Solutions to the problem sets of the course "Beyond the Standard Model"

Sol. 1 $SU(N)$ in fundamental representation: $U = e^{i\vartheta^a T^a}$ ($\vartheta^a \in \mathbb{R}$) $N\times N$ matrix, with $UU^\dagger = 1$ and $\det U = 1$.

(i) $UU^\dagger = e^{i\vartheta^a T^a} e^{-i\vartheta^b T^{b\dagger}} = 1 \quad \forall\, \vartheta^1,\dots,\vartheta^n \Rightarrow$ for each value of a we have $(T^a)^\dagger = T^a$, i.e. the generators are hermitian.

$\det U = e^{Tr(\log U)}$ according to basic linear algebra $\Rightarrow$ in this case

$\det U = e^{Tr(i\vartheta^a T^a)} = 1 \quad \forall\, \vartheta^1,\dots,\vartheta^n \Rightarrow$ for each value of a we have $Tr(T^a) = 0$, i.e. the generators are traceless.

(ii) Take $N\times N$ matrices: $2N^2$ d.o.f. $\xrightarrow[(i)]{\text{herm.}}$ N^2 d.o.f. $\xrightarrow[(i)]{Tr=0}$ (N^2-1) d.o.f.
Hence, based on (i) we know that $SU(N)$ has N^2-1 independent generators.

(iii) Fundamental commutation relation for $SU(N)$: $[T^a, T^b] = i f^{abc} T^c$.
Consequence: $[T^a, T^b]^\dagger \overset{T^a = T^{a\dagger}}{=} -[T^a, T^b] = -i f^{abc} T^c$

$\Vert\ (i f^{abc} T^c)^\dagger = -i (f^{abc})^* (T^c)^\dagger \overset{T^c = T^{c\dagger}}{=} -i (f^{abc})^* T^c$.

Hence, the $SU(N)$ structure constants f^{abc} are real.

(iv) Rotation subgroup $SO(N) \subset SU(N)$, again in the fundamental representation:
$O = e^{i\vartheta^a T^a}$ $N\times N$ matrix, with $OO^\dagger = 1$, $\det O = 1$ and $OO^T = 1$.
Because of (i), $(T^a)^\dagger = T^a$ and $Tr(T^a) = 0$. However, this time we have the additional condition $(T^a)^T = -T^a$ as a result of $OO^T = 1$
$\Rightarrow T^a = (T^a)^\dagger = (T^{aT})^* = -(T^a)^*$, i.e. T^a has purely imaginary components.

Again take $N\times N$ matrices: $2N^2$ d.o.f. $\xrightarrow{\text{herm.}}$ N^2 d.o.f. $\xrightarrow{T^a \text{ purely imaginary}}$ $\frac{1}{2}(N^2 - N)$ d.o.f., i.e. $SO(N)$ has $\frac{1}{2}N(N-1)$ independent generators.
(diagonal elements vanish in this case)

This number coincides with the number of planes of rotation in N dim.

Sol. 2 $SU(N)$ gauge theory for $N > 1$, as given on p. 8-11 of the lecture notes.

(i) Let's try a QED-style charge scaling: $\vartheta^a(x) \to Q\vartheta^a(x)$, $g \to Qg$.
Demanding that W_μ^a transforms in the same way as before charge scaling, i.e. $W_\mu^a \to W_\mu^a - \frac{1}{g}\partial_\mu \vartheta^a - f^{abc}\vartheta^b W_\mu^c$, forces us to also scale $f^{abc} \to \frac{1}{Q} f^{abc}$. However, in view of $[T^a, T^b] = i f^{abc} T^c$, this corresponds to performing the scaling $T^a \to \frac{1}{Q} T^a$. The net effect of all these scalings is that the gauge and Yang-Mills interactions remain unchanged, as gT^a and gf^{abc} are invariant under the scaling. So, the conclusion is that the interaction strength of a non-abelian gauge theory is fixed!

(ii) Next we consider the subset of global $SU(N)$ transformations and try to find the corresponding N^2-1 Noether currents (belonging to T^a).
Infinitesimal $SU(N)$ transformations of the various fields:

$$\psi_j(x) \to \psi_j(x) + \vartheta^a \underbrace{[i(T^a)_{jk}\psi_k]}_{\equiv \Delta^a \psi_j}, \quad \bar{\psi}_j(x) \to \bar{\psi}_j(x) + \vartheta^a \underbrace{[-i\bar{\psi}_k(T^a)_{kj}]}_{\equiv \Delta^a \bar{\psi}_j},$$

$$W^b_\nu(x) \to W^b_\nu(x) \underbrace{\cancel{-\frac{1}{g}\partial_\nu \vartheta^b}}_{\text{global transf.}} + \vartheta^a \underbrace{[-f^{bac}W^c_\nu(x)]}_{\equiv \Delta^a W^b_\nu}.$$

The corresponding Noether currents are (for each independent ϑ^a):

$$J^{a,\mu}(x) \underset{\mathcal{L}\text{ inv.}}{\overset{G^{a,\mu}=0}{=\!=}} \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_j)}\Delta^a\psi_j + \Delta^a\bar{\psi}_j\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi}_j)} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu W^b_\nu)}\Delta^a W^b_\nu$$

$$= -\bar{\psi}_j\gamma^\mu(T^a)_{jk}\psi_k - W^{b,\mu\nu}f^{abc}W^c_\nu = \underbrace{-\bar{\psi}\gamma^\mu T^a\psi}_{-j^{a,\mu}(x)} - \underbrace{W^{b,\mu\nu}f^{abc}W^c_\nu}_{\text{extra gauge-boson terms}}.$$

The combination $g W^a_\mu(x) J^{a,\mu}(x)$ then reads:

$$\underbrace{-g j^{a,\mu}(x)W^a_\mu(x)}_{\text{gauge int.}} \underbrace{- g f^{abc}(\partial^\mu W^{b,\nu}(x) - \partial^\nu W^{b,\mu}(x))W^a_\mu(x)W^c_\nu(x)}_{2* \text{ triple gauge-boson int.}} + \underbrace{g^2 f^{abc}f^{bde}W^a_\mu(x)W^c_\nu(x)W^{d,\mu}(x)W^{e,\nu}(x)}_{4* \text{ quartic gauge-boson int.}}.$$

Sol. 3 Property of Pauli spin matrices: $(\tau^1)^* = \tau^1$, $(\tau^2)^* = -\tau^2$, $(\tau^3)^* = \tau^3$

$$\Rightarrow \tau^2(\tau^j)^* = -\tau^j\tau^2 \quad (j=1,2,3) \quad (1).$$

Consider the conjugate doublet $\Phi^c(x) = i\tau^2\Phi^*(x)$, with $\Phi(x)$ a doublet under $SU(2)$. Then the conjugate doublet transforms as

$$\Phi^c(x) \to \Phi'^c(x) = i\tau^2(\Phi'(x))^* = i\tau^2[U(x)\Phi(x)]^* = i\tau^2 e^{-\frac{i}{2}\vec{\tau}^*\cdot\vec{\vartheta}(x)}\Phi^*(x)$$

$\in \mathbb{R}^3$

$$\overset{(1)}{=} e^{\frac{i}{2}\vec{\tau}\cdot\vec{\vartheta}(x)}\, i\tau^2\Phi^*(x) = U(x)\Phi^c(x).$$

Hence, $\Phi^c(x)$ transforms just like $\Phi(x)$ under $SU(2)$!

this will be used in
- the Higgs sector of the Standard Model
- supersymmetry

!

hint: $2\cos(\varphi)\sqrt{|H_1|^2|H_2|^2-|H_1^\dagger H_2|^2}$

Sol. 4 Consider $V_H = \mu_1^2|H_1|^2+\mu_2^2|H_2|^2-\mu_3^2(H_1\times H_2+h.c.)+\frac{m_Z^2}{2v^2}(|H_1|^2-|H_2|^2)^2+\frac{2m_W^2}{v^2}|H_1^\dagger H_2|^2$.

(i) V_H is guaranteed to be bounded from below by the quartic scalar interactions unless these terms vanish, i.e. if $|H_1|=|H_2|$ and $H_1^\dagger H_2=0$ (orthogonal H_1 and H_2). In that case we get for the quadratic scalar terms the expression
$V_H = \mu_1^2|H_1|^2+\mu_2^2|H_2|^2-2\mu_3^2\cos(\varphi)\sqrt{|H_1|^2|H_2|^2-|H_1^\dagger H_2|^2} = (\mu_1^2+\mu_2^2-2\mu_3^2\cos(\varphi))|H_1|^2$,
which is bounded from below if $\mu_1^2+\mu_2^2-2\mu_3^2\cos(\varphi)\geq 0$
$\Rightarrow$ V_H is always bounded from below if $\mu_1^2+\mu_2^2\geq 2\mu_3^2\max(\cos(\varphi)) = 2\mu_3^2$.

(ii) Take fixed values for $|H_1|$ and $|H_2|$, and subsequently minimize V_H given those fixed values. Since $|H_{1,2}|$ are fixed, the function to minimize during the first step is $\frac{2m_W^2}{v^2}|H_1^\dagger H_2|^2-2\mu_3^2\cos(\varphi)\sqrt{|H_1|^2|H_2|^2-|H_1^\dagger H_2|^2}$, resulting in $-2\mu_3^2|H_1||H_2|$ for $H_1^\dagger H_2=0$ and $\cos(\varphi)=1$ $\leftarrow$ ($H_1\times H_2\in\mathbb{R}$ and positive)

(iii) Finally we have to minimize the following function of $|H_{1,2}|$:

$$f(|H_1|,|H_2|) = \mu_1^2|H_1|^2+\mu_2^2|H_2|^2-2\mu_3^2|H_1||H_2|+\frac{m_Z^2}{2v^2}(|H_1|^2-|H_2|^2)^2.$$

A minimum away from $|H_1|=|H_2|=0$ can be obtained if the quadratic terms $\mu_1^2|H_1|^2+\mu_2^2|H_2|^2-2\mu_3^2|H_1||H_2| = (|H_1|\sqrt{\mu_1^2}-|H_2|\sqrt{\mu_2^2})^2+2(\sqrt{\mu_1^2\mu_2^2}-\mu_3^2)|H_1||H_2|$ can become negative, i.e. $\mu_3^4>\mu_1^2\mu_2^2$.

*) if $|H_1|=|H_2|$, then $f(|H_1|,|H_1|) = |H_1|^2\overbrace{(\mu_1^2+\mu_2^2-2\mu_3^2)}^{\geq 0}$ has its minimum at $|H_1|=|H_2|=0$ $\Rightarrow$ no electroweak symmetry breaking (EWSB) !

*) if $|H_1|\neq|H_2|$, then the minimalization conditions $\frac{\partial f}{\partial|H_1|}=\frac{\partial f}{\partial|H_2|}=0$ read

$$\boxed{\begin{aligned}&2\mu_1^2|H_1^{min}|+\frac{2m_Z^2}{v^2}|H_1^{min}|(|H_1^{min}|^2-|H_2^{min}|^2)-2\mu_3^2|H_2^{min}|=0\\&2\mu_2^2|H_2^{min}|-\frac{2m_Z^2}{v^2}|H_2^{min}|(|H_1^{min}|^2-|H_2^{min}|^2)-2\mu_3^2|H_1^{min}|=0\end{aligned}}$$

Sol. 5 RGE: $\frac{\partial}{\partial \log(Q)} \alpha_i^{-1}(Q) = -\frac{b_i}{2\pi} \quad (i=1,2,3).$

electroweak mixing angle

Experimentally measured quanties: $\alpha_s^{-1}(Q) = \alpha_3^{-1}(Q)$, $\sin^2\theta_W(Q) = \frac{3\alpha_1(Q)}{3\alpha_1(Q)+5\alpha_2(Q)}$

and $\alpha^{-1}(Q) = \frac{\alpha_2^{-1}(Q)}{\sin^2\theta_W(Q)} = \frac{\frac{5}{3}\alpha_1^{-1}(Q)}{1-\sin^2\theta_W(Q)}$ ← electromagnetic coupling

(i) unification @ GUT-scale M_X $\Rightarrow$ $\alpha_1^{-1}(M_X) = \alpha_2^{-1}(M_X) = \alpha_3^{-1}(M_X) = \alpha_G^{-1}$,

$$\sin^2\theta_W(M_X) = \frac{3\alpha_G}{3\alpha_G+5\alpha_G} = \frac{3}{8}$$ ← GUT prediction

(ii)
$$\left.\begin{aligned} I_\alpha &= (b_2-b_3)\alpha_1^{-1}(Q) + (b_3-b_1)\alpha_2^{-1}(Q) + (b_1-b_2)\alpha_3^{-1}(Q) = \, ? \\ Q = M_X &: (b_2-b_3+b_3-b_1+b_1-b_2)\alpha_G^{-1} = 0 \\ \frac{\partial}{\partial \log(Q)}[\,\cdots] &= -\frac{1}{2\pi}\left[(b_2-b_3)b_1 + (b_3-b_1)b_2 + (b_1-b_2)b_3\right] = 0 \end{aligned}\right\} \Rightarrow \; ? = 0$$

I_α is independent of Q: this is called a <u>renormalization group invariant (RGI)</u>!

(iii) Rewriting I_α:
$$\begin{aligned} 0 = I_\alpha &= (b_2-b_3)\frac{3}{5}\alpha^{-1}(Q)(1-\sin^2\theta_W(Q)) + (b_3-b_1)\alpha^{-1}(Q)\sin^2\theta_W(Q) \\ &\quad + (b_1-b_2)\alpha_3^{-1}(Q) \\ &= \frac{3}{5}(b_2-b_3)\alpha^{-1}(Q) + (b_1-b_2)\alpha_3^{-1}(Q) \\ &\quad - \sin^2\theta_W(Q)\left[\frac{3}{5}(b_2-b_3)\alpha^{-1}(Q) + (b_1-b_3)\alpha^{-1}(Q)\right] \end{aligned}$$

$$\Rightarrow \sin^2\theta_W(Q) = \frac{1}{5b_1+3b_2-8b_3}\left[3(b_2-b_3) + 5(b_1-b_2)\,\alpha_3^{-1}(Q)/\alpha^{-1}(Q)\right].$$

(iv)
$$\left.\begin{aligned} Q = m_Z: \; \alpha^{-1}(m_Z) &= 127.916 \pm 0.015 \\ \alpha_3^{-1}(m_Z) = \alpha_s^{-1}(m_Z) &= 8.45 \pm 0.05 \end{aligned}\right\}$$ experimental values

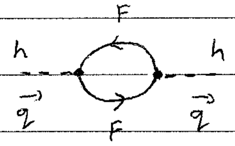
$\overset{(iii)}{\Longrightarrow}$ GUT predicts $\sin^2\theta_W^{GUT}(m_Z) = \begin{cases} 0.2075 \pm 0.0002 & \text{in the SM} \\ 0.2308 \pm 0.0002 & \text{in the MSSM} \end{cases}$

$\Rightarrow$ <u>only the MSSM GUT is compatible with the experimental value $\sin^2\theta_W^{exp}(m_Z) = 0.23131 \pm 0.00007$</u> !

used: $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 41/10 \\ -19/6 \\ -7 \end{pmatrix}$ in the SM, $\begin{pmatrix} 33/5 \\ 1 \\ -3 \end{pmatrix}$ in the MSSM.

Sol. 6 The exercise basically follows the steps on p. 50 and 52 of the lecture notes.

fermion loop

① 

$$= \ominus\, (-i g_F)^2\, (i)^2 \int \frac{d^4\ell}{(2\pi)^4} \frac{Tr\left([\not{\ell} + \not{q} + m_F][\not{\ell} + m_F]\right)}{[\ell^2 - m_F^2][(\ell+q)^2 - m_F^2]}$$

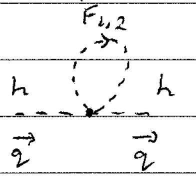
$$= \frac{-2g_F^2}{(2\pi)^4} \left\{ \int \frac{d^4\ell}{\ell^2 - m_F^2} + \int \frac{d^4\ell}{(\ell+q)^2 - m_F^2} + \int \frac{d^4\ell\,(4m_F^2 - q^2)}{[\ell^2 - m_F^2][(\ell+q)^2 - m_F^2]} \right\},$$

QD the same QD LD

using that $Tr([\not{\ell} + \not{q} + m_F][\not{\ell} + m_F]) = 4m_F^2 + 4\ell\cdot(\ell+q)$

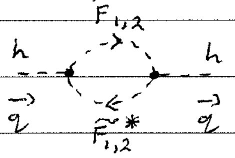
$$= 2[(\ell+q)^2 - m_F^2] + 2[\ell^2 - m_F^2] + 8m_F^2 - 2q^2.$$

boson loop

② 

$$= \oplus\, \underbrace{(-2ih_F)\, i}_{+2h_F} \int \frac{d^4\ell}{(2\pi)^4} \left\{ \frac{1}{\ell^2 - m_{\tilde{F}_1}^2} + \frac{1}{\ell^2 - m_{\tilde{F}_2}^2} \right\}.$$

QD QD

③ 

$$= + \underbrace{(-2ih_F v)^2 (i)^2}_{+4h_F^2 v^2} \int \frac{d^4\ell}{(2\pi)^4} \left\{ \frac{1}{[\ell - m_{\tilde{F}_1}^2][(\ell+q)^2 - m_{\tilde{F}_1}^2]} + \frac{1}{[\ell^2 - m_{\tilde{F}_2}^2][(\ell+q)^2 - m_{\tilde{F}_2}^2]} \right\}$$

LD LD

QD = quadratically divergent

LD = logarithmically divergent

One can immediately read off that the quadratically divergent terms in ① and ② cancel each other if $h_F = g_F^2$ according to

$$\int d^4\ell \left\{ \frac{1}{\ell^2 - m_{\tilde{F}_i}^2} - \frac{1}{\ell^2 - m_F^2} \right\} = (m_{\tilde{F}_i}^2 - m_F^2) \int d^4\ell \frac{1}{[\ell - m_{\tilde{F}_i}^2][\ell^2 - m_F^2]} \leftarrow \text{LD}$$

If also $m_{\tilde{F}_1}^2 = m_{\tilde{F}_2}^2 = m_F^2$, then all terms cancel except the q^2-term in ① (using that $h_F^2 v^2 = g_F^4 v^2 = g_F^2 m_F^2$). This remaining term is proportional to m_h^2 for $q^2 \to m_h^2$, implying that <u>the Higgs mass is protected in this theory</u>!

Sol. 7 Let's project the RHS of $\{\hat{Q}_\gamma, \hat{\bar{Q}}_\beta\} = \{\hat{Q}_\gamma, \hat{Q}^\dagger_\delta\}(\gamma^0)_{\delta\beta} = 2(\gamma^\mu)_{\gamma\beta}\hat{P}_\mu$ onto $\hat{P}_0 = \hat{H}$.

(i) $\frac{1}{8}(\gamma^0)_{\beta\gamma} * 2(\gamma^\mu)_{\gamma\beta}\hat{P}_\mu = \frac{1}{4}\mathrm{Tr}(\gamma^0\gamma^\mu)\hat{P}_\mu = g^{0\mu}\hat{P}_\mu = \hat{P}^0 = \hat{P}_0$

$\parallel$

LHS $\Rightarrow$ $\frac{1}{8}(\gamma^0)_{\delta\beta}(\gamma^0)_{\beta\gamma}\{\hat{Q}_\gamma, \hat{Q}^\dagger_\delta\} = \frac{1}{8}\underbrace{(\gamma^0\gamma^0)}_{I}{}_{\delta\gamma}\{\hat{Q}_\gamma, \hat{Q}^\dagger_\delta\} = \frac{1}{8}\{\hat{Q}_\delta, \hat{Q}^\dagger_\delta\}$.

(ii) $\forall_{|\psi\rangle}\ \langle\psi|\hat{H}|\psi\rangle = \frac{1}{8}\Big(\underbrace{\langle\psi|\hat{Q}_\delta\hat{Q}^\dagger_\delta|\psi\rangle}_{\langle\hat{Q}^\dagger_\delta\psi|\hat{Q}^\dagger_\delta\psi\rangle \geq 0} + \underbrace{\langle\psi|\hat{Q}^\dagger_\delta\hat{Q}_\delta|\psi\rangle}_{\langle\hat{Q}_\delta\psi|\hat{Q}_\delta\psi\rangle \geq 0}\Big) \geq 0.$

no states with negative norm

(iii) Take $|\psi\rangle$ = ground state $|\Omega\rangle$, such that $\hat{H}|\Omega\rangle = E_0|\Omega\rangle \Rightarrow \langle\Omega|\hat{H}|\Omega\rangle = E_0$.

If $E_0 = 0$, then according to (ii) $\forall_\delta\ \langle\hat{Q}_\delta\Omega|\hat{Q}_\delta\Omega\rangle = \langle\hat{Q}^\dagger_\delta\Omega|\hat{Q}^\dagger_\delta\Omega\rangle = 0$

$\Rightarrow \forall_\delta\ \hat{Q}_\delta|\Omega\rangle = \hat{Q}^\dagger_\delta|\Omega\rangle = 0$. Vice versa: $\forall_\delta\ \hat{Q}_\delta|\Omega\rangle = \hat{Q}^\dagger_\delta|\Omega\rangle = 0 \xrightarrow{(ii)} E_0 = 0$.

$\Rightarrow$ <u>an unbroken SUSY ground state has $E_0 = 0$, a broken SUSY ground state necessarily has $E_0 > 0$</u> !