

Proof:  $P_L = \frac{1}{2}(I_4 - \gamma^5) = P_L^2$ ,  $P_R = \frac{1}{2}(I_4 + \gamma^5) = P_R^2$ ,  $P_L + P_R = I_4$  with  $(\gamma^5)^2 = I_4$ ,  $\{\gamma^5, \gamma^\mu\} = 0$  and  $(\gamma^5)^\dagger = \gamma^5$

$P_{L/R}$  project on  $L/R$  chiral modes:

$$\psi_{L/R} = P_{L/R} \psi, \quad \bar{\psi}_{L/R} = (P_{L/R} \psi)^\dagger \gamma^0 = \psi^\dagger P_{L/R} \gamma^0 = \bar{\psi} P_{R/L}$$

$\Rightarrow \bar{\psi} \gamma_\mu \psi = \bar{\psi} \gamma_\mu (P_L^2 + P_R^2) \psi = \bar{\psi} P_R \gamma_\mu P_L \psi + \bar{\psi} P_L \gamma_\mu P_R \psi$   
vector current  $= \bar{\psi}_L \gamma_\mu \psi_L + \bar{\psi}_R \gamma_\mu \psi_R$  (conserving parity).

under the corresponding local gauge transformations chiral states transform in the same way, i.e.  $\psi_{L/R} \rightarrow \psi'_{L/R} = u(x) \psi_{L/R}$ .

\* chiral theories treat different chiral modes differently and thereby implicitly violate parity. under the corresponding local gauge transformations chiral states transform differently.

Example: under  $SU(2)_L$ ,  $\psi_L \rightarrow \psi'_L = u(x) \psi_L$  and  $\psi_R \rightarrow \psi'_R = \psi_R$ .

A chiral  $SU(2)_L$  gauge theory for the weak interactions requires three massive gauge bosons, two responsible for the charge-current (CC) weak interactions [ $W^\pm$  corresponding to the generators  $T^\pm = T^1 \pm iT^2 = \frac{1}{2}(\sigma^1 \pm i\sigma^2)$ ] and one predicted to be responsible for the neutral-current (NC) weak interactions [ $W^3$  corresponding to the generator  $T^3 = \sigma^3/2$ ]. The existence of this third massive vector boson was confirmed experimentally by the not photon, since  $SU(2)_L$  is chiral and  $U(1)_{e.m.}$  is not!

\* observation of weak NC effects in 1973 @ CERN:

e.g.  $\bar{\nu}_\mu e^- \rightarrow \bar{\nu}_\mu e^-$ ,  $\nu_\mu N \rightarrow \nu_\mu X$ ,  $\bar{\nu}_\mu N \rightarrow \bar{\nu}_\mu X$ ,  
↑ nucleon ↑ anything ↓ neutral!

\* discovery of the massive gauge bosons  $W^\pm$  and  $Z$  in 1983 at the CERN  $p\bar{p}$  experiments UA1 and UA2.

Including electromagnetic interactions: the most economic gauge group for the electroweak sector (= weak + e.m. interactions) would therefore be  $SU(2)_L \times U(1)_Y$ . The generator  $T_Y$  belonging to this  $U(1)$  group has to commute with the  $SU(2)$  generators  $T^+, T^-$  and  $T^3 \Rightarrow [T_Y, T^\pm] = 0$ , i.e. the  $Y$  quantum number is not affected by raising/lowering  $\Rightarrow$  each  $SU(2)_L$  multiplet has one  $Y$  quantum number, i.e.  $U(1)_Y$  does not refer to the electromagnetic gauge group as the neutrino and electron have different charges!

$y(\psi_L) \neq y(\psi_R)$ : another sign that the SM is a chiral theory!

$U(1)_Y$ ,  $Y$  = hypercharge: QED-like (Abelian), one gauge boson  $B_\mu$ .

Gauge coupling:  $g'$ , which can be scaled by a quantum number  $Y/2$  that specifies the int. strength.

$(\begin{smallmatrix} \nu_{eL} \\ e_{eL} \end{smallmatrix}), (\begin{smallmatrix} \nu_{\mu L} \\ \mu_{\mu L} \end{smallmatrix}), (\begin{smallmatrix} \nu_{\tau L} \\ \tau_{\tau L} \end{smallmatrix})$

$\bar{e}_R, \bar{\mu}_R, \bar{\tau}_R$

$\nu_{eR}, \nu_{\mu R}, \nu_{\tau R}$

$y(\text{lepton doublet } L) = -1, y(\bar{l}_R) = -2, y(\nu_R) = 0$ ,  
 pure singlet (no strong/electroweak int.)

$y(\text{quark doublet } Q) = 1/3, y(\text{up-type } q_R) = 2/3, y(\text{down-type } q_R) = -2/3$ .

$(\begin{smallmatrix} u_L \\ d_L \end{smallmatrix}), (\begin{smallmatrix} c_L \\ s_L \end{smallmatrix}), (\begin{smallmatrix} t_L \\ b_L \end{smallmatrix})$

$u_R, c_R, t_R$

$d_R, s_R, b_R$

3rd comp. w/ospin

e.m. charge

Relation between  $I_3, Y$  and  $Q$  (Gell-Mann - Nishijima relation):  $Q = I_3 + Y/2$ ,  
 with  $I_3 = \pm 1/2$  for left-handed matter and  $I_3 = 0$  for right-handed matter.  
 since photon = 1/n. combination of  $W^3$  and  $B$ !

The fact that  $N_c = 3$  and that the leptons and quarks come in a family structure with at least three generations (such as  $\nu_e, e; u, d$ ) will turn out to be a crucial ingredient of the Standard Model.

Additional theoretical input: the issue of anomaly cancellation.

What happens to the gauge symmetry (Ward-Takahashi identities) in the presence of chiral fermions in the loop corrections?

Consider to this end a chiral  $U(1)_Y$  theory with massless fermions:  
 not essential

\* The impact on the gauge-boson self-energy:

$$= -g^2 \int \frac{d^4 \ell}{(2\pi)^4} \frac{\text{Tr}(\not{\ell} \gamma^\nu P_L (k+\not{\ell}) \gamma^\mu P_L)}{\ell^2 (k+\ell)^2} \not{\ell} \gamma^\nu (k+\not{\ell}) \gamma^\mu P_L$$

totally antisymmetric

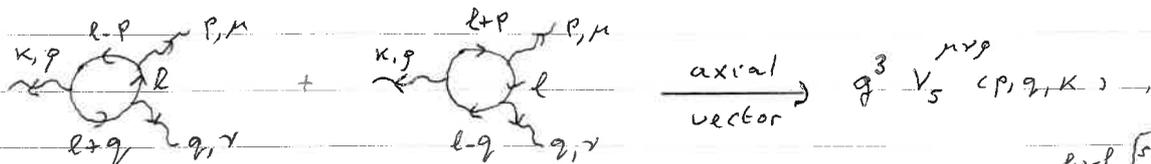
$$= \frac{i}{2} * \text{QED self-energy} + \frac{g^2}{2} \int \frac{d^4 \ell}{(2\pi)^4} \frac{\text{Tr}(\not{\ell} \gamma^\nu (k+\not{\ell}) \gamma^\mu \gamma^5)}{\ell^2 (k+\ell)^2}$$

0, since  $\int d^4 \ell \frac{\ell_\nu k_\mu}{\ell^2 (k+\ell)^2} \propto k_\nu k_\mu$   
 due to Lorentz covariance!

So, the contribution from chiral fermions equals half the QED self-energy, which is consistent with  $\psi = \psi_L + \psi_R$ . Nothing spectacular is happening at this point, since the QED self-energy respects the Ward-Takahashi identity.

†  $W^{1,2,3}$  do not satisfy this, but  $W^+, W^3, W^-$  do since  $Y=0$  and  $Q = I_3 = +1, 0, -1$   
 $I=1$  isospin triplet

\*). The impact on the triple gauge - boson couplings: yet again we obtain half the QED vertex correction (which happens to be 0) and an axial-vector contribution



same integral with all matrices in reverted order = same integral

$$\text{with } V_5^{\mu\nu\rho}(p, q, k) = \frac{1}{2} \int \frac{d^4\ell}{(2\pi)^4} \frac{\text{Tr}([\ell-p]\gamma^\mu \not{\ell} \gamma^\nu [\ell+q]\gamma^\rho \gamma^5)}{(\ell-p)^2 \ell^2 (\ell+q)^2} + (p, \mu \leftrightarrow q, \nu)$$

$$= \int \frac{d^4\ell}{(2\pi)^4} \frac{\text{Tr}([\ell-p]\gamma^\mu \not{\ell} \gamma^\nu [\ell+q]\gamma^\rho \gamma^5)}{(\ell-p)^2 \ell^2 (\ell+q)^2}$$

Ward identities:  $k_\rho V_5^{\mu\nu\rho}(p, q, k)$ ,  $q_\nu V_5^{\mu\nu\rho}(p, q, k)$  and  $p_\mu V_5^{\mu\nu\rho}(p, q, k)$  should all vanish, otherwise gauge invariance and renormalizability are lost.

$$k_\rho V_5^{\mu\nu\rho}(p, q, k) = \int \frac{d^4\ell}{(2\pi)^4} \frac{\text{Tr}([\ell-p]\gamma^\mu \not{\ell} \gamma^\nu [\ell+q]k\gamma^5)}{(\ell-p)^2 \ell^2 (\ell+q)^2} \quad \text{and use that}$$

$$k = -\not{p} - \not{q} = (\ell-p) - (\ell+q) \Rightarrow k\gamma^5 = -\gamma^5(\ell-p) - (\ell+q)\gamma^5$$

$$\text{and } \text{Tr}([\ell-p]\gamma^\mu \not{\ell} \gamma^\nu [\ell+q]k\gamma^5) = -(\ell-p)^2 \text{Tr}(\gamma^\mu \not{\ell} \gamma^\nu [\ell+q]\gamma^5) - 4i\epsilon^{\mu\nu\rho\beta} \ell_\rho q_\beta$$

$$- (\ell+q)^2 \text{Tr}([\ell-p]\gamma^\mu \not{\ell} \gamma^\nu \gamma^5) - 4i\epsilon^{\mu\nu\rho\beta} \ell_\rho p_\beta$$

$$\text{Hence, } k_\rho V_5^{\mu\nu\rho}(p, q, k) = 4i\epsilon^{\mu\nu\rho\beta} \int \frac{d^4\ell}{(2\pi)^4} \left[ \frac{\ell_\rho q_\beta}{\ell^2 (\ell+q)^2} + \frac{\ell_\rho p_\beta}{\ell^2 (\ell-p)^2} \right] \rightarrow 0 \text{ (as before).}$$

$\hookrightarrow \propto q_\rho q_\beta \quad \hookrightarrow \propto p_\rho p_\beta$

So far so good, but what about the 3<sup>rd</sup> Ward identity?

$$p_\mu V_5^{\mu\nu\rho}(p, q, k) = \int \frac{d^4\ell}{(2\pi)^4} \frac{\text{Tr}([\ell-p]\not{p}\not{\ell}\gamma^\nu[\ell+q]\gamma^\rho\gamma^5)}{(\ell-p)^2 \ell^2 (\ell+q)^2}$$

$$= 4i\epsilon^{\nu\rho\beta\gamma} \int \frac{d^4\ell}{(2\pi)^4} \left[ \frac{\ell_\gamma (q+\ell)_\beta}{\ell^2 (\ell+q)^2} - \frac{(\ell-p)_\gamma (p+q+\ell)_\beta}{(\ell-p)^2 (\ell+q)^2} \right]$$

$\hookrightarrow q_\gamma q_\beta$ , which vanishes

$\underbrace{\hspace{10em}}_{\text{would also vanish if the shift } \ell \rightarrow \ell+p \text{ is allowed in the integral}}$

So, the Ward identity  $p_\mu V_5^{\mu\nu\rho}(p, q, k) = 0$  is satisfied, provided that we are allowed to shift the integration momentum by a constant 4-vector without paying a price. The same remark applies to the

remaining Ward identity  $g_V V_5^{\mu\nu\rho}(p, q, k) = 0$ .

We would be allowed to perform the indicated shift if the integral would be convergent. In fact this is not the case. Since the integrand behaves as  $|k|^{-3}$  for  $|k| \rightarrow \infty$ , we are dealing with a linearly divergent integral!

Consider the Euclidean  $n$ -dimensional integral  $I(y) = \int d^n x [f(x+y) - f(x)]$ , with  $f(x) = \mathcal{O}(|x|^{-n})$  for  $|x| \rightarrow \infty$ . The behaviour at  $|x| \rightarrow \infty$  will determine whether this integral will vanish or not  $\Rightarrow$  perform a Taylor expansion:

n-dim. Euclidean vectors

$$I(y) = \lim_{R \rightarrow \infty} \int_{|\vec{x}| \leq R} d^n x \left( y^i \frac{\partial f}{\partial x^i} + \frac{1}{2} y^i y^j \frac{\partial^2 f}{\partial x^i \partial x^j} + \dots \right) \stackrel{\text{Gauss}}{=} \lim_{R \rightarrow \infty} \int_{S_n(R)} d\sigma \frac{x^i}{R} [f(x) y^i]$$

d $\sigma$  pointing outward  
area  $\propto R^{n-1}$

+ negligible terms.

In the case of a linearly divergent integral the surface term survives in general and  $\int d^n x f(x)$  depends on the reference point of the integration!!! Using Wick rotation to turn the actual Minkowskian integral into a Euclidean one, it is indeed found that

- not all three Ward identities can be satisfied simultaneously, not even by redefining the reference point of the loop integral. This is a problem since we couple the axial current to a gauge boson;
- the Ward identities are violated by terms  $\propto \epsilon^{\mu\nu\rho\beta} p_\mu q_\nu$ ;
- no further Ward-identity violating terms occur at higher loop order or in the corrections to vertices involving four or more gauge bosons. (typical for U(1))

Consequence: renormalizability spoiled, predictive power lost!

This phenomenon of quantum (loop) effects invalidating the gauge invariance of a gauge theory is called a gauge anomaly.

What about a chiral  $SU(N)_2$  theory?

linked to the momenta  $p, q, k$

In that case the  $1^{st}$  vertex correction on p. 19 gets a factor  $\text{Tr}(T^a T^b T^c)$  and the  $2^{nd}$  one a factor  $\text{Tr}(T^b T^a T^c) \Rightarrow$  overall factor  $\frac{1}{2} \text{Tr}(\{T^a, T^b\} T^c)$  in front of  $g_V^3 V_5^{\mu\nu\rho}(p, q, k)$ , which vanishes for  $SU(2)$ , since  $\{\sigma^a, \sigma^b\} = 2\delta^{ab} I_2$  and  $\text{Tr}(\sigma^c) = 0$ .



Hence, in order to make the  $SU(3)_c \times SU(2)_L \times U(1)_Y$  gauge theory anomaly free, the following two theoretical constraints should be imposed:

1)  $\sum_{f_L} \gamma(f_L) = 0$  for one  $U(1)_Y$  boson and two  $SU(2)_L$  bosons,

$$\rightarrow \underbrace{2\gamma(L_L)}_{-1} + 2N_c \underbrace{\gamma(Q_L)}_{+1/3} = -2 + \frac{2}{3} N_c \stackrel{N_c=3}{=} 0 \text{ for each fermion generation!}$$

2)  $\sum_f [\gamma^3(f_L) - \gamma^3(f_R)] = 0$  for three  $U(1)_Y$  bosons.

$$\rightarrow \underbrace{2\gamma^3(L_L)}_{-2} - \underbrace{\gamma^3(e_R)}_{-8} + N_c \left[ \underbrace{2\gamma^3(Q_L)}_{2/27} - \underbrace{\gamma^3(u_R)}_{64/27} - \underbrace{\gamma^3(d_R)}_{-8/27} \right] = 6 - 2N_c \stackrel{N_c=3}{=} 0$$

for each fermion generation!

Therefore,  $N_c=3$  and the lepton-quark family structure are crucial for making the Standard Model anomaly free  $\Rightarrow$  predictive power restored!

The mass issue: the above-constructed gauge theory only allows massless particles. A gauge-boson mass term  $+\frac{1}{2}m^2 W_\mu^a W^{a\mu}$  would clearly break gauge invariance and result in problems with the high-energy behaviour and renormalizability of the theory (see Ex. 5)! Actually the same holds for the matter fermions of the theory. Gauge theories based on vector currents, such as QED and QCD, are democratic w.r.t. chiral states and the chiral states transform in the same way under the corresponding local gauge transformations (see p. 17). In that case a mass term  $m\bar{\psi}\psi = m(\bar{\psi}_R\psi_L + \bar{\psi}_L\psi_R)$  would be gauge invariant. However, in a chiral gauge theory, such as the  $SU(2)_L \times U(1)_Y$  gauge theory, different chiral states transform differently and a mass term  $m\bar{\psi}\psi$  would break gauge invariance!

### Chapter 3: Particle masses in the Standard Model

In order to have non-zero particle masses in the Standard Model we have to break gauge invariance without actually breaking it!

Way out: spontaneous symmetry breaking. The Lagrangian is invariant under local gauge transformations, but the dynamics is such that the ground state ('vacuum') is NOT a singlet of the symmetry group, i.e. the symmetry is not apparent in the ground state  $\Rightarrow$  "choosing" one of the degenerate set of ground states