Quantum Field Theory 2: exercises for week 1

Exercise 1: generators for $SU(N)$ and $SO(N)$

Consider the Lie group SU(N) with elements $g = e^{i\alpha^a T^a}$, where α^a are arbitrary real scalar parameters and T^a are the independent generators of the associated Lie algebra with fundamental commutation relations $\left[T^a, T^b\right] = i f^{abc} T^c$. In the defining (fundamental) representation the group elements are given by $N \times N$ matrices U, with $UU^\dagger = 1$ and $\det U = 1$.

(a) Show that the $N \times N$ matrices T^a are hermitian and traceless. You might need that for a matrix M it holds that $\det M = \exp\left(\mathrm{Tr}\left(\ln M\right)\right)$ and don't forget that the scalar parameters α^a can take on any real value.

(b) Count the independent degrees of freedom of the $N \times N$ matrices T^a to argue that SU(N) has N^2-1 independent generators, which implies that a runs from 1 to N^2-1.

(c) Prove that the structure constants f^{abc} are real.

Consider the subgroup SO(N) $\subset$ SU(N), with group elements that are given by $N \times N$ matrices O in the fundamental representation. These matrices additionally satisfy $OO^T = 1$.

(d) Deduce that T^a is purely imaginary and that SO(N) has $\frac{1}{2}N(N-1)$ independent generators.

Each generator is linked to a gauge field in the Lagrangian of a local gauge theory. Knowing the number of generators for a group therefore immediately tells you the number of gauge fields (and therefore the number of new particles!) that you get when implementing that group in a local gauge theory.

Exercise 2: some handy $SU(2)$ properties

Consider $U(x) = \exp\left(\frac{i}{2}\,\vec{\sigma}\cdot\vec{e}_n\,\theta(x)\right) \in SU(2)$, with $\theta(x) \in \mathbb{R}$, $\vec{e}_n$ a unit vector in $\mathbb{R}^3$, and

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

the usual 2×2 Pauli spin matrices.

(a) Use the anticommutation (normalization) identity $\{\sigma^j, \sigma^k\} = 2\delta^{jk} I_2$, with I_2 the 2×2 identity matrix, to show that $(\vec{\sigma}\cdot\vec{e}_n)^2 = I_2$ and subsequently that

$$U(x) = I_2 \cos\bigl(\theta(x)/2\bigr) + i\vec{\sigma}\cdot\vec{e}_n \sin\bigl(\theta(x)/2\bigr).$$

(b) Introduce a doublet field $\Phi(x)$ that transforms under $SU(2)$ according to

$$\Phi(x) \to \Phi'(x) = U(x)\Phi(x).$$

Prove that the conjugate doublet $\tilde{\Phi}(x) \equiv i\sigma^2\Phi^*(x)$, with * denoting complex conjugation and σ^2 the second Pauli spin matrix, has the same $SU(2)$ transformation property as $\Phi(x)$.

Hint: first figure out what happens if you bring σ^2 to the other side of $(\sigma^j)^*$ for all three values of j.

We will make explicit use of this observation during the discussion of the Higgs mechanism in the Standard Model.