Quantum Field Theory 2: exercises for week 4 and 5

Exercise 6: a few covariant derivatives in the Standard Model

By now you have seen that the Standard Model is a local gauge theory based on the gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$, that the generators of each of these three different subgroups are defined to commute with the generators of another subgroup, and that various matter fermions (i.e. leptons and quarks) feature in the theory. In order to combine everything you have learned so far you are requested to write down the covariant derivative that is needed for the left-handed up- and down-quark fields, and the one that is needed for the right-handed electron field. For indicating the different gauge fields and generators you can use the notation adopted in the lecture notes. Please also indicate the relevant hypercharge quantum numbers and mention on which multiplets the various generators will act.

Exercise 7: The relativistic superconductor (A Higgs toy model)

In the first lecture we considered the Lagrangian for the free Klein-Gordon theory

$$\mathcal{L}(x) = [\partial_\mu \phi(x)]^* [\partial^\mu \phi(x)] - m^2 \phi^*(x)\phi(x), \tag{1}$$

where $\phi(x)$ is a complex scalar field with mass m. We have proven that this Lagrangian is invariant under the global gauge transformation $\phi(x) \to \phi'(x) = \mathrm{e}^{i\alpha}\phi(x)$ and derived the corresponding conserved current.

(a) Prove that you can make the Lagrangian in equation (1) invariant under the *local* gauge transformation

$$\phi(x) \to \phi'(x) = \mathrm{e}^{i\alpha(x)}\phi(x) \qquad \text{with} \qquad \alpha(x) \in \mathbb{R} \tag{2}$$

by replacing the derivatives ∂_μ in $\mathcal{L}$ by covariant derivatives $D_\mu \equiv \partial_\mu + igA_\mu(x)$. Derive the required transformation characteristic of $A_\mu(x)$.

(b) Upon replacing the ordinary derivatives by the covariant ones, new interaction terms appear in the Lagrangian. Identify these interaction terms and draw the corresponding Feynman diagrams, using dashed lines for scalar particles and wiggly ones for gauge bosons.

(c) There is no mass term for A^μ in the Lagrangian you considered in part (b). Explain why you are not allowed to just add it manually.

Let us change our Lagrangian by kicking out the scalar mass term and by adding a potential for the ϕ field. We include the scalar mass term by means of this potential and simplify the notation by letting $\phi(x) \to \phi$.

$$\mathcal{L}(x) = [\partial_\mu \phi]^* [\partial^\mu \phi] - V(\phi) \qquad \text{where} \qquad V(\phi) = c_2 \phi^* \phi + c_4 (\phi^* \phi)^2 \tag{3}$$

The coefficients c_2 and c_4 are real and can have either sign. The first potential term resembles an ordinary scalar mass term, but its sign may be opposite when $c_2 < 0$. For a certain choice of signs for c_2 and c_4 the potential has a local maximum for $|\phi| = 0$ and global minima for a non-zero value of $|\phi| = \sqrt{\phi^*\phi}$.

(d) Indicate for which signs of c_2 and c_4 this happens, and make a drawing of the corresponding potential as a function of $|\phi|$.

(e) Show that the location of the minima is given by $|\phi| = \frac{v}{\sqrt{2}}$, where we take $v^2 = -\frac{c_2}{c_4}$.

Because we determined the minima for $|\phi|$, there is a choice in the location of the minima for ϕ. We choose this to be $\frac{v}{\sqrt{2}}$ and reparametrize ϕ as

$$\phi(x) = \frac{1}{\sqrt{2}} \left(v + h(x)\right) \mathrm{e}^{i\omega(x)/v} \qquad \text{where} \qquad h(x), \omega(x) \in \mathbb{R}. \tag{4}$$

(f) What happens to the real and imaginary degree of freedom of ϕ? Where do they go in the reparametrization?

(g) Show what this reparametrization implies for the Lagrangian in equation (3). Identify kinetic, mass and interaction terms and draw the corresponding Feynman diagrams for the interaction terms, indicating explicitly which of the scalar modes (h and/or ω) feature in the interaction.

(h) Determine, based on the Lagrangian you just found, which fields give rise to massive particles and which don't.

Upon replacing the derivatives ∂_μ by the covariant derivatives D_μ in exercise (a), also the Lagrangian in equation (3) is left invariant under the local gauge transformation in (2).

(i) Find all terms arising from the kinetic term in the Lagrangian, when the derivatives are replaced by covariant derivatives. Continue using the reparametrization in equation (4).

(j) One of the terms containing the ω field is not categorizable as a kinetic, mass or interaction term. Which one is it and can you explain what this term would imply phenomenologically?

We can solve this problem by choosing a specific gauge to work in, the *unitary gauge*:

$$\phi^{(u)}(x) = \mathrm{e}^{-i\omega(x)/v}\phi(x) = \frac{1}{\sqrt{2}}\left(v + h(x)\right) \tag{5}$$

$$A_\mu^{(u)}(x) = A_\mu(x) + \frac{1}{gv}\partial_\mu \omega(x) \tag{6}$$

(k) Use this specific choice of fields to rewrite the kinetic term of the Lagrangian including covariant derivatives in terms of $A_\mu^{(u)}(x)$ and $h(x)$.

(l) Identify the different terms (kinetic, mass, interaction) in the rewritten full Lagrangian and draw the corresponding Feynman diagrams for the interaction terms, indicating explicitly which of the scalar modes (h and/or ω) feature in the interaction. Compare your answer to your answers to parts (g) and (b). What are the differences?

(m) *Challenge:* degrees of freedom never just disappear. What happened to the degree of freedom corresponding to $\omega(x)$ and what does this imply phenomenologically?

Although a bit obscured by the mathematics, you might already have seen that due to the potential chosen in equation (3) a mass term for A^μ was generated, even though it was not allowed to add this term by hand to the Lagrangian without breaking gauge invariance. We chose this potential not randomly of course. What you did in this exercise is nothing more, nothing less than using the Higgs mechanism in its full glory: by introducing a specific scalar potential you generated a gauge-boson mass term through its interactions with the scalar field! The potential we chose is the Higgs potential as we currently believe exists in nature.

The only difference between this exercise and the true Higgs mechanism is that the A^μ field, which we interpreted as the photon field, now has a mass, while in vacuum this is not the case. Actually, this idea was invented for the description of the Meissner effect in superconductors (Anderson, 1962), where the electromagnetic field becomes effectively massive inside superconductors through the interactions with the condensate of scalar Cooper pairs of conduction electrons. In the context of the Standard Model we will apply the Higgs mechanism in lecture 6 to give mass to the $W^\pm$ and Z gauge bosons.