

ART

①

College 1

1.1

$$a) \quad B = \left(1 + \frac{aT}{N}\right)^N$$

$$\Rightarrow \ln B = N \ln \left(1 + \frac{a}{N} T\right) = N \left(\frac{aT}{N} - \frac{a^2}{2N^2} T^2 + O\left(\frac{1}{N^3}\right) \right)$$

$$\approx aT - \frac{a^2 T^2}{2N} + \dots$$

$$\Rightarrow B = e^{aT} e^{-\frac{a^2}{2N^2} T^2 \dots} = e^{aT} \left(1 - \frac{a^2}{2N} T^2 \dots\right)$$

$$b) \quad B = \left(1 + \frac{aT}{N}\right)^N = \sum \binom{N}{k} \frac{a^k}{N^k} T^k$$

$$\binom{N}{k} \frac{1}{N^k} = \frac{N(N-1)(N-2)\dots(N-k+1)}{k! N^k} =$$

$$= \frac{1}{k!} \left(1 - \frac{1}{N}\right) \left(1 - \frac{2}{N}\right) \dots \approx$$

$$\approx \frac{1}{k!} \left(1 - \frac{1}{N} \frac{(k-1)k}{2}\right) = \frac{1}{k!} - \frac{1}{2N} \frac{1}{(k-2)!}$$

$$\Rightarrow B = \sum_{k \geq 0} \left\{ \frac{(aT)^k}{k!} - \frac{1}{2N} \frac{(aT)^k}{(k-2)!} \right\} \rightarrow$$

leads to the same result.

1.2

$$1. \quad \Lambda(p \rightarrow q)^{\mu}_{\nu} = \delta^{\mu}_{\nu} - \frac{2}{(p+q)^2} (p+q)^{\mu} (p+q)_{\nu} + \frac{2}{p^2} q^{\mu} p_{\nu}$$

$$\begin{aligned} (a) \quad \Lambda^{\mu}_{\nu} p^{\nu} &= p^{\mu} - \frac{2}{(p+q)^2} (p+q)^{\mu} \underbrace{(p^2 + q \cdot p)}_{//} + \frac{2}{p^2} q^{\mu} p^2 \\ &= \frac{1}{2} (2p^2 + 2q \cdot p) = \\ &= \frac{1}{2} (p^2 + 2q \cdot p + q^2) = \frac{1}{2} (p+q)^2 \\ &= p^{\mu} - (p+q)^{\mu} + 2q^{\mu} = q^{\mu} \end{aligned}$$

$$(b) \quad \Lambda(p \rightarrow q)^{\mu\nu} = \delta^{\mu\nu} - \frac{2}{(p+q)^2} \underbrace{(p+q)^{\mu} (p+q)^{\nu}}_{\substack{\text{symmetric} \\ \text{in } \mu \leftrightarrow \nu}} + \frac{2}{p^2} \underbrace{q^{\mu} p^{\nu}}_{\substack{\mu \leftrightarrow \nu \\ p \leftrightarrow q}}$$

$$\Rightarrow \Lambda(p \rightarrow q)^{\mu\nu} = \Lambda(q \rightarrow p)^{\nu\mu}$$

(c) If $p \cdot r = q \cdot r = 0$
 $\Lambda(p \rightarrow q)^{\mu}_{\nu} r^{\nu} = r^{\mu}$
 (Trivial)

We can also see that because

$$\eta_{\mu\nu} \Lambda^{\mu}_{\alpha} \Lambda^{\nu}_{\beta} = \eta_{\alpha\beta}$$

$$\eta_{\mu\nu} \Lambda^{\mu}_{\alpha} \Lambda^{\nu}_{\beta} (\Lambda^{-1})^{\beta}_{\rho} = \eta_{\alpha\beta} (\Lambda^{-1})^{\beta}_{\rho}$$

$$\Lambda_{\nu\alpha} \delta^{\nu}_{\rho} = (\Lambda^{-1})_{\alpha\rho}$$

$$\Lambda_{\rho\alpha} = (\Lambda^{-1})_{\alpha\rho}$$

2. If i.e. $p = e_x$
 $q = e_x + s \cdot e_y$

leaves z unaffected
 \Rightarrow rotation

If $p = (m, \vec{0})$
 $q = (E, \vec{q})$ } doesn't affect any vector
 orthogonal to $q \Rightarrow$
 \Rightarrow boost.

3. $q^\mu = p^\mu + \epsilon t^\mu$, $p \cdot t = 0$, $\epsilon^2 \rightarrow 0$

$$q^2 = p^2 + 2 \epsilon \cdot (p \cdot t) + \epsilon^2 t^2$$

\downarrow \downarrow
 0 0

$$q^2 = p^2$$

$$(p \cdot q) = p^2$$

$$(p+q)^2 = 4p^2$$

$$\Lambda^\mu_\nu = \delta^\mu_\nu - \frac{1}{2p^2} (2p + \epsilon t)^\mu (2p + \epsilon t)_\nu$$

$$+ \frac{2}{p^2} (p + \epsilon t)^\mu p_\nu$$

$$= \delta^\mu_\nu + p^\mu p_\nu + \frac{\epsilon}{p^2} \{ -t^\mu p_\nu - p^\mu t_\nu + 2t^\mu p_\nu \} =$$

$$= \delta^\mu_\nu + \frac{\epsilon}{p^2} \{ t^\mu p_\nu - p^\mu t_\nu \}$$

Take i.e

$$\left. \begin{aligned} p^t &= e_x^t \\ t^t &= e_y^t \end{aligned} \right\} \Rightarrow p^2 = -1$$

$$\text{Then } T^t{}_v = e^t{}_x e_{y v} - e^t{}_y e_{x v}$$

Same for a boost. i.e $p^\mu = (m, \vec{0})$

$$q^t = \left(m + \frac{\vec{p}^2}{2m}, \vec{p} \right) \Rightarrow$$

$$\begin{array}{c} \swarrow \\ \Rightarrow (m, \vec{p}) \end{array}$$

$$\Rightarrow t^t = (0, \vec{p})$$

$$0 \text{ (because } p \sim 0(c) \\ p^2 \sim 0(c^2)$$

1.3

$$\Lambda(t \rightarrow p) \cdot \Lambda(q \rightarrow t) \cdot \Lambda(p \rightarrow q)$$

$$(p \rightarrow q \rightarrow t \rightarrow p)$$

} from this transformation momentum p stays the same.

$$\rightarrow \left[\delta^{\beta}_{\alpha} - \frac{2}{(p+t)^2} (p+t)^{\beta} (p+t)_{\alpha} + \frac{2}{p^2} p^{\beta} t_{\alpha} \right]$$

$$\circ \left[\delta^{\alpha}_{\mu} - \frac{2}{(q+t)^2} (q+t)^{\alpha} (q+t)_{\mu} + \frac{2}{p^2} t^{\alpha} q_{\mu} \right]$$

$$\circ \left[\delta^{\tau}_{\nu} - \frac{2}{(p+q)^2} (p+q)^{\tau} (p+q)_{\nu} + \frac{2}{p^2} q^{\tau} p_{\nu} \right]$$

It is a transformation. We can find 2 momenta that are not affected by that. First is p^{τ} .

For the second find r :

$$r \cdot q = r \cdot t = r \cdot p = 0$$

$$\text{i.e. } r^{\tau} = \epsilon^{\tau\nu\rho\sigma} p_{\nu} q_{\rho} t_{\sigma}$$

\Rightarrow It is a transformation in the (\vec{q}, \vec{t}) plane.

But it is not a boost because p^{τ} is not affected \Rightarrow Rotation

In case one wants to find the angle ^⑥ of the rotation:

Generally

$$\Lambda^T_v = \int^T_v - \frac{2}{(a+b)^2} (a+b)^T (a+b)_v + \frac{2}{a^2} b^T a_v$$

$$\Rightarrow \Lambda^T_t = 4 - 2 + \frac{2(a \cdot b)}{a^2} = \frac{(a+b)^2}{a^2}$$

$$\boxed{\frac{(a+b)^2}{a^2} = 2(1 + \cos \vartheta)}$$

$$b = a \cos \vartheta + \hat{a} \sin \vartheta$$

$$a+b = a(1 + \cos \vartheta) + \hat{a} \sin \vartheta$$

$$(a+b)^2 = \dots = -2(1 + \cos \vartheta)$$

$$\Rightarrow \text{Find } \Lambda^T_t = 2(1 + \cos \vartheta) \quad \left(\begin{array}{l} a^2 = -1 \\ \hat{a}^2 = -1 \\ \hat{a} \cdot a = 0 \end{array} \right)$$

and solve for ϑ .

In our case (FORM result)

$$\Lambda^T_t = \frac{16(1 + p \cdot q + p \cdot t + q \cdot t)}{(p+q)^2 (p+t)^2 (q+t)^2}$$

1.4

$$\Lambda^{\mu}_{\nu} = \delta^{\mu}_{\nu} - \frac{2}{\Delta} S^{\mu} S_{\nu} + \frac{2}{p^2} q^{\mu} p_{\nu}, \quad S = p + q$$

$$\Delta = S^2$$

$$\det(\Lambda) = -\frac{1}{4!} \epsilon_{\mu_1 \mu_2 \mu_3 \mu_4} \epsilon^{\nu_1 \nu_2 \nu_3 \nu_4} \Lambda^{\mu_1}_{\nu_1} \Lambda^{\mu_2}_{\nu_2} \Lambda^{\mu_3}_{\nu_3} \Lambda^{\mu_4}_{\nu_4}$$

$$= -\frac{1}{4!} \left\{ \frac{\epsilon_{\mu\nu\rho\sigma} \epsilon^{\mu\nu\rho\sigma}}{-24} - 4 \cdot \frac{2}{\Delta} \underbrace{\epsilon_{\mu\nu\rho\sigma} \epsilon^{\mu\nu\rho\alpha}}_{-6 \delta^{\alpha}_{\sigma}} S^{\beta} S_{\alpha} + \right.$$

$$\left. + 4 \cdot \frac{2}{p^2} \epsilon_{\mu\nu\rho\sigma} \epsilon^{\mu\nu\rho\alpha} q^{\beta} p_{\alpha} - \frac{4 \cdot 12}{p^2 \Delta} \underbrace{\epsilon_{\mu\nu\rho\sigma} \epsilon^{\mu\nu\rho\alpha}}_{\substack{\cdot S^{\beta} \\ \cdot S^{\sigma} \\ \cdot q^{\beta} p_{\alpha}}} \right\}$$

$$\downarrow$$

$$-2(\delta^{\sigma}_{\rho} \delta^{\alpha}_{\sigma} - \delta^{\sigma}_{\sigma} \delta^{\alpha}_{\rho})$$

$$= -\frac{1}{4!} \left\{ -24 + \frac{48}{\Delta} \cdot \Delta - 48 \frac{(p \cdot q)}{p^2} + \right.$$

$$\left. + \frac{96}{p^2 \Delta} \left(\Delta (p \cdot q) - \frac{1}{4} \Delta^2 \right) \right\} =$$

$$= -\frac{1}{24} \left(-24 + 48 - 48 \frac{(p \cdot q)}{p^2} + 96 \frac{(p \cdot q)}{p^2} - 24 \frac{\Delta}{p^2} \right)$$

$$= -\frac{1}{24} \left(-24 + 48 - 48 \frac{(p \cdot q)}{p^2} + 96 \frac{(p \cdot q)}{p^2} - 24 \frac{(2p^2 + 2p \cdot q)}{p^2} \right) =$$

$$= 1$$

$$\begin{aligned} p \cdot S &= p^2 + p \cdot q = \\ &= \frac{S^2}{2} = \frac{\Delta}{2} \end{aligned}$$