

remaining Ward identity $g_F V_S^{MVP} (p_1, q_1, k_1) = 0$.

We would be allowed to perform the indicated shift if the integral would be convergent. In fact this is not the case. Since the integrand behaves as $1/\ell^3$ for $|\ell| \rightarrow \infty$, we are dealing with a linearly divergent integral!

Consider the Euclidean n -dimensional integral $I(\gamma) = \int d^n x [f(x + \gamma) - f(x)]$, with $f(x) = O(|\vec{x}|^{-n})$ for $|\vec{x}| \rightarrow \infty$. The behaviour at $|\vec{x}| \rightarrow \infty$ will determine whether this integral will vanish or not \Rightarrow perform a Taylor expansion:

$$I(\gamma) = \lim_{R \rightarrow \infty} \int_{|\vec{x}| \leq R} d^n x \left(\underbrace{\gamma^i \frac{\partial f}{\partial x^i}}_{\vec{\gamma} \cdot (\vec{f}' \vec{\gamma})} + \underbrace{\frac{1}{2} \gamma^i \gamma^j \frac{\partial^2 f}{\partial x^i \partial x^j}}_{\vec{\gamma} \cdot (\frac{1}{2} \vec{\gamma}^i \frac{\partial^2 f}{\partial x^i \partial x^j} \vec{\gamma})} + \dots \right) \xrightarrow{\text{Gauss}} \lim_{R \rightarrow \infty} \int_{S_n(R)} d\sigma \frac{x^i}{R} [f(x \gamma^i)]$$

(d σ pointing outward)
(area $\propto R^{n-1}$)

+ negligible terms.

In the case of a linearly divergent integral the surface term survives in general and $\int d^n x f(x)$ depends on the reference point of the integration!!! Using Wick rotation to turn the actual Minkowskian integral into a Euclidean one, it is indeed found that

- not all three Ward identities can be satisfied simultaneously, not even by redefining the reference point of the loop integral. This is a problem since we couple the axial current to a gauge boson;
- the Ward identities are violated by terms $\propto e^{MVP} p_\mu q_\beta$;
- no further Ward-identity violating terms occur at higher loop order or in the corrections to vertices involving four or more gauge bosons. \leftarrow (typical for $U(1)$)

Consequence: renormalizability spoiled, predictive power lost!

This phenomenon of quantum (loop) effects invalidating the gauge invariance of a gauge theory is called a gauge anomaly.

What about a chiral $SU(N)_L$ theory?

linked to the momenta p_1, q_1, k_1

In that case the 1^{st} vertex correction on $p_1 q_1 g$ gets a factor $\text{Tr}(T^a T^b T^c)$ and the 2^{nd} one a factor $\text{Tr}(T^a T^b T^c)$ \Rightarrow overall factor $\frac{1}{2} \text{Tr}(\{T^a, T^b\} T^c)$ in front of $g^3 V_S^{MVP} (p_1, q_1, k_1)$, which vanishes for $SU(2)_L$ since $\{\sigma^a, \sigma^b\} = 2\delta^{ab} I_2$ and $\text{Tr}(\sigma^c) = 0$.

Hence, in order to make the $SU(3)_c \times SU(2)_L \times U(1)_Y$ gauge theory anomaly free, the following two theoretical constraints should be imposed:

$$1) \boxed{\sum_{f_L} y_{cf_L} = 0} \text{ for one } U(1)_Y \text{ boson and two } SU(2)_L \text{ bosons.}$$

$$\rightarrow 2y(L_L) + 2N_c y(Q_L) = -2 + \frac{2}{3} N_c \stackrel{N_c=3}{=} 0 \text{ for each fermion generation.}$$

$$2) \boxed{\sum_f [y^3(c_f) - y^3(f_f)] = 0} \text{ for three } U(1)_Y \text{ bosons.}$$

$$\rightarrow \underbrace{y^3(c_L)}_{-2} - \underbrace{y^3(f_L)}_{-8} + N_c [\underbrace{y^3(Q_L)}_{2/27} - \underbrace{y^3(U_R)}_{64/27} - \underbrace{y^3(D_R)}_{-8/27}] = 6 - 2N_c \stackrel{N_c=3}{=} 0$$

for each fermion generation.

Therefore, $N_c=3$ and the lepton-quark family structure are crucial for making the Standard Model anomaly free \Rightarrow predictive power restored!

The mass issue: the above-constructed gauge theory only allows massless particles. A gauge-boson mass term $+ \frac{1}{2} m^2 W_\mu^\alpha W^{\alpha\mu}$ would clearly break gauge invariance and result in problems with the high-energy behaviour and renormalizability of the theory (see Ex. 5)! Actually the same holds for the matter fermions of the theory. Gauge theories based on vector currents, such as QED and QCD, are democratic w.r.t. chiral states and the chiral states transform in the same way under the corresponding local gauge transformations (see p. 17). In that case a mass term $m \bar{\psi} \psi = m (\bar{\psi}_R \psi_L + \bar{\psi}_L \psi_R)$ would be gauge invariant. However, in a chiral gauge theory, such as the $SU(2)_L \times U(1)_Y$ gauge theory, different chiral states transform differently and a mass term $m \bar{\psi} \psi$ would break gauge invariance!

Chapter 3 : Particle masses in the Standard Model

In order to have non-zero particle masses in the Standard Model we have to break gauge invariance without actually breaking it!

Way out: spontaneous symmetry breaking. The Lagrangian is invariant under local gauge transformations, but the dynamics is such that the ground state ('vacuum') is NOT a singlet of the symmetry group, i.e. the symmetry is not apparent in the ground state \Rightarrow "choosing" one of the degenerate set of ground states

as the physical vacuum breaks the symmetry!

Quantum mechanical concept: if there is symmetry under a group of transformations with elements \hat{u} , then $[H, \hat{u}] = 0$ for the Hamilton operator. For the ground state $|0\rangle$, with $\hat{H}|0\rangle = E_0|0\rangle$, this implies that $\hat{H}\hat{u}|0\rangle = \hat{u}\hat{H}|0\rangle = E_0\hat{u}|0\rangle \Rightarrow \{\hat{u}|0\rangle\}$ forms a degenerate set of ground states if $\exists \hat{u}|0\rangle \neq |0\rangle$, which is equivalent to $|0\rangle$ not being a singlet under the transformation group.

Higgs toy model!

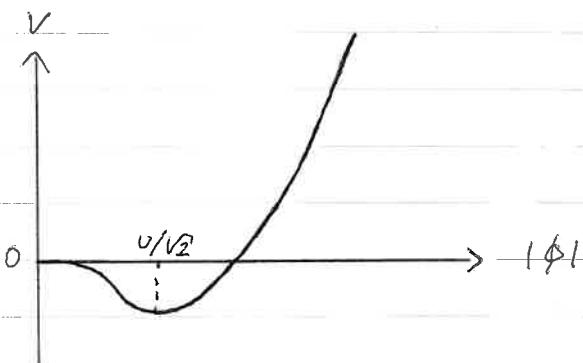
see Ex 7

Let's start with a (u(1) example ... the so-called relativistic superconductor:

$$\mathcal{L} = (\partial_\mu \phi)^+ (\partial^\mu \phi) - V(\phi), \quad V(\phi) = \frac{\mu^2}{2} |\phi|^2 + \frac{\lambda}{4} |\phi|^4 \quad (\mu^2 > 0, \lambda > 0) \text{ and}$$

wrong sign mass term
potential bounded from below

ϕ = complex scalar field.



V has a maximum at $|\phi| = 0$

and a ring of minima at $|\phi|^2 = -\mu^2/2\lambda \equiv v^2/2 > 0$ linked by global u(1) phase transformations.

Consequence: $\phi=0$ is not suitable as ground state, around which the quantum fluctuations (particle interpretation) should be considered.

One of the degenerate ring of minima should be used for that purpose!

vacuum expectation value (even) $\langle \phi \rangle$

Choose the ground state such that $\langle 0 | \phi | 0 \rangle = v/\sqrt{2} \Rightarrow$ global u(1) phase symmetry broken! Reparametrize $\phi(x) = \frac{1}{\sqrt{2}} (v + h(x)) e^{i w(x)/v}$, with $h(x)$ and $w(x)$ real scalar fields, in order to separate classical and quantum fluctuation parts

$$\Rightarrow -V(\phi) = -\frac{1}{2} \mu^2 (v+h)^2 - \frac{\lambda}{4} (vh)^4 = \frac{1}{2} \mu^2 v^2 - \underbrace{\lambda v^2 h^2}_{\text{mass term} \equiv -\frac{1}{2} m_h^2 h^2} - \lambda v h^3 - \frac{\lambda}{4} h^4,$$

$+ \frac{\mu^2 h^2}{2} \text{ interactions}$

$$\begin{aligned} (\partial_\mu \phi)^+ (\partial^\mu \phi) &= \frac{1}{2} [\partial_\mu h - i(v+h) \frac{\partial_\mu w}{v}] e^{-i w/v} [\partial^\mu h + i(v+h) \frac{\partial^\mu w}{v}] e^{+i w/v} \\ &= \frac{1}{2} (\partial_\mu h) (\partial^\mu h) + \frac{1}{2} (1 + h/v)^2 (\partial_\mu w) (\partial^\mu w). \end{aligned}$$

\square interactions

Combined: - the boson described by the h -field has acquired a mass $m_h^2 = \sqrt{2} \mu^2 = \sqrt{2} \mu^2$
 ↳ massive d.o.f. describing physical oscillations.

The w -field corresponds to a massless boson (Goldstone boson) \rightarrow massless d.o.f. describing the symmetry of the true ground states (flat direction along the rims of minima); \rightarrow for each generator belonging to a broken symmetry there will be one such Goldstone boson. \rightarrow generator that does not annihilate the vacuum

going beyond $U(1)$ toy model

Next we turn this into a local $U(1)$ gauge theory and local symmetry breaking:

$$\mathcal{L} = (D_\mu \phi)^T (D^\mu \phi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - V(\phi), \text{ with } D_\mu \phi = (\partial_\mu + ig A_\mu) \phi, F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

This Lagrangian is invariant under local gauge transformations

$$\phi(x) \rightarrow \phi'(x) = e^{i\vartheta(x)} \phi(x), \quad A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) - \frac{i}{g} \partial_\mu \vartheta(x)$$

using reparametrization of ϕ

$$\Rightarrow (D_\mu \phi)^T (D^\mu \phi) = \frac{1}{2} [\partial_\mu h - i(v+h)(gA_\mu + \partial_\mu w/v)] e^{-i\omega(x)/v} [\partial^\mu h + i(v+h)(gA^\mu + \partial^\mu w/v)] e^{+i\omega(x)/v}$$

$$= \frac{1}{2} (\partial_\mu h)(\partial^\mu h) + \frac{1}{2} (1+h/v)^2 [(\partial_\mu w)(\partial^\mu w) + g^2 v^2 A_\mu A^\mu + 2vgA^\mu \partial_\mu w]$$

Interaction

\hookrightarrow bilinear A - w mixing \Rightarrow be careful in interpreting \mathcal{L} and its particle content!

In fact $w(x)$ can be removed by means of a specific gauge transformation

(switching to the unitary gauge): $\phi^{(u)}(x) = e^{-i\omega(x)/v} \phi(x) = \frac{i}{\sqrt{2}} (v+h(x))$,
 $A_\mu^{(u)}(x) = A_\mu(x) + \frac{i}{gv} \partial_\mu w(x)$

$$\Rightarrow \mathcal{L} = \frac{1}{2} [\partial_\mu h - ig A_\mu (v+h)] [\partial^\mu h + ig A^\mu (v+h)] - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \mu^2 (v+h)^2 - \frac{\lambda}{4} (v+h)^4$$

$$= \frac{1}{2} (\partial_\mu h)(\partial^\mu h) + \mu^2 h^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \frac{\mu^2}{\lambda} A_\mu A^\mu [g^2 v^2 + 2gh + g^2 h^2] - \lambda h^3 - \frac{\lambda}{4} h^4 + \frac{1}{4} v^4.$$

Interaction

Hence: - the gauge boson has acquired a mass $m_A = gv$

\Rightarrow this gauge boson has acquired an extra (longitudinal) d.o.f.

[massless: 2 d.o.f. \rightarrow massive: 3 d.o.f.]

- the Goldstone boson w has supplied this extra d.o.f., as the unitary-gauge expression has revealed (by gauging away w)

This is called the Higgs mechanism \hookrightarrow invented for describing the Meissner effect in superconductors

Subsequently we adapt this to the standard model electroweak sector.

To ensure that the ground state is Lorentz invariant

$SU(2)_L \times U(1)_Y$: we need a set of scalar fields Φ that develops a $U(1)_{\text{e.m.}}$ symmetric vacuum expectation value $\langle \Phi \rangle_0$, so that $SU(2)_L \times U(1)_Y \xrightarrow{\langle \Phi \rangle_0} U(1)_{\text{e.m.}}$

\Rightarrow three out of four $SU(2)_L \times U(1)_Y$ gauge bosons acquire mass (3 broken generators), the photon remains massless by taking the chosen ground state to be a singlet under $U(1)_{\text{e.m.}}$ (i.e. $\langle \Phi \rangle_0$ is chosen to be neutral).

↑ meaning: $I_3 + Y/2 = 0$

Most economic solution: $\Phi = (\phi^+, \phi^0)$ is a complex scalar doublet of Higgs fields, with weak isospin $I=1/2$ (doublet) and hypercharge $Y(\Phi) = 1 \Rightarrow I_3 = +1/2$ component has positive charge and $I_3 = -1/2$ component is neutral

added by hand!

↑ needed for $\langle \Phi \rangle_0$

$$\Rightarrow \mathcal{L}_H = (D_\mu \Phi)^+ (D^\mu \Phi) - \frac{1}{2} (\Phi^+ \Phi) - \lambda (\Phi^+ \Phi)^2, \quad \Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 - i\phi_2 \\ \phi_0 + i\phi_3 \end{pmatrix}$$

$$D_\mu \Phi = (\partial_\mu + \frac{i}{2} g \vec{\sigma} \cdot \vec{w}_\mu + \frac{i}{2} g' Y B_\mu) \Phi \stackrel{Y=1}{=} (\partial_\mu + \frac{i}{2} [g \vec{\sigma} \cdot \vec{w}_\mu + g' B_\mu]) \Phi.$$

↑ $SU(2)_L$ ↑ $U(1)_Y$

Transformation property: $\Phi(x) \rightarrow \Phi'(x) = e^{\frac{i}{2} \vec{\sigma} \cdot \vec{\theta}(x)} e^{\frac{i}{2} q_y(x)} \Phi(x)$.

choose $\langle \Phi \rangle_0 = (v/\sqrt{2})$, $v^2 = -\mu^2/\lambda$, reparametrize $\Phi(x) = e^{\frac{i}{2} \vec{\sigma} \cdot \vec{w}(x) + i\theta(x)/\sqrt{2}}$ and $\sum_{i \in L} \epsilon_i \in SU(2)$

switch to the unitary gauge $\Phi(x) \rightarrow \tilde{\Phi}(x) = \sum_i \epsilon_i \Phi(x) = \frac{1}{\sqrt{2}} (v + h(x))$

$\Rightarrow D_\mu \tilde{\Phi}^{(u)} = (\partial_\mu + \frac{i}{2} [g \vec{\sigma} \cdot \vec{w}_\mu^{(u)} + g' B_\mu^{(u)}]) \frac{1}{\sqrt{2}} (v + h)$, h = physical Higgs-boson field,

$$-V(\tilde{\Phi}^{(u)}) = \mu^2 h^2 - \lambda v h^3 - \frac{\lambda}{4} h^4 + \frac{1}{4} \lambda v^4 \quad (\text{as on p.22}).$$

Consequence: the three would-be Goldstone bosons represented by $\vec{w}(x)$ are "eaten" by the gauge bosons to form three massive mass eigenstates

starting from unitary gauge

$$Z_\mu = c_w w_\mu^3 - s_w B_\mu \quad (\text{Z boson}), \quad A_\mu = s_w w_\mu^3 + c_w B_\mu \quad (\text{photon}),$$

$$\tan \Theta_W = g'/g$$

$w_\mu = \frac{1}{\sqrt{2}} (w_\mu^+ + i w_\mu^-)$ (w^\pm bosons), with $c_w = \cos \Theta_W$, $s_w = \sin \Theta_W$ and $\Theta_W = \text{weak mixing angle}$,

$$\Delta m_{\text{mass}}^2 = m_w^2 w_\mu^+ w_\mu^- + \frac{1}{2} m_Z^2 Z_\mu Z_\mu + 0 \cdot A_\mu A_\mu,$$

$$m_w^2 = \frac{1}{4} v^2 g^2, \quad m_Z^2 = \frac{1}{4} v^2 (g^2 + g'^2) = m_w^2/c_w^2, \quad m_A^2 = m_Z^2 = 0, \quad v = 246 \text{ GeV}.$$

see Ex. 8