Standard Model and Beyond Exercises week 6

Exercise 10: How many generators do you see?

Consider the Lie group SU(N) with elements $g = \mathrm{e}^{i\alpha^a T^a}$, where α^a are real parameters and T^a are the generators of the associated Lie algebra with fundamental commutation relations $\left[T^a, T^b\right] = i f^{abc} T^c$. In the defining (fundamental) representation the group elements are given by $N \times N$ matrices U, with $UU^\dagger = 1$ and $\det U = 1$.

(a) Show that T^a is hermitian and traceless. You might need that for a matrix M it holds that $\det M = \exp\left(\mathrm{Tr}\left(\log M\right)\right)$.

(b) Argue that SU(N) has $N^2 - 1$ independent generators.

(c) Prove that the structure constants f^{abc} are real.

Consider the subgroup SO(N) $\subset$ SU(N), with group elements that are given by $N \times N$ matrices O in the fundamental representation. These matrices satisfy $OO^T = 1$.

(d) Deduce that T^a is purely imaginary and that SO(N) has $\frac{1}{2}N(N-1)$ independent generators.

Each generator is linked to a gauge field in the lagrangian of a local gauge theory. Knowing the number of generators for a group therefore immediately tells you the number of gauge fields (and therefore the number of new particles!) that you get when implementing that group in a local gauge theory.

Exercise 11: Why scaled couplings are problematic in the non-abelian case

Consider the local SU(N) gauge theory for $N > 1$, as described in the lecture notes.

(a) Why is the QED-like charge scaling $\alpha^a(x) \to Q\alpha^a(x)$, $g \to Qg$ not an effective means of changing the interaction strength?
Hint: You have to demand that after charge scaling W^a_μ transforms in the same way as before charge scaling. Only then one and the same gauge field could mediate the SU(N) interaction between differently charged particles.

(b) Consider the subset of global SU(N) gauge transformations, for which $\mathcal{L}_{\mathrm{SU}(N)}$ is invariant. Derive the conserved Noether currents $J^{a,\mu}(x)$ for each of the independent global transformations labeled by $\alpha^a \in \mathbb{R}$ and link the combination $gW^a_\mu(x)J^{a,\mu}(x)$ to the three types of interactions contained in the SU(N) gauge theory.

Not being able to change the coupling strength is a strong statement! It immediately tells you that the coupling strength between the gauge bosons and other particles is fixed, given by the theory.