Interaction picture: in order to make efficient use of the time evolution of the system, we switch to the interaction picture. In that case states only evolve due to the interaction term $\hat{V}$ in the Hamilton operator, i.e. the asymptotic free states do not change: (taking the pictures to coincide at time t_-)

$$|\psi_I(t)\rangle = \hat{U}_I(t,t_-)\,|\psi_I(t_-)\rangle = \underbrace{e^{i\hat{H}_0(t-t_-)}}_{\text{removes free evolution!}}\,e^{-i\hat{H}(t-t_-)}\,|\psi_I(t_-)\rangle .$$

$\hat{H}_0$ and $\hat{H}$ are time-independent

The time-evolution operator in the interaction picture satisfies the differential equation $\quad i\frac{\partial}{\partial t}\hat{U}_I(t,t_-) = \hat{V}_I(t)\,\hat{U}_I(t,t_-)$, with $\hat{U}_I(t_-,t_-) = \hat{1}$

$\hat{V}_I(t) = e^{i\hat{H}_0(t-t_-)}\hat{V}e^{-i\hat{H}_0(t-t_-)}$ involving free fields only

$$\Rightarrow \underbrace{\int_{t_-}^{t} dt_1 \frac{\partial}{\partial t_1}\hat{U}_I(t_1,t_-)}_{LHS} = \hat{U}_I(t,t_-) - \hat{1} = \underbrace{-i\int_{t_-}^{t} dt_1\, \hat{V}_I(t_1)\,\hat{U}_I(t_1,t_-)}_{RHS} .$$

This results in the integral equation $\hat{U}_I(t,t_-) = \hat{1} - i\int_{t_-}^{t} dt_1\, \hat{V}_I(t_1)\,\hat{U}_I(t_1,t_-)$, which can be solved iteratively

$$\hat{U}_I(t,t_-) = \hat{1} - i\int_{t_-}^{t} dt_1\,\hat{V}_I(t_1) + (-i)^2\int_{t_-}^{t} dt_1\int_{t_-}^{t_1} dt_2\,\overbrace{\hat{V}_I(t_1)\,\hat{V}_I(t_2)}^{\text{ordered in time}} + \ldots .$$

Here the product $\hat{V}_I(t_1)\hat{V}_I(t_2)$ is duly ordered in time (i.e. $t_1 \geq t_2$). This so-called Dyson series can be used as perturbative expansion around the free case in terms of powers of the interaction $\hat{V}$. It can be written as a time-ordered exponential:

$$\boxed{\hat{U}_I(t,t_-) = T\left(e^{-i\int_{t_-}^{t} dt'\,\hat{V}_I(t')}\right) = \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{t_-}^{t} dt_1 \cdots \int_{t_-}^{t} dt_n\, T\left(\hat{V}_I(t_1)\cdots\hat{V}_I(t_n)\right)} .$$

time ordering from late to early

Transition amplitudes revisited: the transition amplitudes mentioned on the previous page simply reflect the time evolution of the initial state on the time interval $(t_-, t_+) \to (-\infty, +\infty)$:

$$S_{fi} = \langle f|\hat{U}_I(t_+,t_-)|i\rangle \equiv \langle f|\hat{S}|i\rangle = \langle f|(\hat{1} + i\hat{T})|i\rangle .$$

S-matrix (scattering matrix) $\qquad$ transition operator

The part involving the transition operator is the one we are interested in, since for decays obviously $|f\rangle \neq |i\rangle$ and the same holds for scattering as detectors will not overlap with the beam pipe:

$\langle f|i\hat{T}|i\rangle \equiv (2\pi)^4 \underbrace{\delta^{(4)}(P_{in}-P_{out})}_{\text{4-momentum conservation in terms of on-shell (free) initial/final-state momenta}}\, i\mathcal{M}(i\to f)$, with $\mathcal{M}$ called the matrix element.

Ingredients of the matrix elements:
- *) free-particle initial/final states,
- *) free fields in $\hat{U}_I(t_+,t_-)-\hat{1}$,
- *) time ordering in $\hat{U}_I(t_+,t_-)-\hat{1}$.

Let's discuss these ingredients and their consequences without going too deep into their QFT origin.

Free theories: the equations of motion of free theories are linear wave equations, with all quantum fields as well as their individual components satisfying the KG equation. This is needed in order to implement particle-wave duality in the right way by giving rise to the correct relation between energy and momentum for the free particles described by the free theory

$\Rightarrow$ solutions to the eqns. of motion of free theories can be expanded in plane-wave modes and polarization vectors (which span spin space): $\boxed{\text{on-shell: } p^2=m^2}$

$v(p,\lambda)\, e^{-ip\cdot x}$ $\leftrightarrow$ positive-energy solutions with $p^\mu = (\sqrt{\vec{p}^{\,2}+m^2}, \vec{p}) \equiv (E_{\vec{p}}, \vec{p})$ and polarization mode labeled by λ $\leftrightarrow$ particles,

$v_c(p,\lambda)\, e^{+ip\cdot x}$ $\leftrightarrow$ negative-energy solutions $\leftrightarrow$ antiparticles

to guarantee that the vacuum (0-particle) state $|0\rangle$ has the lowest energy, the energy spectrum should be bounded from below
$\Rightarrow$ in the negative-energy solutions the quantum numbers should be inverted when switching to the particle picture!

As such, free fields have the following generic form:

$$\hat{F}(x) = \sum_\lambda \int \frac{d\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \Big(v(p,\lambda)\,\hat{a}(\vec{p},\lambda)\, e^{-ip\cdot x} + v_c(p,\lambda)\,\hat{a}_c^\dagger(\vec{p},\lambda)\, e^{+ip\cdot x} \Big)\Big|_{p^0=E_{\vec{p}}},$$

with $\hat{a}^\dagger(\vec{p},\lambda)$ creating (adding) the particle modes, $\hat{a}(\vec{p},\lambda)$ annihilating (removing) them, $\hat{a}_c^\dagger(\vec{p},\lambda)$ creating the antiparticle modes and $\hat{a}_c(\vec{p},\lambda)$ annihilating them. For bosons (B) / fermions (F) these operators satisfy commutation/anticommutation relations:

(B) $[\hat{a}(\vec{p},\lambda), \hat{a}^\dagger(\vec{p}\,',\lambda')] = [\hat{a}_c(\vec{p},\lambda), \hat{a}_c^\dagger(\vec{p}\,',\lambda')] = (2\pi)^3 \delta_{\lambda\lambda'}\delta(\vec{p}-\vec{p}\,')\,\hat{1}$, all other commutators 0

(F) $\{\hat{a}(\vec{p},\lambda), \hat{a}^\dagger(\vec{p}\,',\lambda')\} = \{\hat{a}_c(\vec{p},\lambda), \hat{a}_c^\dagger(\vec{p}\,',\lambda')\} = (2\pi)^3 \delta_{\lambda\lambda'}\delta(\vec{p}-\vec{p}\,')\,\hat{1}$, all other anticommutators 0

The corresponding adjoint field is given by

$$\hat{\bar{F}}(x) = \sum_{\lambda} \int \frac{d\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \Big(\bar{v}_c(p,\lambda)\, \hat{a}_c(\vec{p},\lambda)\, e^{-ip.x} + \bar{v}(p,\lambda)\, \hat{a}^{\dagger}(\vec{p},\lambda)\, e^{+ip.x} \Big)\Big|_{p^0 = E_{\vec{p}}}$$

← role of particles and antiparticles are interchanged w.r.t. $\hat{F}(x)$

Polarization vectors : scalars $\rightarrow$ $v = \bar{v} = v_c = \bar{v}_c = 1$

Dirac fermions $\rightarrow$ $v = u^{\lambda}(p)$ with $p_{\mu}\gamma^{\mu} u^{\lambda}(p) \equiv \not{p}\, u^{\lambda}(p) = m u^{\lambda}(p)$

spinors $\rightarrow$
$$\begin{cases} v_c = v^{\lambda}(p) \text{ with } \not{p}\, v^{\lambda}(p) = -m v^{\lambda}(p) \\ \bar{v} = \bar{u}^{\lambda}(p) = u^{\lambda\dagger}(p)\gamma^0 \text{ with } \bar{u}^{\lambda}(p)\not{p} = m\bar{u}^{\lambda}(p) \\ \bar{v}_c = \bar{v}^{\lambda}(p) = v^{\lambda\dagger}(p)\gamma^0 \text{ with } \bar{v}^{\lambda}(p)\not{p} = -m \bar{v}^{\lambda}(p) \end{cases}$$

Gauge bosons $\rightarrow$ $v = \bar{v}_c = \epsilon^{\lambda}_{\mu}(p)$, $\bar{v} = v_c = \epsilon^{\lambda *}_{\mu}(p)$,
with $p^{\mu}\epsilon^{\lambda}_{\mu}(p) = 0$ — polarization vector

Extra remark: a massless gauge boson has one d.o.f. less, which results in the extra condition $n^{\mu}\epsilon^{\lambda}_{\mu}(p) = 0$ for $n^{\mu} = (1, \vec{0})$. In this way only the transverse d.o.f. remain: $\epsilon^{\lambda}_0(p) = 0$ and $\vec{p}\cdot\vec{\epsilon}^{\,\lambda}(p) = 0$.

Particle states and vacuum in the free theory : the free-particle state corresponding to a particle with on-shell 4-momentum p^{μ} and polarization mode λ is given by

$$|\vec{p},\lambda\rangle \equiv \sqrt{2E_{\vec{p}}}\; \hat{a}^{\dagger}(\vec{p},\lambda)\,|0\rangle \, .$$

Lor. inv. normalization : $\langle \vec{p}\,',\lambda' | \vec{p},\lambda\rangle = 2E_{\vec{p}}(2\pi)^3 \delta_{\lambda\lambda'}\delta(\vec{p}-\vec{p}\,')$

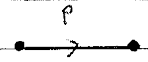
For the antiparticle state $\hat{a}^{\dagger}(\vec{p},\lambda)$ should be simply replaced by $\hat{a}^{\dagger}_c(\vec{p},\lambda)$. The state $|0\rangle$ in these expressions is the vacuum (0-particle) state. This state has the obvious property $\hat{a}(\vec{p},\lambda)|0\rangle = \hat{a}_c(\vec{p},\lambda)|0\rangle = 0$ ("annihilating the vacuum") since particles that are not present in a state cannot be annihilated. Similarly, $\langle 0|\,\hat{a}^{\dagger}(\vec{p},\lambda) = \langle 0|\,\hat{a}^{\dagger}_c(\vec{p},\lambda) = 0$.

Feynman diagrams and rules for $iM(i\to f)$: the latter aspect forms the basis for deriving the non-vanishing contributions to $\langle f|(\hat{S}-\hat{1})|i\rangle$ --- only paired creation and annihilation operators (i.e. referring to the same particles and quantum numbers) give rise to non-vanishing (anti) commutators.
Any unpaired creation/annihilation operator can be (anti) commutated past all other operators all the way to the left/right, where it will annihilate the vacuum. If we indicate paired operators by a line, the non-vanishing contributions can be represented by all possible continuous drawings (Feynman diagrams). The rules for associating analytic expressions with specific pieces of diagrams are called the Feynman rules of the theory.

Momentum-space Feynman rules: $i\mathcal{M}(i\to f)$ = sum of all Feyman diagrams.

- Propagators: a line that connects a free field and an adjoint free field in the expansion of $\hat{S}-\hat{1}$ is called an internal line or propagator

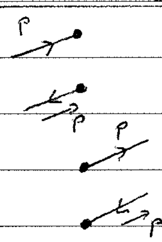
pole in on-shell case $p^2=m^2$ | projects on polarization ~~vectors in~~ on-shell case

↳ drawing:  , rule: $\dfrac{i}{p^2-m^2+i\epsilon}\sum_\lambda v(p,\lambda)\bar{v}(p,\lambda)$,

with $\sum_\lambda v(p,\lambda)\bar{v}(p,\lambda) = \begin{cases} 1 & \text{(scalars)} \\ \not{p}+m & \text{(Dirac fermions)} \\ -g_{\mu\nu}+p_\mu p_\nu/m^2 & \text{(massive gauge bosons)} \\ -g_{\mu\nu}+\dfrac{p_\mu n_\nu+p_\nu n_\mu}{p.n}-\dfrac{p_\mu p_\nu}{(p.n)^2} & \text{(massless gauge bosons)} \end{cases}$.

Key ingredient: the scalar case $\langle 0|T(\hat{F}(x)\hat{\bar{F}}(y))|0\rangle = \int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}e^{-ip.(x-y)}$.

- External lines: a line that connects the initial or final state to a free field in the expansion of $\hat{S}-\hat{1}$ is called an external line

$|i\rangle$	drawing	rule		
incoming particle :		$v(p,\lambda)$	← $\hat{a}$ in $\hat{F}$	all on-shell, i.e. $p^0=E_{\vec{p}}$
incoming antiparticle :		$\bar{v}_c(p,\lambda)$	← $\hat{a}_c$ in $\hat{\bar{F}}$	
outgoing particle :		$\bar{v}(p,\lambda)$	← $\hat{a}^\dagger$ in $\hat{\bar{F}}$	
outgoing antiparticle :		$v_c(p,\lambda)$	← $\hat{a}_c^\dagger$ in $\hat{F}$	
$|f\rangle$				

- Vertices: each interaction (involving more than two fields) that occurs in the expansion of $\hat{S}-\hat{1}$ is called a vertex and is indicated by a dot (e.g. the dots connected by the propagators mentioned above).

↳ drawing: • , rule: depends on the theory considered (an example is given below)

Additional Feynman rule: energy and momentum are conserved at each vertex, due to $\int\frac{d^4x_n}{(2\pi)^4}e^{-ix_n.(p_1+\cdots+p_K)}=\delta^{(4)}(p_1+\cdots+p_K)$.

origin: $\int_{t_-}^{t_+}dt_n\,\hat{V}_I(t_n) = \int d^4x_n\,\mathcal{H}_I^{int}(x_n)$

- Some (internal) momenta are not fixed by energy-momentum conservation, these momenta are called loop momenta ⟹ Feynman rule: $\int\frac{d^4l}{(2\pi)^4}$ (l = loop momentum).

integral to be performed

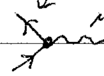
- Arrow convention: arrows are used to indicate the direction of flow of particles, with antiparticles going against the flow.
  ↳ additional rule for fermions: insert γ-matrices and spinors while going against the arrow!

- Fermionic minus signs: *) diagrams with two identical fermions interchanged differ by a minus sign (due to Fermi statistics) ← e.g. for $e^-e^- \to e^-e^-$
  *) each closed loop of fermion propagators (fermion loop) receives a minus sign and involves a trace in spinor space.
  ↑ no open indices

- Symmetry factor: to avoid double counting, each diagram should be divided by the so-called symmetry factor, i.e. the number of ways in which diagram components can be interchanged without changing the actual diagram
  ↑ this only happens for interactions involving identical fields ⇒ not needed for QED!

Additional Feynman rules specific for QED: — for fermions, ∿ for photons
↑ e.g. $e^\pm$ ↑ γ

continuous flow ↓ matrix in spinor space

QED vertex:  $= -iq\gamma^\mu \longleftrightarrow \mathcal{L}^{int}_{QED} = -\mathcal{H}^{int}_{QED} = -q\bar{\psi}\gamma^\mu\psi A_\mu$ for Dirac fermions with charge q.

Photon propagator:  $= \dfrac{-i}{p^2+i\epsilon}\left(g_{\mu\nu} - \dfrac{p_\mu n_\nu + p_\nu n_\mu}{p\cdot n} + \dfrac{p_\mu p_\nu}{(p\cdot n)^2}\right) \xrightarrow{\text{effectively}} \dfrac{-ig_{\mu\nu}}{p^2+i\epsilon}$.
↑ consequence of gauge inv.: photon couples to conserved currents

From transition amplitudes to transition rates.

① Decay: we start with the probability density for the decay of particle A with momentum $\vec{k}_A$ to a final state consisting of n particles with momenta $\vec{p}_1, \dots, \vec{p}_n$, which reads

$$\frac{|\langle \vec{p}_1 \cdots \vec{p}_n | i\hat{T} | \vec{k}_A\rangle|^2}{\langle \vec{k}_A|\vec{k}_A\rangle\langle \vec{p}_1\cdots\vec{p}_n|\vec{p}_1\cdots\vec{p}_n\rangle} = \frac{|\mathcal{M}(i\to f)|^2 \overbrace{\left((2\pi)^4\delta^{(4)}(k_A - \textstyle\sum_j p_j)\right)^2}^{(2\pi)^4\delta^{(4)}(k_A-\sum_j p_j)\,*\,(2\pi)^4\delta^{(4)}(0)}}{2E_{\vec{k}_A}(2\pi)^3\delta(\vec{0})\prod_{j=1}^n\left(2E_{\vec{p}_j}(2\pi)^3\delta(\vec{0})\right)}$$

$$= \frac{|\mathcal{M}(i\to f)|^2}{2E_{\vec{k}_A}V}(2\pi)^4\delta^{(4)}(k_A - \sum_j p_j)\frac{VT}{\prod_{j=1}^n(2E_{\vec{p}_j}V)}.$$
← linear in the time span T (cf. Fermi's Golden Rule)

used: $(2\pi)^3\delta(\vec{0}) = \lim_{\vec{p}\to\vec{0}}\int_V d\vec{x}\, e^{i\vec{p}\cdot\vec{x}} = V$ and similarly $(2\pi)^4\delta^{(4)}(0) = VT$ with $T = \int_{t_-}^{t_+} dt$.

The differential decay rate for decay into an n-particle final state with momenta in the bin $d\vec{p}_1 \cdots d\vec{p}_n$ around $\vec{p}_1, \ldots, \vec{p}_n$ is obtained by dividing the previous result by T and multiplying by the density of states:

$$\boxed{\begin{aligned} d\Gamma_n &= \text{previous result} * \frac{\prod_{j=1}^{n} \left(V \frac{d\vec{p}_j}{(2\pi)^3}\right)}{T} = \frac{|\mathcal{M}(i\to f)|^2}{2E_{\vec{k}_A}} d\Pi_n , \\ \text{with } d\Pi_n &= \prod_{j=1}^{n} \left(\frac{d\vec{p}_j}{(2\pi)^3} \frac{1}{2E_{\vec{p}_j}} \right) (2\pi)^4 \delta^{(4)}(k_A - \textstyle\sum_j p_j) \text{ the n-body phase-space element} \end{aligned}}$$

Lor. inv.

frame dependent

↳ Total partial decay rate : $\Gamma_n = C_f \int d\Gamma_n$, with C_f taking into account that identical particles are indistinguishable.

Total decay rate : $\Gamma = \Sigma$ all partial decay rates $= 1/\tau$, with τ the lifetime of the decaying particle.

Remark : mostly the decay rate (width) Γ is calculated in the rest frame of the decaying particle ($\Rightarrow E_{\vec{k}_A} = m_A$ in that case).

② Scattering : we can switch quite easily to the transition rate for the scattering process of two initial state particles A and B with momenta $\vec{k}_A$ and $\vec{k}_B$ producing the (binned) n-particle final state considered above. To this end we simply have to divide by the normalization of the $|\vec{k}_B\rangle$ state $\Rightarrow$ factor $1/2VE_{\vec{k}_B}$.

The corresponding differential cross section then reads

effective area of a chunk taken out of the beam by each target particle

$$d\sigma = \frac{\text{scattering rate}}{N_A * \text{beam flux}} \equiv \frac{\dot{N}_S}{\mathcal{L}} ,$$

with N_A = # target particles relevant for scattering
beam flux F = flux of particles $\perp$ beam
$\Big\} \to$ luminosity $\mathcal{L}$

Typically we calculate this for a single target particle (i.e. $N_A = 1$) and a single beam particle [in a fixed-target set-up] or one particle in each beam [in a beam-on-beam set-up]

$\Rightarrow$ take $N_A = 1$ and $F = \dfrac{|\vec{v}_B - \vec{v}_A|}{V} = \dfrac{|\vec{k}_A/E_{\vec{k}_A} - \vec{k}_B/E_{\vec{k}_B}|}{V}$,

$$\boxed{d\sigma = \frac{|\mathcal{M}(i\to f)|^2}{4|E_{\vec{k}_B}\vec{k}_A - E_{\vec{k}_A}\vec{k}_B|} d\Pi_n , \text{ with the "flux factor" } 4|E_{\vec{k}_B}\vec{k}_A - E_{\vec{k}_A}\vec{k}_B| \text{ inv. under boosts along the beam direction}}$$

↳ Total cross section $\sigma = C_f \int d\sigma$ (see above).