

To ensure that the ground state is Lorentz invariant

$SU(2)_L \times U(1)_Y$: we need a set of scalar fields Φ that develops a $U(1)_{\text{e.m.}}$ symmetric vacuum expectation value $\langle \Phi \rangle_0$, so that $SU(2)_L \times U(1)_Y \xrightarrow{\langle \Phi \rangle_0} U(1)_{\text{e.m.}}$

\Rightarrow three out of four $SU(2)_L \times U(1)_Y$ gauge bosons acquire mass (3 broken generators), the photon remains massless by taking the chosen ground state to be a singlet under $U(1)_{\text{e.m.}}$ (i.e. $\langle \Phi \rangle_0$ is chosen to be neutral).

meaning: $I_3 + Y/2 = 0$

Most economic solution: $\Phi = (\phi^+, \phi^0)$ is a complex scalar doublet of Higgs fields, with weak isospin $I=1/2$ (doublet) and hypercharge $Y(\Phi) = 1 \Rightarrow I_3 = +1/2$ component has positive charge and $I_3 = -1/2$ component is neutral

added by hand!

needed for $\langle \Phi \rangle_0$

$$\Rightarrow D_\mu \Phi = (D_\mu \Phi)^+ (D^\mu \Phi) = \underbrace{\mu^2 (\Phi^+ \Phi)}_{<0>} - \underbrace{\lambda (\Phi^+ \Phi)^2}_{>0}, \quad \Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 - i\phi_2 \\ \phi_0 + i\phi_3 \end{pmatrix}$$

$$D_\mu \Phi = (\partial_\mu + \frac{i}{2} g \vec{\sigma} \cdot \vec{w}_\mu + \frac{i}{2} g' Y B_\mu) \Phi \stackrel{Y=1}{=} (\partial_\mu + \frac{i}{2} [g \vec{\sigma} \cdot \vec{w}_\mu + g' B_\mu]) \Phi.$$

$\uparrow SU(2)_L$ $\uparrow U(1)_Y$

Transformation property: $\Phi(x) \rightarrow \Phi'(x) = e^{\frac{i}{2} \vec{\sigma} \cdot \vec{\theta}(x)} e^{\frac{i}{2} \theta_y(x)} \Phi(x)$.

choose $\langle \Phi \rangle_0 = (v/\sqrt{2})$, $v^2 = -\mu^2/\lambda$, reparametrize $\Phi(x) = e^{\frac{i}{2} \vec{\sigma} \cdot \vec{w}(x) + i\theta(x)/\sqrt{2}}$ and $\sum_{x \in E} \in SU(2)$

switch to the unitary gauge $\Phi(x) \rightarrow \tilde{\Phi}(x) = \sum_{x \in E} \Phi(x) = \frac{1}{\sqrt{2}} (v + h(x))$

$\Rightarrow D_\mu \tilde{\Phi}^{(u)} = (\partial_\mu + \frac{i}{2} [g \vec{\sigma} \cdot \vec{w}_\mu^{(u)} + g' B_\mu^{(u)}]) \frac{1}{\sqrt{2}} (v + h)$, h = physical Higgs-boson field,

$$-V(\tilde{\Phi}^{(u)}) = \mu^2 h^2 - \lambda v h^3 - \frac{\lambda}{4} h^4 + \frac{1}{4} \lambda v^4 \text{ (as on p.22).}$$

Consequence: the three would-be Goldstone bosons represented by $\vec{w}(x)$ are "eaten" by the gauge bosons to form three massive mass eigenstates

starting from unitary gauge

$$Z_\mu = c_w w_\mu^3 - s_w B_\mu \quad (\text{Z boson}), \quad A_\mu = s_w w_\mu^3 + c_w B_\mu \quad (\text{photon}),$$

$$\tan \Theta_W = g'/g$$

$w_\mu = \frac{1}{\sqrt{2}} (w_\mu^+ + i w_\mu^-)$ (w^\pm bosons), with $c_w = \cos \Theta_W$, $s_w = \sin \Theta_W$ and $\Theta_W = \text{weak mixing angle}$,

$$\Delta m_{\text{mass}} = m_w^2 w_\mu^+ w_\mu^- + \frac{1}{2} m_Z^2 Z_\mu Z_\mu + 0 \cdot A_\mu A_\mu,$$

see Ex. 8

$$m_w^2 = \frac{1}{4} v^2 g^2, \quad m_Z^2 = \frac{1}{4} v^2 (g^2 + g'^2) = m_w^2/c_w^2, \quad m_A^2 = m_Z^2 = 0, \quad v = 246 \text{ GeV.}$$

§ 3.1 Giving mass to fermions and intergenerational mixing

As mentioned on p. 21, a mass term $m\bar{\psi}\psi = -m(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L)$ would break gauge invariance in the Standard Model. We will therefore use the Higgs mechanism to also give mass to fermions. To this end we introduce $SU(2)_L \times U(1)_Y$ invariant interactions between the fermions and scalar doublets, which compensate the $SU(2)_L$ transformations of $\bar{\psi}_L$. Moreover, $U(1)_Y$ gauge invariance requires Higgs doublets with hypercharge $-Y(\bar{\psi}_L\psi_R) = Y(\psi_L) - Y(\psi_R) = 2(Q - I_3) = 2Q - 2I_3$

- => use $\tilde{\Phi}$, $Y(\tilde{\Phi}) = -Y(\bar{\psi}_L\psi_R) = +1$ to give mass to fermions with $I_3 = -\frac{1}{2}$,
- use $\tilde{\tilde{\Phi}}$, $Y(\tilde{\tilde{\Phi}}) = -Y(\bar{\psi}_L\psi_R) = -1$ to give mass to fermions with $I_3 = +\frac{1}{2}$.

For one generation of fermions (e, ν_e, u, d), these so-called yukawa interactions read

$$\mathcal{L}_{\text{Yuk}} = -f_e \bar{L}_L \tilde{\Phi} e_R - f_d \bar{Q}_L \tilde{\Phi} d_R - f_{\nu_e} \bar{L}_L \tilde{\tilde{\Phi}} \nu_R - f_u \bar{Q}_L \tilde{\tilde{\Phi}} u_R + \text{h.c.},$$

$\begin{matrix} (\bar{\psi}_{e_L}, \bar{\psi}_{\nu_L}) \\ (\bar{\psi}_{u_L}, \bar{\psi}_{d_L}) \end{matrix}$

$$\text{with } \tilde{\tilde{\Phi}} \equiv i\sigma^2 \tilde{\Phi}^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tilde{\Phi}^* = \begin{pmatrix} \phi^+ \\ -\phi^- \end{pmatrix} \xrightarrow[\text{unitary gauge}]{} \frac{1}{\sqrt{2}} \begin{pmatrix} u^+ \\ 0 \end{pmatrix},$$

$I(\tilde{\tilde{\Phi}}) = \frac{1}{2}$, $Y(\tilde{\tilde{\Phi}}) = -1 \Rightarrow \tilde{\tilde{\Phi}}$ has the same $SU(2)_L$ and opposite $U(1)_Y$ transformation properties as $\tilde{\Phi}$ (see Ex. 2),
 $f_e, f_d, f_{\nu_e}, f_u \in \mathbb{R}$ (called "yukawa couplings").

\hookrightarrow in the "standard" version of the Standard Model

The fermion mass terms now follow directly from the non-zero vev's $\langle \tilde{\Phi} \rangle_0 = (v/\sqrt{2})$ and $\langle \tilde{\tilde{\Phi}} \rangle_0 = (0/v)$ $\Rightarrow m_i = v f_i / \sqrt{2}$ for $i = e, d, \nu_e, u$.

\hookrightarrow mass lower doublet comp. \swarrow mass upper doublet comp.

The mass generation for the other fermion species proceeds in a similar way.

Gauge eigenstates need not be identical to mass eigenstates: eigenstates with the same quantum numbers can mix, as observed in the weak decays

$$\begin{array}{c} n \rightarrow p \bar{e} \bar{\nu}_e \text{ as well as } l \rightarrow p \bar{e} \bar{\nu}_e \text{ ?} \\ (\text{nucleus}) \qquad \qquad (\text{nucleus}) \end{array}$$

$\begin{matrix} \uparrow \\ \text{d} \rightarrow u \text{ transition} \end{matrix} \qquad \begin{matrix} \uparrow \\ s \rightarrow u \text{ transition} \end{matrix}$

Hence, we need a more general version of the Yukawa sector in the actual Standard Model.

Introduce therefore several generations of leptons and quarks, and indicate the gauge eigenstates (which feature in/are produced by weak interactions) as generation label

$$\nu_{AR}^i = (\nu_{eR}^i, \nu_{\mu R}^i, \dots), e_{AR}^i = (e_R^i, \mu_R^i, \dots), u_{AR}^i = (u_R^i, c_R^i, \dots), d_{AR}^i = (d_R^i, s_R^i, \dots),$$

$$L_{AL}^i = \begin{pmatrix} \nu_{AL}^i \\ e_{AL}^i \end{pmatrix}, Q_{AL}^i = \begin{pmatrix} u_{AL}^i \\ d_{AL}^i \end{pmatrix} \quad \text{gauge doublets}$$

$$\bar{\nu}_{AR}^i = \bar{\nu}_{eR}^i, \bar{e}_{AR}^i = \bar{e}_R^i, \bar{u}_{AR}^i = \bar{u}_R^i, \bar{d}_{AR}^i = \bar{d}_R^i \quad \text{gauge singlets}$$

The matter part of the Lagrangian, incl. electroweak covariant derivatives, then reads

$$S_F = \bar{L}_{AL}^i \partial^\mu [\partial_\mu + \frac{i}{2} g \vec{\sigma} \cdot \vec{w}_\mu - \frac{i}{2} g' B_\mu] L_{AL}^i + \bar{e}_{AR}^i \partial^\mu [\partial_\mu - ig' B_\mu] e_{AR}^i + \bar{\nu}_{AR}^i \partial^\mu \partial_\mu \nu_{AR}^i + \dots$$

Furthermore we allow for Yukawa interactions with non-diagonal complex couplings:

$$S_{Yuk} = - (f_e)_{AB} \bar{L}_{AL}^i \Phi e_{BR}^i - (f_d)_{AB} \bar{Q}_{AL}^i \Phi d_{BR}^i - (f_{\nu_e})_{AB} \bar{\nu}_{AL}^i \Phi \nu_{BR}^i - (f_u)_{AB} \bar{u}_{AL}^i \Phi u_{BR}^i + h.c.$$

with $(f_i)_{AB}$ $i=e, d, \nu_e, u$ complex $n \times n$ matrices in family space with n generations of leptons and quarks. This results in mass terms

$$-\frac{v}{\sqrt{2}} [(f_e)_{AB} \bar{e}_{AL}^i e_{BR}^i + (f_d)_{AB} \bar{d}_{AL}^i d_{BR}^i + (f_{\nu_e})_{AB} \bar{\nu}_{AL}^i \nu_{BR}^i + (f_u)_{AB} \bar{u}_{AL}^i u_{BR}^i + h.c.],$$

involving mass matrices $M_{AB}^{(i)} = \frac{v}{\sqrt{2}} (f_i)_{AB} \quad (i=e, d, \nu_e, u)$ for the gauge eigenstates!

Next we want to bring this into diagonal form with positive entries on the diagonal, thereby switching to the mass eigenstates. For an arbitrary complex matrix $M^{(i)}$ we know from linear algebra that

$$\exists_{T_i \text{ unitary}} s_i M^{(i)} T_i = M_{\text{diag}}^{(i)} = \text{diagonal matrix with } (M_{\text{diag}})^{(i)}_{AA} \geq 0. \quad \text{no summation}$$

Consequence: gauge eigenstates contain a mixture of mass eigenstates

$$\left. \begin{array}{l} u_{AL}^i = (S_u)_{AB} u_{BL} \text{ etc.} \\ \bar{u}_{AR}^i = (T_u)_{AB} \bar{u}_{BR} \text{ etc.} \end{array} \right\} \Rightarrow \bar{u}_{AL}^i M_{AB}^{(u)} u_{BR}^i = \bar{u}_{AL}^i (S_u T_u)_{AB} u_{BR}^i = \bar{u}_{AL}^i (M_{\text{diag}})^{(u)}_{AB} u_{BR}^i \text{ etc.}$$

mass eigenstates Property diagonalized

Next we check whether the electroweak interactions can mix the generations (flavours).

Flavour-changing mixing effects: we start at lowest-order level

- No lowest order flavour-changing neutral currents (FCNC):

$$\bar{u}_{A_L}^i \gamma_\mu u_{A_L}^j = \bar{u}_{A_L}^i \gamma_\mu (\underbrace{S_u S_u^\dagger}_{I_n})_{AB} u_{B_L}^j = \bar{u}_{A_L}^i \gamma_\mu u_{A_L}^j \text{ etc., and similarly for } R \text{ modes.}$$

- Mixing does occur in the charged-current (CC) interactions:

$$\bar{u}_{A_L}^i \gamma_\mu d_{A_L}^j = \bar{u}_{A_L}^i \gamma_\mu (\underbrace{S_u S_d^\dagger}_{I_n})_{AB} d_{B_L}^j \equiv \bar{u}_{A_L}^i \gamma_\mu (V_{CKM})_{AB} d_{B_L}^j,$$

V_{CKM} = unitary quark-mixing matrix (Cabibbo - Kobayashi - Maskawa, 1973)

↳ Nobel Prize 2008

$$= \begin{pmatrix} V_{ud} & V_{us} & \dots \\ V_{cd} & V_{cs} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \quad \leftarrow \begin{array}{l} \text{can be assigned to the down-type quarks,} \\ \dots \text{but there is no strong need for that} \end{array}$$

A similar thing can be done in the lepton sector, although that is not seen as a standard procedure in many texts on the Standard Model. We will come back to this.

How many independent parameters will determine V_{CKM} , bearing in mind that some phases can be absorbed into a redefinition of the quark fields?

Concept: $\psi \rightarrow e^{i\phi} \psi \Rightarrow \pi \psi = i\psi^\dagger \rightarrow e^{-i\phi} \pi \psi$ and therefore all fundamental anticommutation relations stay the same \Rightarrow physics unaltered!

In expressions like $\bar{u}_{A_L}^i \gamma_\mu (V_{CKM})_{AB} d_{B_L}^j$, one phase can be absorbed for each quark flavour, with the exception of an overall phase. Assuming V_{CKM} to be a complex $n \times n$ matrix, the d.o.f. counting goes as follows:

$2n^2$ d.o.f. $\xrightarrow{V^\dagger V = I_n}$ $2n^2 - n^2 = n^2$ d.o.f. $\xrightarrow{\text{absorb unphysical quark phases}}$ $n^2 - (2n-1) = (n-1)^2$ d.o.f., out of which $\binom{n}{2}$ d.o.f. correspond to real n -dim. rotations (i.e. angles) and $(n-1)^2 - \binom{n}{2} = \frac{1}{2}(n-1)(n-2)$ d.o.f. correspond to independent physical phases.

Hence:

- for $n=2$ one rotation (Cabibbo) angle determines V_{CKM} , which is effectively real since all phases are absorbable;
- for $n=3$ three rotation angles and one unabsorbable phase determine V_{CKM} , opening up the possibility that $V_{CKM} \notin \mathbb{R}$

\Rightarrow at least three generations are needed for having $V_{CKM} \notin \mathbb{R}$ in the CC interactions!

The experimental results on V_{CKM} can be summarized in the Wolfenstein parametrization:

$$V_{CKM} \approx \begin{pmatrix} 1 - \lambda^2/2 & \lambda & A\lambda^3(\varphi - i\beta) \\ -\lambda & 1 - \lambda^2/2 & A\lambda^2 \\ A\lambda^3(1 - \rho - i\beta) & -A\lambda^2 & 1 \end{pmatrix},$$

good approximation

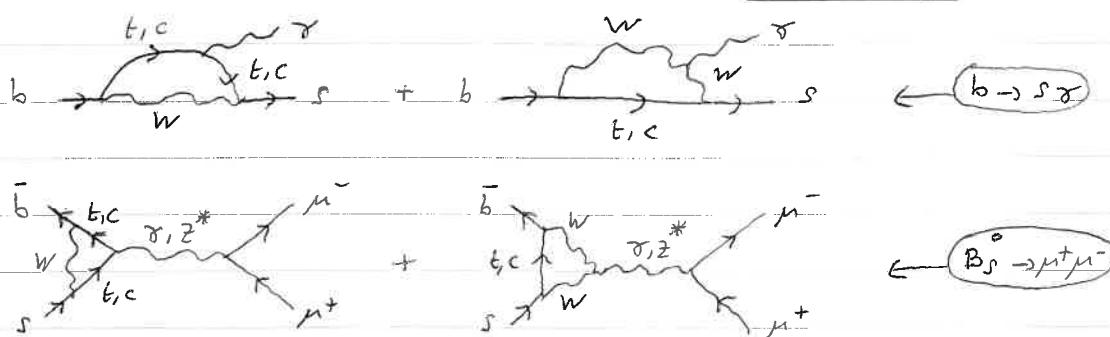
$$\text{in fact: } |V_{CKM}^{\text{exp.}}| \approx \begin{pmatrix} 0.97 & 0.23 & 0.004 \\ 0.23 & 0.97 & 0.04 \\ 0.009 & 0.04 & 1.00 \end{pmatrix}_{AB}$$

with $\lambda = 0.226 \pm 0.001$, $A = 0.81 \pm 0.02$, $\varphi = 0.14 \pm 0.02$, $\beta = 0.35 \pm 0.02$.

↳ pronounced hierarchy among components?

The mixing in the quark sector gives rise to some interesting phenomena.

* FCNC occur in higher loop order, inducing rare decays such as $b \rightarrow s\gamma$ and $B_s^0 \rightarrow \mu^+ \mu^-$:



The associated amplitudes are small due to the GIM mechanism (Glashow-Iliopoulos-Maiani, 1970):

$$\begin{aligned} & \text{shorthand for } V_{CKM} \\ & \propto \sum_j V_{kj}^+ V_{ji}^* \bar{u}_k \gamma_\mu (I_4 - \gamma^5) \frac{p_1 + m_j}{p_1^2 - m_j^2} \gamma_\mu \frac{p_2 + m_i}{p_2^2 - m_i^2} \gamma_\nu (I_4 - \gamma^5) u_i \\ & = \sum_j \frac{2 V_{kj}^+ V_{ji}^*}{(p_1^2 - m_j^2)(p_2^2 - m_i^2)} \bar{u}_k \gamma_\mu [p_1 \gamma_\mu p_2 + m_j^2] \gamma_\nu (I_4 - \gamma^5) u_i \end{aligned}$$

⇒ suppression factors $V_{kj}^+ V_{ji}^*$ for $k \neq i$, $g^2 m_j^2 / m_w^2$ if j light, and $g^2 m_i^2 / m_z^2$ if $j = G$.

needed: additional j dependence in order to avoid that $\sum_j V_{kj}^+ V_{ji}^* = \delta_{ki}$ (which would mean no FCNC)

Such FCNC processes put severe constraints on physics beyond the SM: models beyond the SM might give rise to additional generational mixing, involving new non-SM particles that feature in the FCNC loop diagrams
⇒ constraints on the associated mixing matrices and/or the masses of the new particles of the model, in order not to exceed the amount of FCNC effects that we observe in experiment!