Quantum Field Theory 2: exercises for week 7

Exercise 10: Neutrino oscillations

With the discovery of neutrino oscillations, for which the nobel prize was awarded in 2015, it was proven that at least two of the three known neutrinos should have a non-zero mass. In this exercise we will investigate why this statement holds true by looking at a neutrino with definite momentum $\vec{p}$ that is created a distance L away from the detector. At its creation also a lepton ℓ_α^+ is created. At detection the neutrino creates a lepton ℓ_β^-. This indicates that the flavour eigenstate at creation is α and at detection β, using greek indices $\alpha, \beta = (e, \mu, \tau)$ to label the three flavour eigenstates through their direct link to the corresponding charged-lepton mass eigenstate. We will use roman indices $j = (1, 2, 3)$ to indicate the three neutrino mass eigenstates (= energy eigenstates) with masses m_j. In this exercise we will represent the PMNS matrix simply by U to avoid unnecessary notational clutter. To make the results more transparent and avoid errors, we will explicitly write all summations rather than rely on the summation convention.

(a) During the lecture we defined the relation between the flavour-eigenstate quantum fields and the mass-eigenstate quantum fields as

$$\bar{\nu}_{\alpha_L}(x) = \sum_j \bar{\nu}_{j_L}(x)(U^\dagger)_{j\alpha} \tag{1}$$

Argue that this implies that the particle states are related through

$$|\nu_\alpha\rangle = \sum_j U^*_{\alpha j}\, |\nu_j\rangle \tag{2}$$

(b) At time $t = 0$ the produced neutrino is in the flavour state

$$|\nu(t=0)\rangle = |\nu_\alpha\rangle \tag{3}$$

and has fixed momentum $\vec{p}$. Show that the quantum mechanical probability amplitude for oscillation to the flavour state $|\nu_\beta\rangle$ at time $t > 0$ is given by

$$\langle \nu_\beta | \nu(t)\rangle = \sum_j U_{\beta j} U^*_{\alpha j} \exp(-iE_j t), \qquad \text{with} \quad E_j = \sqrt{\vec{p}^{\,2} + m_j^2} \tag{4}$$

You may assume the mass eigenstates $|\nu_j\rangle$ to be mutually orthogonal and normalized.

(c) Assume the neutrino to be ultrarelativistic, i.e. $m_j \ll E_j$, to derive the following oscillation probability at time $t > 0$:

$$|\langle \nu_\beta | \nu(t)\rangle|^2 \approx \left| \sum_j U_{\beta j} U^*_{\alpha j} \exp\Big[-i m_j^2 \frac{t}{2|\vec{p}\,|}\Big] \right|^2 \tag{5}$$

where $|\vec{p}\,|$ can be approximated by a fixed (average) energy value E in the subsequent expressions.

(d) The ultrarelativistic neutrino travels a distance L to the detector before being detected in a particular flavour state $|\nu_\beta\rangle$. Show that the corresponding oscillation probability can be expressed as:

$$\begin{aligned} P(\nu_\alpha \to \nu_\beta) \approx\ & \delta_{\alpha\beta} - 4\sum_{j>k} \mathrm{Re}\left(U_{\beta j} U^*_{\alpha j} U^*_{\beta k} U_{\alpha k}\right) \sin^2(\Delta m^2_{jk} \frac{L}{4E}) \\ & + 2\sum_{j>k} \mathrm{Im}\left(U_{\beta j} U^*_{\alpha j} U^*_{\beta k} U_{\alpha k}\right) \sin(\Delta m^2_{jk} \frac{L}{2E}) \end{aligned} \tag{6}$$

with $\Delta m^2_{jk} = m_j^2 - m_k^2$. Hint: use that $\sum_j U_{\beta j} U^*_{\alpha j} = \delta_{\alpha\beta}$.

(e) What happens to this expression if all neutrinos would have the same mass?

(f) Explain why such oscillation experiments can at best tell you that two of the three known neutrinos should have a non-zero mass.

(g) Show that equation (6) is not sensitive to the interchange of j and k. What does this imply?

It turns out that if you take the oscillation phase of this phenomenon (i.e., the argument of the $\sin^2$-term) and rewrite it to SI units, you get

$$\Phi \approx 1.27\,\Delta m^2_{jk}(\mathrm{eV}^2)\,\frac{L(\mathrm{km})}{E(\mathrm{GeV})} \tag{7}$$

(h) What will be the typical size of the squared-mass splittings one is sensitive to in an experiment on the Earth's surface looking at neutrinos of 1 GeV originating from the atmosphere on the other side of the Earth? And what about 1 MeV neutrinos produced in the Sun and detected at Earth?

The fact that equation (6) is invariant under the interchange of neutrino type is an important notion. It leaves behind the question what the signs of the mass differences between the different neutrinos actually are. This ordering is known as the *neutrino mass hierarchy* and is one of the largest open questions in neutrino physics.