms of the velocity v, therefore, the Lorentz transform matrix takes the well-known form

$$\Lambda = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad , \quad \beta = \frac{v}{c} \quad , \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} \ . \tag{2.25}$$

2.1.8 Velocity addition

Consider three observers whose spatial frame axes are parallel. Let B move along A's 1-axis with velocity $\beta_1 c$, and let C move along B's 1-axis with velocity $\beta_2 c$. By matrix multiplication, the Lorentz transform relating C's frame to that of A has the form (with obvious notation):

$$\Lambda = \begin{pmatrix} \gamma_1\gamma_2(1+\beta_1\beta_2) & (\beta_1+\beta_2)\gamma_1\gamma_2 & 0 & 0 \\ (\beta_1+\beta_2)\gamma_1\gamma_2 & \gamma_1\gamma_2(1+\beta_1\beta_2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ . \tag{2.26}$$

The velocity of C in A's frame is therefore βc, with

$$\beta = \frac{\beta_1+\beta_2}{1+\beta_1\beta_2} \ . \tag{2.27}$$

Since this is smaller than unity for any $\beta_{1,2}$ smaller than unity, we see that it is impossible to exceed the speed of light by adding any number of Lorentz boosts.

2.1.9 Kinematics in special relativity

Consider a particle with mass m, moving with velocity $\vec{v}$. In a time interval dt is travels over a distance $d\vec{x} = \vec{v}\ dt$. In spacetime, therefore, it traverses a vector dx^μ, with

$$dx^0 = c\ dt \quad , \quad dx^k = v^k\ dt \ \ , \ \ k = 1,2,3 \ \ . \tag{2.28}$$

The proper time interval taken is

$$d\tau = \sqrt{1 - v^2/c^2}\ dt \ \ , \tag{2.29}$$

and is a real invariant. This suggests the definition of a proper vectorial four-momentum:

$$p^\mu \equiv m\frac{dx^\mu}{d\tau} \ \ . \tag{2.30}$$

This momentum transforms as a well-behaved contravariant vector. Its individual components are

$$p^0 = \frac{mc}{\sqrt{1 - v^2/c^2}} \quad , \quad p^k = \frac{mv^k}{\sqrt{1 - v^2/c^2}} \ \ . \tag{2.31}$$

This definition has the important attractive property that, *if* momentum in conserved in any one inertial frame, this conservation law is automatically true in any other inertial frame. We may also define the relativistic energy, as

$$E = cp^0 = \frac{mc^2}{\sqrt{1 - v^2/c^2}} \ \ . \tag{2.32}$$

Even at rest, therefore, a particle has an amount of kinetic energy:

$$E(v = 0) = mc^2 \ \ . \tag{2.33}$$

The low-velocity approximation,

$$\begin{aligned} \vec{p} &\approx m\vec{v} + \mathcal{O}(v^3/c^2) \ \ , \\ E &\approx mc^2 + \frac{1}{2}mv^2 + \mathcal{O}(v^4/c^2) \ \ , \end{aligned} \tag{2.34}$$

conform to the Newtonian definition, except for the rest energy. In Newtonian mechanics, where particles are not created or destroyed, this rest energy is just an inert, additive constant and does not influence the physics.

2.1.10 A note on a famous formula

The formula '$E = mc^2$' is perhaps the most well-known formula of physics, but it is also widely misunderstood. Let us digress for a moment to inspect it more closely.

In the first place, it is easy to see that, given the fact that c (with dimension m/sec) appears to be a universal constant, a formula linking mass (in kg) to energy (in Joule = kg m^2/sec^2) *may* hold true. In the second place, this energy, as appaears from the previous section, is *kinetic* in nature, that is, it describes an amount of motion. In Newtonian mechanics, motion is always *in* time and *through* space; but in Minkowski space, the time-like four-space component x^0 is equally something *through* which a particle (and all of us) move, *in eigen*time. The denomination 'rest energy' is therefore somewhat misleading: even at perfect Newtonian standstill, any body moves through time, and it is the kinetic energy associated with *this* motion that is described by the formula. As a matter of fact, all bodies at relative rest move through time x^0 at the same speed, namely c. The occurence of m just follows from the commonplace observation that, at equal speed, a heavier body contains more 'motion' than a light one. The only non-obvious ingredient of the formula is the numerical constant of proportionality, which turns out to be just 1.

In the second place, it is often stated that 'matter and energy are equivalent'. This is, strictly speaking, not true: matter is a *thing*, a *substans*[5], and energy is a *property*, an *accidens*. There is no such thing as 'pure energy': energy must always be carried by particles. The situation probably closest to a notion of 'pure energy' is a collection of photons or other massless particles. It would probably be more correct to say that any system can be characterized by various numerical quantities (*accidentes*), and that the quantity 'energy' and the quantity 'mass' are the same up to a factor c^2. Water gets heavier if it is heated.

In the third place, in the public imagination Einstein's famous formula is linked to nuclear reactions and atomic bombs, and even a well-known scientist has been heard to remark that 'the nearest place where $E = mc^2$ is at work is in the center of the sun, where energy is generated by fusion reactions'[6]. This is totally misguided: *any* chemical, atomic, or nuclear reaction in which particles are transformed or rearranged, results in a change of mass. If the reaction is exothermic, the rest mass of the reaction products is less than the rest mass of the original ingredients[7]. Of course, since c is pretty large in everyday units, the change of mass is usually too minute to be easily detected.

In the fourth place, it is appealing to make statements like 'if we could transform this [fill in commonplace object] into energy, we would have enough energy to [fill in desirable goal]'. Turning, say, an amount of water into photons would involve violation of fermion number on a massive scale, as well as probably some amount of violation of charge conservation, if the water is not strictly electrically neutral[8]. Someone willing to revoke these fundamental conservation laws really has no right to insist that in such an exotic world energy conservation

[5] In the language of Aristotle.

[6] C.B. in 'Nieuwslicht', April 15, 2005.

[7] Note that, in order do determine the rest masses of the reaction products, we have to let them come to rest, that is, take away their kinetic energy. There is, therefore, no conflict with energy conservation in such an observation.

[8] Interestingly, angular momentum might remain conserved even if the water has some nonzero total angular momentum, because the photons can carry orbital angular momentum in addition to intrinsic spin

still ought to hold.[9]

2.2 Newtonian gravitation from general relativity

2.2.1 Newtonian gravitation

The Newtonian theory of gravity can be described by a few postulates. In the first place, space and time are assumed to be absolute, and space is flat. The notions of coordinate transformations and covariance are therefore much less fundamental in Newtonian mechanics than in relativity: in Newtonian mechanics there *are* preferred coordinate systems, namely those that are at rest in absolute space. The motion, in this absolute space, of point particles under the influence of gravity, is then described by two postulates:

1. The momentum of a particle is given by $m\vec{v}$ where $\vec{v}$ is the velocity: $v^k = dx^k/dt$ $(k = 1, 2, 3)$. The coordinates $\vec{x}$ and t are given in the absolute space. The rate of change of momentum equals the force, which can sometimes be defined as the gradient of a potential:

$$\frac{d}{dt}\,p^k = m\,\frac{d}{dt}\,v^k = \frac{d^2}{(dt)^2}\,x^k = -\,\frac{\partial}{\partial x^k}\,V \quad . \tag{2.35}$$

2. The potential energy responsible for the gravitational force between two point masses is given by

$$V = -\,G_N\,\frac{m_1 m_2}{|\vec{r}|} \quad . \tag{2.36}$$

   Here $\vec{r}$ is the vector between the two point masses, $m_{1,2}$ are their respective masses, and G is Newton's universal gravitational constant:

$$G_N \approx 6.67\,10^{-11}\,\frac{\text{meter}^3}{\text{kg sec}^2} \quad . \tag{2.37}$$

A few points are in order here. In the first place, the acceleration $d\vec{v}/dt$ of a point mass in a given gravitational field is independent of its mass. In other words, all point particles fall in the same way, whatever their mass. In the second place, Eq.(2.35) is clearly not suited as a covariant equation because it has upper indices on the left, and lower indices on the right. For Newtonian mechanics, this does not matter, but it makes it a subtle matter to derive Eq.(2.35) from general relativity. Thirdly, the gravitational forces implied by Eq.(2.36)

[9] The most efficient energy-producing process on Earth is the annihilation of a material object and its exact anti-material counterpart into nothing but photons, but the only antimatter available is laboriously produced, in miniscule quantities, using great amounts of energy. Even as a device for simply *storing* appreciable amounts of energy, antimatter would be extremely inefficient, totally unreliable, and very dangerous.

propagate instantaneously, since a change in m_2, say, implies an simultaneous change in V all over space. It is therefore in conflict with special relativity since 'simultaneous' is not a valid operational concept in special (and general) relativity.

2.2.2 The Poisson equation for Newtonian gravity

Newtonian gravity is most efficiently expressed by its defining equation, the Poisson equation. To see how it works, a preliminary excercise is in order. Consider the function $\phi(\vec{x}) = 1/|\vec{x}||$, where $\vec{x}$ is a three-dimensional vector. We can compute its Fourier transform, $\psi(\vec{k})$, as follows:

$$\begin{aligned}
\psi(\vec{k}) &\equiv \int d^3x\ \exp(i\vec{k}\cdot\vec{x})\ \phi(\vec{x}) \\
&= \frac{1}{\sqrt{\pi}} \int d^3x\ \exp(i\vec{k}\cdot\vec{x}) \int\limits_{-\infty}^{\infty} du\ \exp(-u^2\vec{x}^2) \\
&= \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} du \prod_{j=1}^{3} \left\{ \int\limits_{-\infty}^{\infty} dx\ \exp\left(-u^2\left(x - \frac{ik_j}{2u^2}\right)^2 - \frac{k_j^2}{4u^2}\right)\right\} \\
&= \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} du \left(\frac{\pi}{u^2}\right)^{3/2} \exp\left(-\frac{\vec{k}^2}{4u^2}\right) = 2\pi \int\limits_{0}^{\infty} du\ \frac{1}{u^3}\ \exp\left(-\frac{\vec{k}^2}{4u^2}\right) \\
&= 2\pi \int\limits_{0}^{\infty} dw\ \exp\left(-\frac{1}{2}\vec{k}^2 w\right) = \frac{4\pi}{\vec{k}^2} \quad , \qquad (2.38)
\end{aligned}$$

where in the last line we have defined $w = 1/(2u^2)$. We therefore have[10]

$$\phi(\vec{x}) = \frac{1}{|\vec{x}|} \quad \Leftrightarrow \quad \psi(\vec{k}) = \frac{4\pi}{\vec{k}^2} \quad . \qquad (2.39)$$

Now, consider the differential equation

$$\vec{\nabla}^2\ \phi(\vec{x}) = Q\ \delta^3(\vec{x}) \quad . \qquad (2.40)$$

Multiplying both sides by $\exp(i\vec{x}\cdot\vec{k})$ and integrating over $\vec{x}$, we may employ partial integration on the left-hand side to obtain

$$-\vec{k}^2\ \psi(\vec{k}) = Q \quad . \qquad (2.41)$$

Therefore, the Newtonian potential (that is, the potential energy per unit of mass of the test particle) generated by a point mass M situated at the origin,

$$V(\vec{x}) = -\frac{G_N M}{|\vec{x}|} \quad , \qquad (2.42)$$

[10] The inverse Fourier transform can also be performed, but calls for some more fast footwork.

Is the, essentially unique, solution of the Poisson equation,

$$\vec{\nabla}^2 V(\vec{x}) = 4\pi \, G_N \, M \, \delta^3(\vec{x}) \quad . \tag{2.43}$$

2.2.3 Gravitational behaviour as geodesic motion

One of the fundamental tenets of general relativity is that point particles that move under the influence of gravity alone (motion in free fall) do so along geodesics. There are a number of good reasons for this idea. In the first place, the motion is independent of the particle mass, and *may* therefore allow a description in purely geometric terms since it only involves coordinates. In the second place, the geodesic equations are of second order in the evolution parameter s, just as the Newtonian equation (2.35) is of second order in the evolution parameter t. A relation between ds and dt *may* therefore allow us to relate geodesic and Newtonian motion. In the third place, the geodesics are essentially the only curves between two given points that can be defined in a coordinate-independent way, and this is of course precisely what the principle of general relativity suggests we ought to aim at.

2.2.4 Newtonian mechanics from geodesics

Let us consider a particle following some trajectory in spacetime, parametrized by the arc length s: the trajectory is therefore a curve $x^\mu(s)$. We shall use the following assumptions:

1. As an approximation to the absolute, static nature of Newtonian space and time, we shall assume the metric to be static, and use a static coordinate system, so that the metric does not contain any explicit time dependence. The metric is allowed, of course, to depend on $\vec{x}$. We shall assume that the metric differs only slightly from the Minkowski one. In particular, g_{00} is close to 1, and g_{kk} close to -1. We therefore write

$$\begin{aligned} g_{00} &= 1 + \epsilon \, f_{00}(\vec{x}) \quad , \\ g_{kk} &= -1 + \epsilon \, f_{kk}(\vec{x}) \quad , \quad k = 1,2,3 \quad , \\ g_{kn} &= g_{nk} = \epsilon \, f_{kn}(\vec{x}) \quad , \quad k,n = 1,2,3 \quad , \end{aligned} \tag{2.44}$$

   with a small parameter ϵ.

2. We shall assume the particle to move slowly through space in this frame, so that $dx^k/dt \ll c$ (k=1,2,3). We shall assume that dx^k/dt is of order $\epsilon \, c$. In what follows we shall systematically neglect terms of order ϵ^2.

Let us denote the four-velocity dx^μ/ds by u^μ. The four-velocity defined in this way is dimensionless. Since for any massive particle's curve parametrized by s the kinematic condition holds, we find[11]

$$g_{00} \, (u^0)^2 + \mathcal{O}(\epsilon^2) \approx 1 \quad . \tag{2.45}$$

[11] Code `Riemann` with option 'Geodesic'

The geodesic equation for the space coordinates reads, approximately,

$$\frac{d}{ds}u^k = -\frac{1}{2}\, g_{00,k}\, (u^0)^2 + \mathcal{O}(\epsilon^2) \quad = \quad -\frac{1}{2g_{00}}\, g_{00,k} + \mathcal{O}(\epsilon^2) \ . \tag{2.46}$$

Since ds and $c\,dt$ differ only by negligible orders in ϵ, we may write this as

$$\frac{d^2}{dt^2}x^k = -c^2\frac{\partial}{\partial x^k}\sqrt{g_{00}} + \mathcal{O}(\epsilon^2) \ . \tag{2.47}$$

The interpretation is that a nonrelativistic particle with mass m appears, in our chosen coordinate system, to move under the influence of a Newtonian potential

$$V = mc^2\sqrt{g_{00}} \ , \tag{2.48}$$

which has the correct dimension of energy.

We see that, for particles moving at nonrelativistic speeds, only a single component of the metric, g_{00} governs the motion, and the other nine independent components are effectively irrelevant. The form of g_{00} thus provides us with a Newtonian gravitational potential.

A final note: we have somewhat arbitrarily decided to identify the velocities $u^k = \dot{x}^k$, with the Newtonian velocities v^k. This really only makes sense if the metric tensor is close to the Minkowski metric, since we need to rely on the time dt being approximately equal to the proper time ds/c.

2.2.5 The Newtonian gravitational potential

Our search for Newtonian gravity is not finished yet, since although we have established Eq.(2.35) as an approximation, we still have to see how Eq.(2.36) may arise. To this end we have to replace the postulate (2.36) by another one, formulated in a covariant manner. In general relativity, this postulate states that in regions of space free of matter or other types of nongravitational energy, the Einstein tensor vanishes:

$$G_{\mu\nu} = 0 \ . \tag{2.49}$$

Note that since $D \neq 2$, the vanishing of the Einstein tensor and the Ricci tensor are equivalent, so that an equivalent statement is

$$R_{\mu\nu} = 0 \ . \tag{2.50}$$

However, since Eq.(2.48) tells us that the space is *not* flat, the Riemann tensor may not vanish, since that would be overly restrictive. On the other hand, requiring $R = 0$ is not restrictive enough, since $R = 0$ may happen even if $R_{\mu\nu} \neq 0$.

We shall again use the almost-Minkowskian metric of Eq.(2.46). The Christoffel symbols, then, are also of order ϵ, and the Ricci tensor is approximated by the simpler form

$$R_{\mu\nu} \approx \Gamma^\alpha{}_{\mu\nu,\alpha} - \Gamma^\alpha{}_{\mu\alpha,\nu} + \mathcal{O}(\epsilon^2) \ . \tag{2.51}$$

The 10 independent equations contained in $R_{\mu\nu} = 0$ must hold separately. We may choose $\mu = \nu = 0$. In that case the second term in Eq.(2.51) also vanishes, and since we have

$$\Gamma_{\alpha 00} = \frac{-1}{2} g_{00,\alpha} \tag{2.52}$$

for static metrics, we find that

$$R_{00} = \frac{\epsilon}{2} \sum_{k=1}^{3} \frac{\partial^2}{(\partial x^k)^2} f_{00}(\vec{x}) + \mathcal{O}(\epsilon^2) \ . \tag{2.53}$$

For the almost-Minkowskian metric, therefore, we have

$$R_{00} = 0 \ \Rightarrow \ \sum_{k=1}^{3} \frac{\partial^2}{(\partial x^k)^2} g_{00}(\vec{x}) \approx 0 \ . \tag{2.54}$$

This is nothing but the Poisson equation satisfied by the Newtonian gravitational potential (2.43). It holds for the empty space away from a point source. Note that the 00 component of the Ricci tensor is the only one that, to leading order, *only* depends on f_{00}: the other ones mix f_{00} with other f's. These, of course, are also valid, but they just determine the other components of the metric, that are unimportant in the Newtonian approximation.

Some idea of the propriety of the almost-Minkowskian approximation is provided by Earth. In Nijmegen, on the Earth's surface, the acceleration due to gravity is about 10 meter/sec^2. With the results of the previous section, we see that then the gradient of ϵf_{00} must be about 10^{-16} per meter, and the approximation is very good indeed! As a more extreme case, consider a very massive star with the mass of the sun compacted into a radius of 10 kilometers: by a simple scaling argument[12] we arrive at an acceleration due to gravity at the star's surface of about 10^{12} meter/sec^2: a hefty acceleration, but a gradient of ϵf_{00} of only about 10^{-5} per meter.

2.3 The Schwarzschild metric

2.3.1 Spherically symmetric solution

We now search for a more exact solution to the equation $R_{\mu\nu} = 0$. We consider a metric that is spherically symmetric around the origin. This is a model for the non-Newtonian gravitational field around, say, a spherically symmetric massive object such as a star. We still require the metric to be spherically symmetric, but we do not insist on a static solution. The metric therefore depends on r and t. We expect that very far from the origin, the metric approaches the Minkowskian form; and somewhat closer to the origin, the Newtonian approximation should of

[12]The earth's radius is about 6000 kilometers. The mass of the earth is about 6×10^{24} kilograms, that of the sun about 2×10^{30} kilograms. The acceleration is proportional to the mass, and inversely proportional to the square of the radius.

course hold since the gravitational field of a star is well (if not exactly) described by Newtonian gravity.

The spherical symmetry suggests that we use polar coordinates for the space-like part of the metric. Thus, our coordinates will be

$$x^1 \equiv r \ , \ \ x^2 = \vartheta \ , \ \ x^3 = \phi \ , \ \ x^0 = ct \ , \tag{2.55}$$

and the general form of the metric will be diagonal, with

$$(ds)^2 = g(r,ct)(cdt)^2 - f(r,ct)(dr)^2 - r^2\left((d\vartheta)^2 + \sin(\vartheta)^2(d\phi)^2\right) \ . \tag{2.56}$$

The components g_{22} and g_{33} are fixed by our use of polar coordinates, so there are two functions, $f(r,ct)$ and $g(r,ct)$, to be found. Direct computation[13] tells us that the only nonvanishing components of the Ricci tensor are the diagonal ones, and in addition

$$R_{01} = R_{10} = \frac{1}{r\,f}\,\frac{\partial}{\partial x^0}f \ . \tag{2.57}$$

Therefore, $f(r,ct)$ cannot depend on t, but only on r. Once we insert this in the diagonal components, no derivatives with respect to x^0 remain[14], and we find

$$\begin{aligned} R_{00} &= \frac{g'^2}{4g^2} + \frac{f'g'}{4fg} - \frac{g''}{2g} + \frac{f'}{rf} \ , \\ R_{11} &= -\frac{g'^2}{4fg} - \frac{f'g'}{4f^2} + \frac{g''}{2f} + \frac{g'}{rf} \ , \end{aligned} \tag{2.58}$$

where primes denote derivatives with respect to r. We may therefore form the following combination:

$$rfgR_{11} + rf^2R_{00} = f'g + g'f \ , \tag{2.59}$$

so that the product of f and g only depends on x^0. We can write

$$g(r,ct) = k(ct)h(r) \ \ , \ \ f(r) = 1/h(r) \ , \tag{2.60}$$

upon which substitution we have

$$R_{22} = 1 - h - rh' \ , \tag{2.61}$$

with the immediate unique solution

$$h(r) = 1 - \frac{R_s}{r} \ , \tag{2.62}$$

where R_s is a constant, whose magnitude and sign is still to be determined. It is easily checked that this solution indeed solves[15] all components of $R_{\mu\nu} = 0$. Let us inspect the metric that we have obtained so far: it gives

$$\begin{aligned} (ds)^2 &= k(x^0)\left(1 - \frac{R_s}{r}\right)(dx^0)^2 - \left(1 - \frac{R_s}{r}\right)^{-1}(dr)^2 \\ &\quad -r^2\left((d\vartheta)^2 + \sin(\vartheta)^2(d\phi)^2\right) \ . \end{aligned} \tag{2.63}$$

[13] Code `Riemann` with option `Ricci`

[14] The only occurrences of $\partial g/\partial x^0$ are multiplied with $\partial f/\partial x^0$.

[15] If it didn't, there would be *no* spherically symmetric solution at all!

The determinant of this metric is given by

$$g = -k(x^0) r^4 \sin(\vartheta)^2 \quad . \tag{2.64}$$

By assumption, g cannot vanish, except at the obvious coordinate singularities $r = 0$ and $\sin(\vartheta) = 0$. Therefore, $k(x^0)$ must have fixed sign: and if we are to approach the Minkowski metric at all, that sign must be positive. The appearance of $k(x^0)$ is thus seen to be nothing more than the effect of an 'unlucky' choice of time coordinate: by the transformation

$$x^0 \to (x^0)' \equiv \int k(x^0)^{1/2}\, dx^0 \quad , \tag{2.65}$$

it completely disappears from the metric[16]. The constant R_s can be determined by examining the Newtonian limit (for instance, by taking r sufficiently large so that g_{00} differs only slightly from unity): to match with Eq.(2.36), we must have

$$R_s = \frac{2G_N M}{c^2} \quad , \tag{2.66}$$

where M is the mass of the gravitational point source, located at $r = 0$. The final result is called the *Schwarzschild metric*:

$$\begin{aligned}(ds)^2 \quad = \quad & \left(1 - \frac{2G_N M}{c^2}\frac{1}{r}\right)(cdt)^2 - \left(1 - \frac{2G_N M}{c^2}\frac{1}{r}\right)^{-1}(dr)^2 \\ & -r^2\left((d\vartheta)^2 + \sin(\vartheta)^2\,(d\phi)^2\right) \quad . \end{aligned} \tag{2.67}$$

An important observation is in order here. The Scharschild metric is the *only* spherically symmetric solution, as we have seen; and it only depends on a single property of the source, namely its mass. Therefore, whatever happens to this source, provided it remains spherically symmetric, the Schwarzschild metric will remain the same. In particular, no spherically symmetric waves in the metric are possible.

2.3.2 The Schwarzschild radius

Because G_N is small and c is large, the Schwarzschild radius R_s tends to be quite small: in usual units, we have

$$R_s = 1.48\ 10^{-27}\left(\frac{M}{\text{kg}}\right)\ \text{meter} \quad . \tag{2.68}$$

For the sun, with $M \approx 2\ 10^{30}$ kilogram, R_s is about 3 kilometers; for the earth, with $M \approx 6\ 10^{24}$ kilogram, R_s is about 0.9 centimeters; and for a proton with mass $1.67\ 10^{-27}$ kilogram, R_s is a miniscule $2.5\ 10^{-52}$ centimeter. In all these cases, R_s is much less than the actual physical radius of the object. Since

[16]Note that such a trick would not work to get rid of the factor $1 - R_s/r$, since that would destroy the Minkowskian character of the spacelike part.

the field equations $R_{\mu\nu} = 0$ only hold in regions where no mass is present, the Schwarzschild solution should not be expected to describe the metric inside these bodies. On the other hand, in principle nothing forbids the existence of bodies with extremely high density, for which R_s exceeds their physical radius.

2.3.3 Falling in a Schwarzschild metric

The schwarzschild metric becomes singular at $r = R_s$, where $R_s \equiv 2G_N M/c^2$ is the *Schwarzschild radius*. In order to get a physical idea of what this means, let us consider a particle falling towards $r = 0$, starting beyond R_s. For simplicity we assume the motion to be radial, that is, $\dot{\vartheta} = 0$ and $\dot{\phi} = 0$. The equation kinematic condition then becomes

$$\left(1 - \frac{R_s}{r}\right) c^2 \, (\dot{t})^2 - \left(1 - \frac{R_s}{r}\right)^{-1} (\dot{r})^2 = 1 \ , \tag{2.69}$$

and the geodesic equation for x^0 reads

$$\ddot{t} + \frac{2R_s}{r(r - 2R_s)} \, \dot{r} \, \dot{t} = 0 \ . \tag{2.70}$$

This last equation implies that

$$\frac{d}{ds} t = \frac{a}{c} \left(1 - \frac{R_s}{r}\right)^{-1} \ , \tag{2.71}$$

with a a constant. Inserting this in Eq.(2.69), we find that

$$\frac{d}{ds} r = -\left(a^2 - 1 + \frac{R_s}{r}\right)^{1/2} \ , \tag{2.72}$$

where we have taken $\dot{r} < 0$ since the particle is falling inwards. Combining these last two equations, we find a relation between r and t:

$$\frac{dt}{dr} = \frac{\dot{t}}{\dot{r}} = -a \left(1 - \frac{R_s}{r}\right)^{-1} \left(a^2 - 1 + \frac{R_s}{r}\right)^{-1/2} \ . \tag{2.73}$$

When the particle has approached closely to the Schwarzschild radius R_s, we may write $r = R_s + \rho$, with small ρ, and then

$$\frac{dt}{dr} \approx -\frac{R_s}{c \, \rho} + \mathcal{O}(\rho^0) \ . \tag{2.74}$$

Therefore, t goes to inifinity as r approaches R_s: the particle is *observed* to move ever more *slowly*, taking an infinite time to reach the Schwarzschild radius! On the other hand, in the same limit,

$$\frac{ds}{dr} = -\frac{1}{a} + \mathcal{O}(\rho) \ , \tag{2.75}$$

so that the particle's *proper time* is still finite when the Schwarzschild radius is reached.

2.3.4 Falling through the Schwarzschild radius

As we have seen, falling towards the origin in the Schwarzschild metric, a particle takes a finite amount of *proper* time to reach $r = R_s$; what happens afterwards? In fact nothing much. The Schwarzschild metric, indeed, is singular at $r = R_s$, but the spacetime itself is not, as might already be suggested from $R_{\mu\nu} = 0$ and the vanishing Gauss curvature. To exhibit this more clearly, we may change to a coordinate system in which the metric is regular.

The so-called *Eddington-Finkelstein* coordinates are defined as follows: we keep r, ϑ and ϕ, but instead of $x^0 = ct$ we define x^0 by

$$x^0 \equiv ct + r + R_s \log\left|\frac{r}{R_s} - 1\right| \quad . \tag{2.76}$$

Taking differentials, we find that

$$d(ct) = d(x^0) - \frac{1}{1 - R_s/r}dr \quad ; \tag{2.77}$$

and inserting this in the schwarzschild metric we find that the metric in these new variables is given by

$$(ds)^2 = \left(1 - \frac{R_s}{r}\right)(dx^0)^2 - 2d(x^0)(dr) - r^2\left((d\vartheta)^2 + \sin(\vartheta)^2(d\phi)^2\right) \quad . \tag{2.78}$$

In terms of these variables, nothing special happens at $r = R_s$ (but note that x^0 diverges there.

Another choice of new coordinates is due to Dirac. Here we replace r and x^0 by

$$\begin{aligned} \rho &= ct + \frac{2}{3}\frac{r^{3/2}}{R_s^{1/2}} + (rR_s)^{1/2} + R_s \log\left(\frac{\sqrt{r/R_s} - 1}{\sqrt{r/R_s} + 1}\right) \quad , \\ \tau &= ct + (rR_s)^{1/2} + R_s \log\left(\frac{\sqrt{r/R_s} - 1}{\sqrt{r/R_s} + 1}\right) \quad . \end{aligned} \tag{2.79}$$

The metric can be worked out straightforwardly: we have

$$(ds)^2 = (d\tau)^2 - \frac{R_s}{r}(d\rho)^2 - r^2\left((d\vartheta)^2 + \sin(\vartheta)^2(d\phi)^2\right) \quad . \tag{2.80}$$

This metric has the advantage that it is diagonal, but since r depends on ρ and τ:

$$r = \left(\frac{3}{2}R_s^{1/2}(\rho - \tau)\right)^{2/3} \quad , \tag{2.81}$$

it cannot be called static.

We may conclude that the singularity in the metric at $r = R_s$ is a coordinate rather than a physical singularity. Nevertheless, the fact remains that a particle will take an infinite observer time to cross the Schwarzschild radius.

2.3.5 Some other orbits

The possible orbits of bodies in the Schwarzschild metric are more diverse than those in Newtonian gravity (where they are all conic sections). One thing remains, however. The geodesic equation for ϑ reads

$$\ddot{\vartheta} + \frac{2}{r}\dot{r}\dot{\vartheta} - \sin(\vartheta)\cos(\vartheta)\dot{\phi}^2 = 0 \ . \tag{2.82}$$

If, therefore, a body in orbit has $\vartheta = \pi/2$ and $\dot{\vartheta}$ at any given moment, ϑ will remain equal to $\pi/2$ at all times. Since we may orient our coordinate system at will, every orbit in the Schwarzschild metric will be planar, like in Newtonian gravity. Therefore, from now on let us assume $\vartheta = \pi/2$. The geodesic equation for ϕ then reads

$$\ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} = 0 \ , \tag{2.83}$$

which leads to a conservation law:

$$r^2\dot{\phi} = L \ , \tag{2.84}$$

with L a constant. It plays the rôle of angular momentum[17]. Similarly, the geodesic equation for x^0 reads

$$\ddot{x}^0 + \frac{R_s}{r(r-R_s)}\dot{r}\dot{x}^0 = 0 \ , \tag{2.85}$$

so that we have another conservation law,

$$\dot{x}^0 = \frac{K}{(1-r/R_s)} \ , \tag{2.86}$$

where K is another constant, which will turn out to describe an energy[18].

Let us now concentrate on the possibility of circular orbits, that is $r(s) = \rho$, a constant. We then find for the geodesic equation for r that

$$\frac{K^2R_s}{2\rho(\rho-R_s)} - \frac{L^2(\rho-R_s)}{\rho^4} = 0 \ , \tag{2.87}$$

while the kinematic condition gives

$$\frac{K^2\rho}{\rho-Rs} - \frac{L^2}{\rho^2} = 1 \ . \tag{2.88}$$

We may solve this to find:

$$K^2 = \frac{2(\rho-R_s)^2}{\rho(2\rho-3R_s)} \ , \quad L^2 = \frac{\rho^2R_s}{(2\rho-3R_s)} \ . \tag{2.89}$$

[17]Note, however, that is has the dimension of length rather than of action.
[18]Note, however, that it is dimensionless.

Therefore, circular orbits are only allowed for

$$\rho \geq \frac{3}{2}R_s = \frac{3G_N M}{c^2} \quad . \tag{2.90}$$

For such orbits, the angular velocity is ω, with

$$\omega^2 = \left(\frac{d\phi}{dt}\right)^2 = \left(\frac{\dot{\phi}}{\dot{t}}\right)^2 = \frac{L^2(\rho - R_s)^2 c^2}{\rho^6 K^2} = \frac{R_s c^2}{2\rho^2} = \frac{G_N M}{\rho^3} \quad , \tag{2.91}$$

which is in fact Kepler's law! Furthermore, we have

$$\begin{aligned}(ds)^2 &= \left(1 - \frac{R_s}{\rho}\right) c^2 (dt)^2 - \rho^2 (d\phi)^2 \\ &= \left(c^2\left(1 - \frac{R_s}{\rho}\right) - \omega^2\rho^2\right)(dt)^2 = \left(1 - \frac{3R_s}{2\rho}\right)(dt)^2 \quad . \end{aligned} \tag{2.92}$$

The orbit with $\rho = 3R_s/2$ corresponds to $(ds)^2 = 0$, that is, at this distance a beam of light follows a circular path! For massive objects, we must have $(ds)^2 > 0$ and therefore their circular orbits must have $\rho > R_s$.

More general orbits can be discussed in the following way. Let us drop the constraint $r = \rho$. The kinematic condition can be written as

$$\frac{1}{2}(\dot{r})^2 = \frac{K^2 - 1}{2} - V(r) \quad , \quad V(r) = \frac{1}{2}\left(\frac{L^2}{r^2} - \frac{L^2 R_s}{r^3} - \frac{R_s}{r}\right) \quad . \tag{2.93}$$

By differentiating once more to s we the find

$$\ddot{r} = -V'(r) \quad . \tag{2.94}$$

That is, r behaves as the coordinate of a Newtonian particle of unit mass in an effective potential $V(r)$, which follows the usual convention $V(r \to \infty) = 0$; and the total energy of the particle is $(K^2 - 1)/2$. We see immediately that every possible orbit for a particle coming in from infinity must have $K > 1$. Inspection of the potential gives us qualitative information about the various orbits. It is conventient to express all distances in units of R_s, so that we may put $R_s = 1$ for the moment. The effective potential always goes to $-\infty$ as r approaches zero. The possible extrema $r_\pm$ of $V(r)$ obey

$$r_\pm^2 - 2L^2 r_\pm + 3L^2 = 0 \quad \to \quad r_\pm = L^2 \pm L\sqrt{L^2 - 3} \quad . \tag{2.95}$$

We may formulate some conclusions.

- For $L < \sqrt{3}$, the effective potential is monotonically increasing, and any particle coming in from infinity will end up at the origin[19]; no bound states are possible.

[19] There is no contradiction with the previous results: note that we are now considering r as a function of s, not of t!

- For $L > \sqrt{3}$, the effective potential has a maximum at r_-, and a minimum at r_+. This means that for given L two circular orbits are possible, one with $\rho = r_+$ and one with $\rho = r_-$. The circular orbits with $\rho = r_+$ are stable: a small perturbation will cause r to oscillate around r_+ (in Newtonian gravity, these will be elliptic orbits, but not so in Einteinian gravity). The orbits with $\rho - r_-$ are *un*stable: a small perturbation will drive r away from r_-.

- The maximum of the potential at $r = r_-$ is given by

$$V(r_-) = \frac{2 - L^2 + L\sqrt{L^2 - 3}}{2L\left(L^2 - L\sqrt{L^2 - 3}\right)^3} \quad , \tag{2.96}$$

  and vanishes for $L = 2$. Any particle coming in with total energy larger than $V(r_-)$ will disappear towards $r = 0$. For $L < 2$, this means that *any* particle coming in from infinity will be captured. For $L > 2$, $V(r_-) > 0$, and particles coming in from infinity with $K^2 < 1 + V(r_-)$ will emerge again: scattering states are possible, but only up to some energy. This contrasts with the Newtonian case, where any particle with positive total energy is in a scattering state, and any particle with negative total energy is in a bound state.

2.3.6 Perihelion precession

For starters, we present a quick derivation of the shape of the Newtonian closed orbits. The two most important conserved quantities for Kepler orbits (in polar coordinates in the plane of revolution) are the total energy and the total angular momentum:

$$\begin{aligned} E &= \frac{m}{2}\left(\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2\right) - \frac{G_N M m}{r} \quad , \\ L &= m r^2 \frac{d\phi}{dt} \quad . \end{aligned} \tag{2.97}$$

Here m is the mass of the 'planet', and M that of the 'Sun'. We may eliminate dt in favour of $d\phi$, and arrive at

$$\left(\frac{dr}{d\phi}\right)^2 \frac{1}{r^4} + \frac{1}{r^2} = \frac{2G_N M m^2}{L^2}\frac{1}{r} + \frac{2Em}{L^2} \quad . \tag{2.98}$$

Putting $1/r(\phi) = u(\phi)$, we obtain

$$(u')^2 + u^2 = 2\lambda u + \frac{2Em}{L^2} \quad , \quad \lambda = \frac{G_N M m^2}{L^2} \quad , \tag{2.99}$$

where the prime denotes differentiation with repsect to ϕ. Another differentiation then yields

$$u'' + u = \lambda \quad , \tag{2.100}$$

with the solution (chosen to have its maximum at $\phi = 0$):

$$u(\phi) = \lambda(1 + \epsilon\cos(\phi)) \quad . \tag{2.101}$$

The *excentricity* ϵ is determined from the original equation (2.99) to be

$$\epsilon = \sqrt{1 + \frac{2EL^2}{G_N^2 M^2 m^3}} \quad , \tag{2.102}$$

and the shape of the orbit is given by

$$r(\phi) = \frac{1/\lambda}{1 + \epsilon\cos(\phi)} \quad . \tag{2.103}$$

For bound states, with $E < 0$ (hence $\epsilon < 1$), these are ellipses with the 'Sun' in one of the focal points. For given L, the minimum energy is $-G_N M^2 m^3/2L^2$, and the orbit is then a circle. In any case, the orbits are strictly periodic with period 2π. Note that this simple result arises from the fact that the potential is exactly proportional to $1/r$.

We now turn to the general-relativistic case, using the Schwarzschild metric. As before, we may use the assumption that $\theta = \pi/2$, and the geodesic equations for ϕ and x^0 allow us to write

$$\frac{d\phi}{ds} = \frac{A}{r^2} \quad , \quad \frac{dx^0}{ds} = \frac{K}{1 - R_s/r} \quad . \tag{2.104}$$

The kinematic condition then tells us that

$$1 = \frac{K^2}{1 - R_s/r} - \frac{1}{1 - R_s/r}\left(\frac{dr}{ds}\right)^2 - \frac{A^2}{r^2} \quad . \tag{2.105}$$

Exchanging ds for $d\phi$, we may write this as

$$\left(\frac{dr}{d\phi}\right)^2 \frac{1}{r^4} + \frac{1}{r^2} = \frac{K^2 - 1}{A^2} + \frac{R}{A^2}\frac{1}{r} + \frac{R}{r^3} \quad . \tag{2.106}$$

Comparison with the Newtonian case then imposes the following identifications:

$$A = \frac{L}{mc} \quad , \quad \frac{K^2 - 1}{2} mc^2 = E \quad , \tag{2.107}$$

and, again writing $r(\phi) = 1/u(\phi)$, we arrive at the relativistic adaptation of eQ.(2.99):

$$(u')^2 + u^2 = 2\lambda u + \frac{2mE}{L^2} + R_s u^3 \quad . \tag{2.108}$$

The equation of motion for u then becomes

$$u'' + u = \lambda + \frac{3R}{2}u^2 \quad . \tag{2.109}$$

Writing $u(\phi) = \lambda v(\phi)$, we find the more manageable form

$$v'' + v = 1 + \delta v^2 \quad , \quad \delta = \frac{3R_s\lambda}{2} = 3\left(\frac{G_N Mm}{Lc}\right)^2 \quad . \tag{2.110}$$

In usual situations, the dimensionless number δ is extremely small, and we may employ a perturbative approach, by writing

$$v = v_0 + \delta v_1 + \mathcal{O}(\delta^2) \quad . \tag{2.111}$$

To first order in δ, we have

$$\begin{aligned} v_0'' + v_0 &= 1 \ , \\ v_1'' + v_1 &= v_0^2 \ . \end{aligned} \tag{2.112}$$

The general solution that has a minimum at $\phi = 0$ is given to first order by

$$v(\phi) = 1 + \delta\left(1 + \frac{B^2}{2}\right) + B\cos(\phi) + \delta B\phi\sin(\phi) - \frac{B^2\delta}{6}\cos(2\phi) \quad . \tag{2.113}$$

From Eq.(2.108) we find that B is, to first order in δ, given by

$$B = \epsilon + \delta\frac{1 + 3\epsilon^2}{3\epsilon} \quad . \tag{2.114}$$

By construction, $v(\phi)$ has a minimum at $\phi = 0$, but the next minimum is not reached at $\phi = 2\pi$ but rather at $\phi = 2\pi(1 + \delta)$. The orbit is now not closed in upon itself: the angle at which r takes its minimum is shifted by $2\pi\delta$ at each revolution. This is the famous 'perihelion precession' which was one of the first triumphantly confirmed predictions of general relativity, for the case of the planet Mercury[20].

2.3.7 Light deflection

Let us again try to solve Eq.(2.109), for the case of a massless particle, *i.e.* a photon. Since a photon has $(ds)^2 = 0$, the constant L in Eq.(2.83) must be infinite. In that case, $\lambda = 0$ and we find, instead of Eq.(2.109),

$$u''(\phi) + u(\phi) = \frac{3}{2}R_s u(\phi)^2 \quad . \tag{2.115}$$

We cosider the following situation: a photon comes in from infinity, scatters in the gravtiational potential, and exits again towards infinity. If we would put $R_s = 0$ (*i.e.* in flat Minkowski space) we would have a solution u_0, with

$$u_0''(\phi) + u_0(\phi) = 0 \quad , \tag{2.116}$$

[20]This is by no means straightforward, since the perihelion of Mercury is also influenced by the other planets. The observed precession is about 0.16 degree *per century*: after accounting for the effects of Venus, Jupiter, etc, a residue of 0.01194 ± 0.00001 degree per century remains: 0.01194 is exactly what general relativity predicts. We are talking, here about a one-percent correction to a miniscule precession! The Hulse-Taylor binary pulsar, on the other hand, has a precession of about 4 degrees *per year*.

and an appropriate solution, with perihelion at $\phi = 0$:

$$u_0(\phi) = A\cos(\phi) \quad : \tag{2.117}$$

for $\phi = \pm\pi/2$, the photon is at infinity. If R_s is nonzero but small compared to $1/A$, we may write

$$u(\phi) = u_0(\phi) + R_s A u_1(\phi) + \mathcal{O}(R_s^2 A^2) \quad . \tag{2.118}$$

The first-order part of Eq.(2.116) reads

$$u_1''(\phi) + u_1(\phi) = \frac{3}{2} A\cos(\phi)^2 \quad , \tag{2.119}$$

with the following particular solution:

$$u_1(\phi) = A - \frac{1}{2} A\cos(\phi)^2 \quad . \tag{2.120}$$

It is easily checked that the full (approximate) solution

$$u(\phi) = A\cos(\phi) + A^2 R_s \left(1 - \frac{1}{2}\cos(\phi)^2\right) \tag{2.121}$$

indeed satisfies Eq.(2.116) up to terms of second order in R_s. The requirement $u(\phi) = 0$ which places the photon at infinity now no longer corresponds to $|\phi| = \pi/2$ but rather to $|\phi| \approx \pi/2 + AR_s$, as long as this is a small number. As seen from the origin, therefore, the photon coming in from, and exiting towards, infinity subtends an angle of $\pi + 2AR_s$ rather than just π. A light ray from a distant star, grazing the Sun's surface, will therefore be bent over an angle

$$\theta_{\text{Einstein}} \approx \frac{2R_s}{R_{\text{sun}}} = \frac{4G_N M_{\text{sun}}}{R_{\text{sun}} c^2} \quad . \tag{2.122}$$

For our Sun, with a mass of about 2 10^{30} kg and a radius of about 7 10^8 m, the deflection angle is about 1.75 arc seconds.

It is instructive to compare this result with that expected in Newtonian gravity. To this end, we assing the photon a very small mass m, and a velocity at infinity equal to c. It will then have a total energy $E = mc^2/2$, and an angular momentum approximately equal to Bmc, where B is the impact parameter[21]. The Newtonian equation (2.98) then becomes

$$\begin{aligned}(u')^2 + u^2 &= \frac{2G_N M m^2}{L^2} u + \frac{2mE}{L^2} \\ &= \frac{R_s}{B^2} u + \frac{1}{B^2} \quad ,\end{aligned} \tag{2.123}$$

[21]The impact parameter is the closest distance at which the light would pass the origin if there were no deflection.

and we may indeed take the limit $m \to 0$ here. By differentiating we find

$$u''(\phi) + u(\phi) = \frac{R_s}{2B^2} \quad . \tag{2.124}$$

The appropriate solution is therefore

$$u(\phi) = \frac{R_s}{2B^2} + \frac{1}{A}\cos(\phi) \quad , \quad A \approx B\left(1 - \frac{R_s^2}{8B^2}\right) \tag{2.125}$$

if R_s/B is small; then, A and B are very nearly equal. The distance A is again the grazing distance, and we see that the Newtonian result for the deflection at the Sun's surface is

$$\theta_{\text{Newton}} \approx \frac{R_s}{R_{\text{sun}}} = \frac{1}{2}\theta_{\text{Einstein}} \quad ; \tag{2.126}$$

the difference is precisely a factor 2.

2.3.8 A digression: Newton's deflection hoax

In the previous paragraph we have seen that the deflection of a light-particle in general relativity exceeds that in Newtonian gravity by a factor 2. However, in Newtonian gravity the speed of the light particle is not constant. If we take the kinetic energy at infinity to be $mc^2/2$, the kinetic energy at perihelion will be

$$E = \frac{1}{2}mc^2 + \frac{G_N M m}{R_{\text{sun}}} = \frac{1}{2}mc^2\left(1 + \frac{R_s}{R_{\text{sun}}}\right) \quad . \tag{2.127}$$

so that the light's velocity is increased by a factor of approximately $1{+}R_s/2R_{\text{sun}}$. Let us now play out the following scenario. Suppose that Newtonian theory is valid, but that some force is available, acting in the direction of motion, so as to keep the velocity of the light particle always at c, which in this world is just some non-special velocity[22]. In that case, the light particle is moving more slowly at perihelion, and hence will be bent more by the Sun's gravity. We shall compute the deflection in this situation.

Since neither the total energy nor the angular momentum are conserved, we must adopt a different approach. The motion around the sun will still be planar. Let us write the position and velocity in polar coordinates as

$$\vec{x} = r\begin{pmatrix} \cos(\phi) \\ \sin(\phi) \end{pmatrix} \quad , \quad \dot{\vec{x}} = \dot{r}\begin{pmatrix} \cos(\phi) \\ \sin(\phi) \end{pmatrix} + r\,\dot{\phi}\begin{pmatrix} -\sin(\phi) \\ \cos(\phi) \end{pmatrix} \quad . \tag{2.128}$$

The acceleration,

$$\ddot{\vec{x}} = \left(\ddot{r} - r\,\dot{\phi}^2\right)\begin{pmatrix} \cos(\phi) \\ \sin(\phi) \end{pmatrix} + \left(2\dot{r}\dot{\phi} + r\ddot{\phi}\right)\begin{pmatrix} -\sin(\phi) \\ \cos(\phi) \end{pmatrix} \quad , \tag{2.129}$$

[22] Instead of the photon, we may envisage the spaceship *Spirit of St. Isaac*, provided with *Mach energy absorber-desorbers*, that keep the velocity with respect to absolute space constant. The spaceship is manned by a diehard space alien trying to mimic Einstein's deflection result by a clever hoax.

can be decomposed as follows:

$$\begin{aligned} \ddot{\vec{x}} &= C_1\,\vec{x} \;+\; C_2\,\dot{\vec{x}}\ , \\ C_1 &= \frac{\ddot{r}}{r} - 2\frac{\dot{r}^2}{r^2} - \dot{\phi}^2 - \frac{\dot{r}\ddot{\phi}}{r\dot{\phi}}\ , \\ C_2 &= 2\frac{\dot{r}}{r} + \frac{\ddot{\phi}}{\dot{\phi}}\ . \end{aligned} \tag{2.130}$$

The requirement that the magnitude of the velocity is always c implies

$$\dot{r}^2 + r^2\dot{\phi}^2 = c^2\ , \tag{2.131}$$

from which we immediately infer that

$$r^2 + \left(\frac{dr}{d\phi}\right)^2 = \frac{c^2}{\dot{\phi}^2}\ . \tag{2.132}$$

The acceleration along the velocity direction is due to the special force; that along the radius is due to Newtonian gravity, so that (using Eq.(2.131)) we have

$$rC_1 = \frac{\ddot{r}\,c^2}{c^2 - \dot{r}^2} - \frac{c^2}{r} = -\frac{G_N M}{r^2}\ . \tag{2.133}$$

Multiplying this equation by $2\dot{r}/c^2$ and integrating, we find

$$\log\left(\frac{r^4\dot{\phi}^2}{c^2K^2}\right) = -\frac{R_s}{r}\ . \tag{2.134}$$

Here, K is an integration constant. This gives us $\dot{\phi}$:

$$\dot{\phi} = \frac{cK}{r^2}e^{-R_s/2r}\ , \tag{2.135}$$

and the shape of the orbit is governed by the following equation (where again $u = 1/r$):

$$(u')^2 + u^2 = \frac{1}{K^2}e^{R_s u} \approx \frac{1}{K^2} + \frac{R_s}{K^2}u\ , \tag{2.136}$$

working to first order. The solution that has the perihelion at $\phi = 0$ is given by

$$u = \frac{1}{A}\cos(\phi) + \frac{R_s}{2K^2}\quad ,\quad \frac{1}{A^2} = \frac{1}{K^2} + \frac{R_s^2}{4K^4}\ , \tag{2.137}$$

so that K and A are again essentially equal. Infinity is reached ($u = 0$) for $|\phi|$ approximately $\pi/2 + R_s/2A$, and we find the deflection of the light particle in this 'hoax' universe to be

$$\theta_{\text{Newton's hoax}} \approx \frac{R_s}{R_{\text{sun}}} = \frac{1}{2}\theta_{\text{Einstein}}\ . \tag{2.138}$$

The hoax doesn't work: the deflection is larger than in the purely Newtonian case, but only by a tiny higher-order amount. To obtain Einstein's result the light particle would have to move appreciably *slower* than c at perihelion, which would be weird indeed.

2.4 The Einstein equation

2.4.1 Form of the Einstein equations

For matterless space we have used the Einstein equation $G_{\mu\nu} = 0$; in the presence of matter, this has of course to be modified. We therefore propose that the form of the Einstein equation in the presence of matter ought to read

$$G^{\mu\nu} = (\text{object depending on matter})^{\mu\nu} \quad . \tag{2.139}$$

The right-hand side, then, should be a symmetric rank-two tensor. Furthermore, from the fact that $G^{\mu\nu}_{;\nu} = 0$ by construction we may evolve two options. The right-hand side may also be covariantly divergenceless by construction; alternatively, we may use the vanishing divergence as a physical prediction. It is this last alternative that we shall use.

2.4.2 The relativistic dust model

We shall assume that matter enters the picture in the form of a fluid-like dust consisting of point particles of unit mass, that do not interact with each other but only with the gravitational field. Let us denote the coordinates of such a fluid element by $z^\mu(s)$; the velocity of this element is then given by $\dot{z}^\mu$, which we may denote by $v^\mu(x)$ since it may vary from point to point. This constitutes a contravariant vector field. We now introduce an additional scalar function $\rho(x)$ such that

$$\rho\ v^0\ \sqrt{|g|}$$

is the density of the fluid, and

$$\rho\ v^k\ \sqrt{|g|}$$

is the k-component of its flow. The kinematic condition $v_\mu v^\mu = 1$ gives, upon covariant differentiation,

$$v_\mu\ (v^\mu{}_{;\nu}) = 0 \quad . \tag{2.140}$$

In addition, *if* the fluid element moves along a geodesic, we must also have (*cf.* Eq.(1.135)):

$$v^\nu\ (v^\mu{}_{;\nu}) = 0 \quad . \tag{2.141}$$

2.4.3 Qualitative form of the Einstein equation

The matter content of spacetime is described by the scalar function $\rho(x)$ and the vector field $v^\mu(x)$. We therefore propse that the Einstein equation reads

$$G^{\mu\nu} = \kappa\ \rho\ v^\mu\ v^\nu \quad . \tag{2.142}$$

This has direct consequences for the behaviour of matter. For, the vanishing of the divergence of the left-hand side implies that of the right-hand side:

$$G^{\mu\nu}{}_{;\nu} = 0 \quad \Rightarrow \quad (\rho\ v^\mu\ v^\nu)_{;\nu} = 0 \quad . \tag{2.143}$$

By Leibniz' rule, we may write this as

$$v^\mu\ (\rho\ v^\nu)_{;\nu} + \rho\ v^\nu\ v^\mu{}_{;\nu} = 0\ \ . \tag{2.144}$$

Multiplying this by v_μ we get

$$(\rho\ v^\nu)_{;\nu} + \rho\ v^\nu\ \left(v_\mu\ (v^\mu{}_{;\nu})\right) = 0\ \ . \tag{2.145}$$

The second term vanishes because of Eq.(2.140), so we find

$$(\rho\ v^\mu)_{;\mu} = 0\ \ , \tag{2.146}$$

which tells us that matter is conserved. Therefore we also have

$$v^\nu\ v^\mu{}_{;\nu} = 0\ \ , \tag{2.147}$$

which tells us that matter moves along geodesics. These two attractive properties follow directly from the no-divergence property of the Einstein tensor, coupled with the proposed form of the Einstein equation and the fact that the density function ρ occurs with a first power.

Finally, the Einstein equation may be rewritten as an equation for the Ricci tensor. We have

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = \kappa\ \rho\ v^\mu\ v^\nu\ \ . \tag{2.148}$$

Multiplying by $g_{\mu\nu}$ this gives

$$R = -\kappa\ \rho\ \ , \tag{2.149}$$

so that we arrive at

$$R^{\mu\nu} = \kappa\ \rho\ \left(v^\mu\ v^\nu - \frac{1}{2}g^{\mu\nu}\right)\ \ . \tag{2.150}$$

2.4.4 Quantitative form of the Einstein equation

We still have to find the value of κ. We may do this by investigating the Newtonian limit. We therefore assume the matter to consist of a single particle of mass M, at rest at the origin. Moreover, we assume the metric to differ only very slightly from the Minkowskian one, so that $\sqrt{|g|}$ equals 1 to a very good approximation. Therefore, we may write

$$v^0 = 1\ \ ,\ \ \vec{v} = 0\ \ ,\ \ \rho = M\ \delta^3(\vec{x})\ \ . \tag{2.151}$$

The Einstein equation for the Ricci tensor therefore implies

$$R^{00} = \frac{1}{2}\ \kappa\ M\ \delta^3(\vec{x})\ \ . \tag{2.152}$$

In the almost-Minkowskian case, we have see that the left-hand side is approximately equal to

$$R^{00} \approx \frac{1}{2}\vec{\nabla}^2\ g^{00}\ \ . \tag{2.153}$$

Furthermore, the Newtonian gravitational potential V is related to g^{00} by

$$V = c^2 \sqrt{g^{00}} \quad , \tag{2.154}$$

up to an additive constant. The Poisson equation for the Newtonian potential therefore reads

$$\vec{\nabla}^2 V \approx \frac{c^2}{2} \vec{\nabla}^2 g^{00} = \frac{c^2 \kappa M}{2} \delta^3(\vec{x}) = 4\pi G_N M \delta^3(\vec{x}) \quad . \tag{2.155}$$

This fixes the constant κ to be

$$\kappa = 8\pi \frac{G_N}{c^2} \quad . \tag{2.156}$$

Since it is supposed to be constant, it must have this value in all situations. The full Einstein equation now reads

$$G^{\mu\nu} = 8\pi \frac{G_N}{c^2} \rho v^\mu v^\nu \quad . \tag{2.157}$$

2.5 Action principles

2.5.1 The Einstein action

It is always useful to formulate dynamical laws in terms of action principles. In the first place, the notion of extremal action is aesthetically pleasing and easily remembered. In the second place, the action is usually more simple an object than an equation of motion, and there is less *a-priori* freedom in its choice. In the third place, any operation that leaves the action invariant is automatically an invariance of the resulting physics. Lastly, many attempts at quantizing a classical theory are formulated in terms of path integrals emplyoing the classical action[23].

We start by considering pure gravity, without matter or (electromagnetic) energy. Since the action must be a scalar, the essentially unique choice is to take

$$\begin{aligned} S_{\text{grav}} &= \int R \sqrt{|g|}\, d^4x \quad , \\ R &= g^{\mu\nu} \left({\Gamma^\alpha}_{\mu\nu,\alpha} - {\Gamma^\alpha}_{\mu\alpha,\nu} + {\Gamma^\alpha}_{\mu\nu}{\Gamma^\beta}_{\alpha\beta} - {\Gamma^\alpha}_{\mu\beta}{\Gamma^\beta}_{\nu\alpha} \right) \quad . \end{aligned} \tag{2.158}$$

As usual, the integral runs over a volume with boundaries on which the variation of the dynamical quantity, in this case the metric $g_{\mu\nu}$, vanishes. Note, however, that the Gauss curvature R contains second derivatives of the metric, which is unusual for an action, and even undesireable. However, we may employ partial integration to write

$$S_{\text{grav}} = \int g^{\mu\nu} \left({\Gamma^\alpha}_{\mu\beta}{\Gamma^\beta}_{\nu\alpha} - {\Gamma^\alpha}_{\mu\nu}{\Gamma^\beta}_{\alpha\beta} \right) \sqrt{|g|}\, d^4x \quad , \tag{2.159}$$

[23] We shall *not* make the attempt to construct a quantum theory of gravity here...

which does contain only first derivatives of the metric. The proof of this result is straightforward but involves quite a bit of relabelling of indices: an alternative, diagrammatic approach is described in the Appendix.

2.5.2 Varying the metric

The variational principle used in deriving the Einstein equations from the einstein action calls for varying the metric, keeping it fixed at the boundaries of the integration volume. That is, we replace

$$g_{\mu\nu} \quad \to \quad g_{\mu\nu} + \delta g_{\mu\nu} \quad , \tag{2.160}$$

where δg is infinitesimal. This implies, of course, variation in the quantities that depend on the metric:

$$\begin{aligned}
\delta g^{\mu\nu} &= -g^{\mu\alpha}\, g^{\nu\beta}\, \delta g_{\alpha\beta} \quad , \\
\delta\sqrt{|g|} &= \frac{1}{2}\sqrt{|g|}\; g^{\alpha\beta}\, \delta g_{\alpha\beta} \quad , \\
\delta\Gamma_{\alpha\mu\nu} &= \frac{1}{2}\left(\delta g_{\alpha\mu,\nu} + \delta g_{\alpha\nu,\mu} + \delta g_{\mu\nu,\alpha}\right) \quad .
\end{aligned} \tag{2.161}$$

Because of Eq.(2.159), we do not have to worry about the variation of *derivatives* of the Christoffel symbols of the second kind. Carefully working out all the variations, and performing partial integration for terms containing $\delta g_{\mu\nu,\alpha}$ and the like, we arrive at the variation of the action:

$$\delta S_{\text{grav}} = -\frac{1}{2}\int d^4x\; \sqrt{|g|}\; G^{\alpha\beta}\; \delta g_{\alpha\beta} \quad , \tag{2.162}$$

from which we immediately have the Einstein equation for empty space:

$$G^{\alpha\beta} = 0 \quad , \tag{2.163}$$

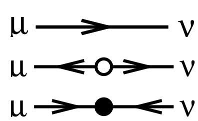
since the (infintesimal) variations are by assumption arbitrary.

Appendix

3.6 Diagrammatic form of tensorial equations

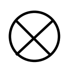
In many results in our theory, summation of many pairs of repeated indices occurs. This can quickly become cumbersome and confusing, while on the other hand the *names* of the indices is of course irrelevant, and only the order in which they are hooked up matters. This argues for the development of a diagrammatic apporach, which we shall now outline. We shall adopt the convention that *lower* indices correspond to lines oriented *outwards*, and *upper* indices to lines oriented *inwards*. Moreover, we need to introduce some symbols. Taking an unfilled dot for the covariant metric and a filled one for the contravariant metric, we have

$\delta^{\mu}{}_{\nu}$ =
$g_{\mu\nu}$ =
$g^{\mu\nu}$ =

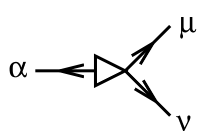


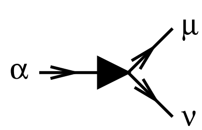
, , . (3.1)

The ubiquitous factor $\sqrt{|g|}$ will be represented by

$\sqrt{|g|}$ =  . (3.2)

Finally, the Christoffel symbols of the first and second kind are denoted by unfilled and filled triangles, respectively:

$\Gamma_{\alpha\mu\nu}$ = 

, $\Gamma^{\alpha}{}_{\mu\nu}$ = 

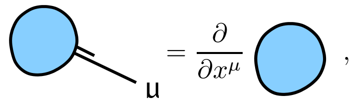
. (3.3)

This notation automatically assigns the correct symmetry to the Christoffel symbols. In any diagram, the orientation of every line is now unambigous, and we will not bother to indicate it explicitly. We trivially have

 . (3.4)

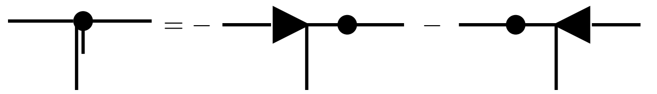
3.6.1 Derivatives

We need an indicator for a partial derivative: we shall define

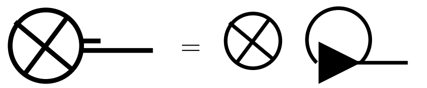


(3.5)

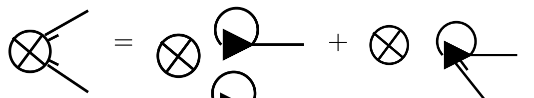
where the blob denotes any diagram: the derivative has to be attached in all possible places in the diagram. It is now trivial to see that


(3.6)

Also we have the result that

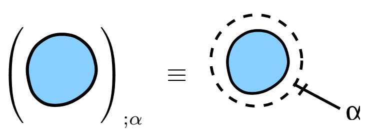

(3.7)

Applying this derivative twice, we see that


(3.8)

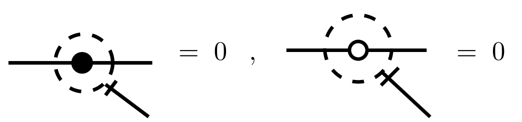
which constitutes the diagrammatic proof that the object ${\Gamma^{\alpha}}_{\mu\alpha,\nu}$ is, in fact, symmetric in its indices μ, ν (*cf.* Eq.(1.206)).

We now turn to the covariant derivative. We shall denote the covariant derivative to α of a diagram by

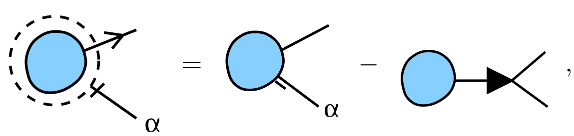


(3.9)

The reason for this notation is that it makes it explicit that covariant differentiations do not commute (as we shall see). From the fact that the covariant derivatives of the metric tensors must vanish,

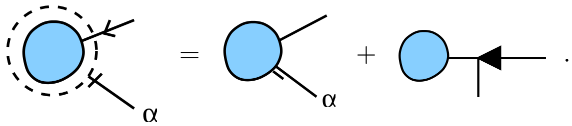

(3.10)

we find immediately the prescription for the covariant derivatives of contravariant and covariant vectors:



(3.11)

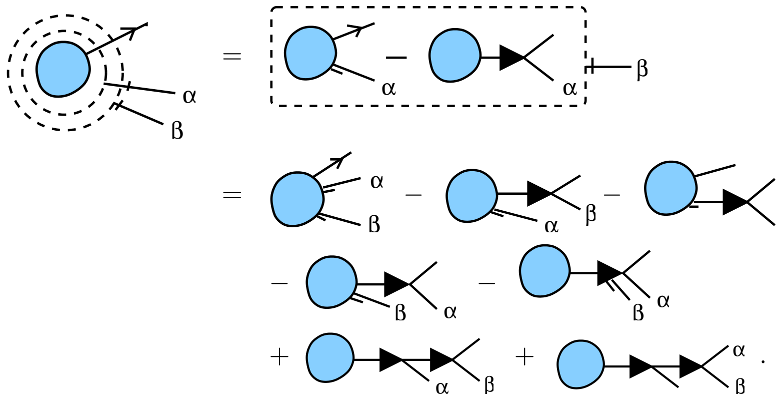
and



(3.12)

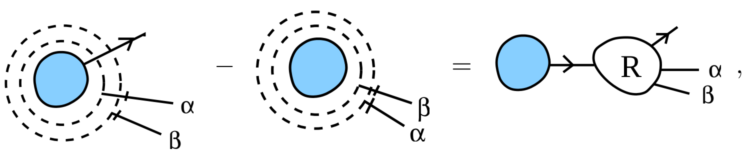
3.6.2 The curvature

We can arrive at the diagrammatic form for the Riemann-Christoffel tensor as follows. Taking a covariant vector, we successively apply covariant differentiation to α and β. This gives (note the explicit non-commutativity!)



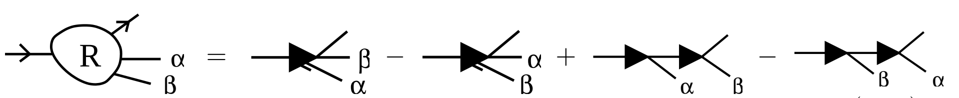
(3.13)

Of these 7 terms, the first, third, and seventh are explicitly symmetric in α, β, and the second and fourth also form a symmetric combination. We therefore find



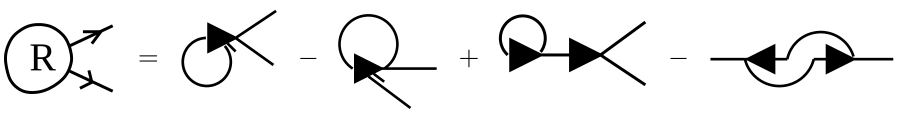
(3.14)

where the Riemann-Christoffel tensor with one contravariant index is defined as



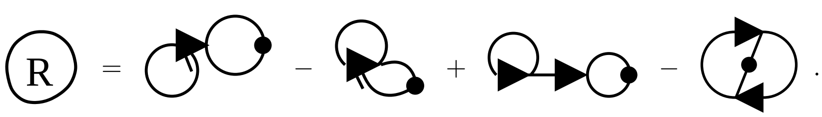
(3.15)

The totally covariant Ricci tensor is given by



(3.16)

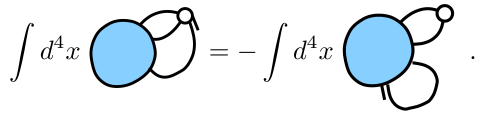
and the Gauss curvature is



(3.17)

3.6.3 Partial integration

The diagrammatic rule for partial integration involving the derivative of a covariant metric tensor can be written as

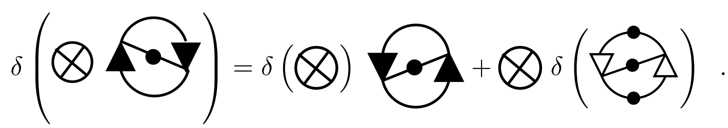


(3.18)

In practice, one of course leaves out the integration indication. It is necessary to remember, however, that the diagrammatic blob also contains the disconnected factor $\sqrt{|g|}$, which is also subject to the differentiation.

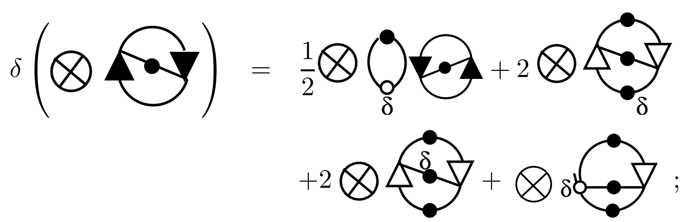
3.6.4 Diagrammatic derivation of the Einstein equation

We now perform the variation of the metric on the action S_{grav}. For the first term in the expression (2.159), we have



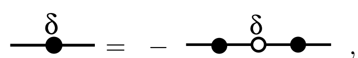
(3.19)

In the last term we have switched to first-kind Christoffel symbols because they contain the covariant metric derivatives. Denoting the infinitesimal variation elements of the metric tensor by a δ symbol, and keeping in mind the definition of the Christoffel symbol as derivatives of the covariant metric, we have



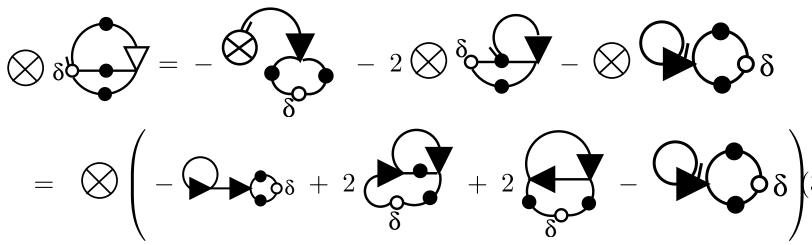
(3.20)

in the last term, we have worked out the first-kind Christoffel symbol: two out of its three contributions cancel. Let us now compute this contribution in some detail. In the first place,



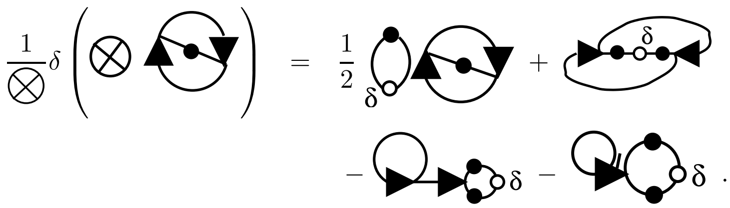
(3.21)

which we may apply to the second and third term in Eq.(3.20). In the second place, we can apply the partial-integration rule, Eq.(3.18), to write



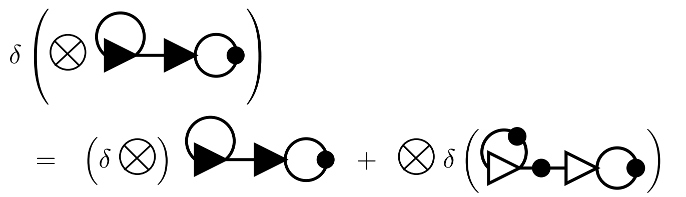
(3.22)

The final result for the first term is, therefore,



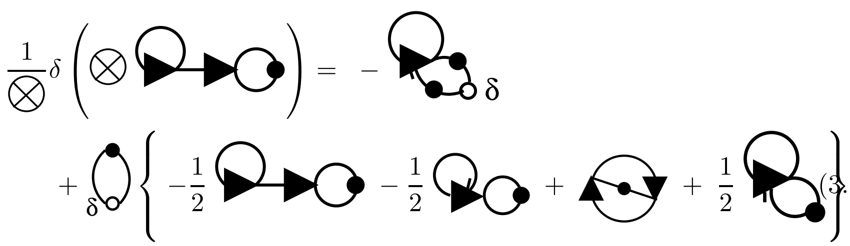
(3.23)

For the second term in Eq.(2.159), we have



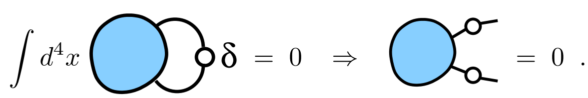
(3.24)

We can now perform the same computations as for the first term. Note that in this case derivatives on the term $\sqrt{|g|}$ also generate some disconnected diagrams. The result, after a considerable number of cancellations, is



(3.25)

The operation of taking out the variation $\delta g_{\alpha\beta}$, and lowering the indices of the tensor it multiplies, is diagrammatically given by



(3.26)

Subtracting the results for the two terms in Eq.(2.159), and eliminating the $\delta g_{\alpha\beta}$ as described above, and finally dropping the overall factor $\otimes$, we find precisely the Einstein equation

$$\frac{1}{2}\, g_{\mu\nu}\, R \;-\; R_{\mu\nu} \;=\; 0 \;, \tag{3.27}$$

with the diagrammatic representation of the Ricci tensor and the Gauss curvature given by Eqs.(3.6.2) and (3.17), respectively.