Quantum Field Theory 2: exercises for week 2

Exercise 3: Absence of scaled couplings in the non-Abelian case

Consider the local $\mathrm{SU}(N)$ gauge theory for $N \geq 2$, as described in the lecture notes.

(a) Why is the QED-like charge scaling $\alpha^a(x) \to Q\alpha^a(x)$, $\mathrm{g} \to Q\mathrm{g}$ not an effective means of changing the interaction strength?

Hint: You have to demand that after charge scaling W^a_μ transforms in the same way as before charge scaling. Only then one and the same gauge field could mediate the $\mathrm{SU}(N)$ interaction between differently charged particles. At some point in your proof you will have to take into account the commutator algebra of the generators.

(b) Consider the subset of global $\mathrm{SU}(N)$ gauge transformations, for which the Lagrangian density $\mathcal{L}_{\mathrm{SU}(N)}$ is invariant.

- Derive the conserved Noether currents $J^{a,\mu}(x)$ for each of the independent global transformations labeled by $\alpha^a \in \mathbb{R}$ and link the combination $\mathrm{g}W^a_\mu(x)J^{a,\mu}(x)$ to the three types of interactions contained in the $\mathrm{SU}(N)$ gauge theory.
- Indicate the fundamental difference with the QED case.

Not being able to change the coupling strength is a strong statement! It immediately tells you that the coupling strength between the gauge bosons and other particles is fixed, given by the theory.

Exercise 4: Feynman rules for the $SU(N)$ interactions

Consider again the local $\mathrm{SU}(N)$ gauge theory for $N \geq 2$. Derive the momentum-space Feynman rules for the three $\mathrm{SU}(N)$ interaction vertices, as given on pages 11 and 12 of the lecture notes. Please take all gauge bosons to be incoming in these Feynman rules.

Hints: in the case of the gauge-boson interactions you should keep a close eye on all possible permutations of the various indices. In the Lagrangian density all indices are repeated and therefore summed over. A summed $\mathrm{SU}(N)$ index a in the Lagrangian density is actually a dummy variable and need not be identical to the open index a of the specific gauge boson in the Feynman rule. To get a feeling for how the derivation works for the triple gauge-boson interactions, you are advised to first consider the fully connected $\mathcal{O}(\mathrm{g})$ contributions to the Green's function

$$\langle 0|\, T\Big(\hat{W}^a_{\mu_I}(x_1)\, \hat{W}^b_{\nu_I}(x_2)\, \hat{W}^c_{\rho_I}(x_3)\, e^{i\int d^4z\, \hat{\mathcal{L}}_{\mathrm{int}_I}(z)}\Big)|0\rangle$$

in terms of contractions. This Green's function is directly linked to the interaction vertex, as can be seen by drawing the associated Feynman diagram. Subsequently, for figuring out the consequences of a field derivative in momentum space, you may use that the gauge-boson quantum field in the interaction picture is given by

$$\hat{W}^e_{\alpha_I}(x) \;=\; \int \frac{d\vec{p}}{(2\pi)^3}\, \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_{r=1}^{2} \Big(\hat{a}^{e,r}_{\vec{p}}\, \epsilon^{e,r}_\alpha(p)\, e^{-ip\cdot x} + \hat{a}^{e,r\dagger}_{\vec{p}}\, \epsilon^{e,r*}_\alpha(p)\, e^{ip\cdot x}\Big)\Big|_{p_0\,=\,E_{\vec{p}}\,=\,|\vec{p}\,|}$$

in terms of the creation and annihilation operators $a^{e,r\dagger}_{\vec{p}}$ and $a^{e,r}_{\vec{p}}$. Here e is the adjoint $\mathrm{SU}(N)$ index of the gauge field, whereas $\vec{p}$ and r represent the momentum and spin quantum numbers of the associated particle modes.

This exercise may feel a bit awkward, but it prepares you for what you will typically encounter in the literature, where summed indices in the Lagrangian density and open indices for the Feynman rules will in general not be taken all different.