Reader for the course Quantum Field Theory

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Contents of the lecture course:

1) The Klein-Gordon field
2) Interacting scalar fields and Feynman diagrams
3) The Dirac field
4) Interacting Dirac fields and Feynman diagrams
5) Quantum Electrodynamics (QED)

This reader should be used in combination with the textbook

"An introduction to Quantum Field Theory" (Westview Press, 1995)
by Michael E. Peskin and Daniel V. Schroeder.

Contents

1 The Klein-Gordon field

The first four lectures cover Chapter 2 of the textbook by Peskin & Schroeder. The relevant conventions are listed on pages xix−xxi in the book, involving the use of so-called natural units ($\hbar = c = \mu_0 = \epsilon_0 = 1$) by absorbing these constants in the relevant fields and quantities. As a result, a single scale remains: mass. Please familiarize yourself with these conventions and treat Chapter 1 as reading material, as recommended by the authors.

Throughout this reader you will encounter circled numbers. These numbers match the markers listed in the course's storyline (`http://www.hef.ru.nl/~wimb/QFT_story.pdf`).

1.1 Arguments in favour of Quantum Field Theory

From particle−wave duality we know that the properties of e.g. electrons and photons are similar: both objects give rise to diffraction phenomena and carry a particle-like punch. Historically electromagnetism was first perceived as a field theory and its particle interpretation (photons) was observed later through the photo-electric effect. The other way around, electrons were first perceived as elementary particles and the field aspects emerged only once relativistic energies were considered.

> 1 *Question: what is more fundamental, the fields (with particles being derived quantities resulting from quantization) or the particles (with the fields being derived quantities resulting from collective many-particle behaviour)?*

There are four observations that support the former point of view.

1. Classical physics: as supported by experiment there should be no "action at a distance", i.e. there should be no forces that are felt everywhere instantaneously. As a result, the instantaneous laws of Newton and Coulomb had to be replaced by the local laws of nature of Einstein and Maxwell, based on field theories! ... However, strictly speaking a locally defined particle approach is still possible.

2. Relativistic quantum mechanics: as supported by any high-energy collision experiment a relativistic one-particle quantum theory is not feasible. The number of particles is not conserved, i.e. particles are not indestructible. This differs strongly from non-relativistic quantum mechanics as formulated by Schrödinger, where massive particles are around forever and can thus be perceived as fundamental. Photons are massless and are therefore always to be treated relativistically, so we have no photon conservation.

Let's recall what happened when we were trying to construct a relativistic quantum mechanical theory for a free particle in flat (Minkowskian) spacetime. The ingredients for the construction were:

- A wave equation that keeps its form under Lorentz-transformations, as required by the relativity principle.

- A correct quantum mechanical probability interpretation.

- The relativistic relation $E = \sqrt{\vec{p}^{\,2} + m^2}$ should be built in, in order to ensure that particle–wave duality is properly incorporated.

The following problems were encountered:

- Negative-energy solutions, leading to an energy spectrum that is unbounded from below. Dirac solved this for fermionic theories by demanding that the sea of negative-energy states (Dirac sea) is occupied. Unwanted transitions are then forbidden provided that the exclusion principle applies, which is the case for fermions. However, that means that the resulting one-particle theory has in fact an infinite number of particles.

- At energies of the order of the particle mass, extra particles can be liberated from the Dirac sea. In Dirac's theory this is called particle–hole creation, which corresponds to particle–antiparticle pair creation in quantum field theory.

In order to see at what length scales the breakdown of one-particle quantum mechanics occurs we use the old units for a moment and consider a particle with mass m in a box with size L. According to Heisenberg's uncertainty relation the momentum of the particle then has an uncertainty $\Delta p = \mathcal{O}(\hbar/L)$. This in turn leads to an uncertainty in the relativistic energy $E = \sqrt{p^2c^2 + m^2c^4} \geq pc$ of roughly $c\Delta p = \mathcal{O}(\hbar c/L)$. If this energy uncertainty exceeds the energy threshold $2mc^2$ then pair creation may occur. This happens at length scales $L \leq \mathcal{O}(\lambda_c)$, with $\lambda_c = \hbar/(mc)$ the Compton wavelength. At these length scales we cannot say anymore that we are dealing with a single particle, since it is accompanied by a swarm of particle–antiparticle pairs, and a description with an unspecified number of particles is required! Note that the Compton wavelength is smaller than the de Broglie wavelength $\lambda_b = h/p$, which is the length scale where the wave-like nature of particles becomes apparent.

> 1 *The Compton wavelength is the length scale where even the concept of a single point-like particle breaks down.*

So, if we were to use a particle approach that is defined locally, it cannot be a single-particle approach since multi-particle objects will unavoidably feature.

3. Many-particle quantum mechanics: the particle interpretation of a quantum mechanical theory can change radically in a different physical environment (cf. particles becoming waves in low-temperature superfluid ^{4}He, coherent states in a driven oscillator system, ...). That means that the nature of particles can change!

4. The observation that all particles of the same type and in the same physical setting are always the same everywhere. This hints at a description of physics that spans all of space and time.

1.2 Lagrangian and Hamiltonian formalism (§ 2.2 in the book)

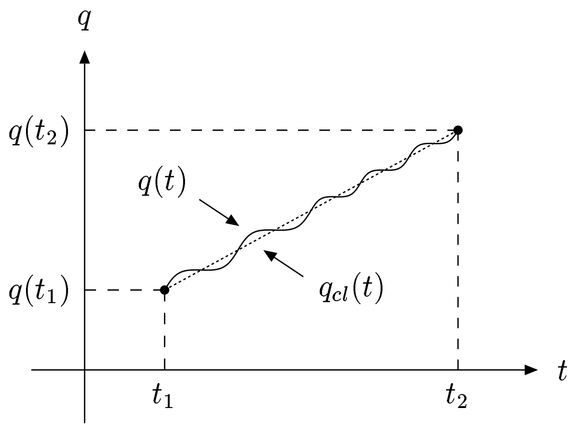
2a *In order to set up quantum field theory we first consider classical field theory in the Lagrangian and Hamiltonian formalism. The philosophy behind this is that wave equations can be viewed as equations of motion for the wave functions, i.e. the fields. This is best formulated in terms of Lagrangians for continuous systems. Such Lagrangians are particularly suitable for discussing symmetries, the cornerstones of relativistic quantum field theory.*

Classical Lagrangian formalism: for a finite number of degrees of freedom the Lagrangian is given by

$$L\big(\{q_j(t)\}, \{\dot{q}_j(t) = \mathrm{d}q_j(t)/\mathrm{d}t\}, t\big) \;=\; T - V \;,$$

where q_j are generalized coordinates, T is the kinetic energy and V the potential energy.

Hamilton's variation principle: classical solutions to the equations of motion (classical paths) are obtained by finding the extrema of the action $S = \int_{t_1}^{t_2} \mathrm{d}t\, L$ under synchronous variations of the paths while keeping the endpoints fixed.



Variation around the classical path for a free particle

The condition for a stationary action reads

$$\delta S \;=\; \delta\Big(\int_{t_1}^{t_2} \mathrm{d}t\, L\Big) \;=\; 0 \quad \text{for} \quad q_j(t) \to q_j(t) + \delta q_j(t) \quad \text{such that} \quad \delta q_j(t_1) = \delta q_j(t_2) = 0 \;.$$

From this it follows that

$$\sum_j \int_{t_1}^{t_2} \mathrm{d}t\, \Big(\frac{\partial L}{\partial q_j}\,\delta q_j + \frac{\partial L}{\partial \dot{q}_j}\,\delta \dot{q}_j\Big) \;=\; \sum_j \Big[\frac{\partial L}{\partial \dot{q}_j}\,\delta q_j\Big]_{t=t_1}^{t=t_2} + \sum_j \int_{t_1}^{t_2} \mathrm{d}t\, \Big(\frac{\partial L}{\partial q_j} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_j}\Big)\delta q_j \;=\; 0\,.$$

This has to be true for all δq_j, so from this the Lagrange equations follow:

$$\forall_j \quad \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial L}{\partial \dot{q}_j}\Big) \;=\; \frac{\partial L}{\partial q_j} \; .$$

These are the equations of motion for a system without boundary conditions.

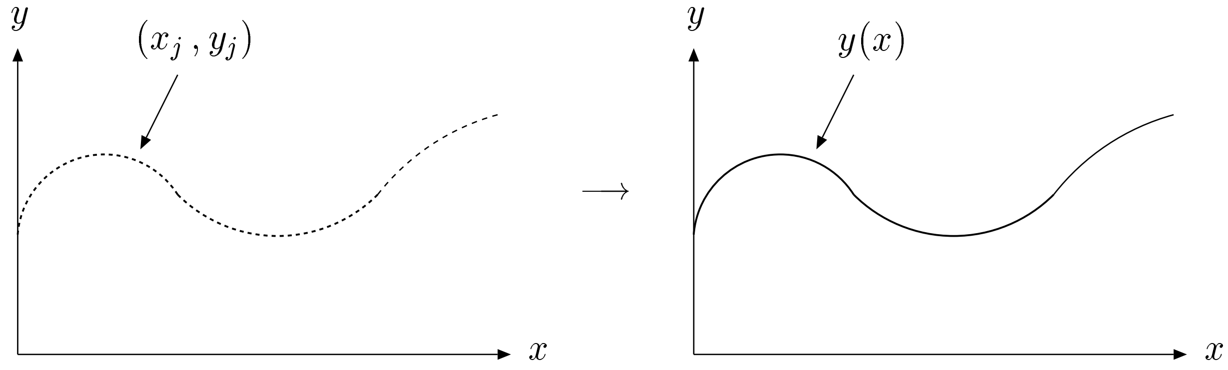


Figure 1: A classical, non-relativistic example of a continuous system.

Now we switch from a discrete set of particles to a field. A field is a dynamical system with a continuous, infinite number of degrees of freedom, i.e. at least one degree of freedom for each point in space. An example is given by the string in figure 1, in which case gradients in x will enter V as elastic energy (see also Ex. 1). In the field-theory case the discrete set of generalized coordinates $\{q_j(t)\}$ is replaced by a continuous generalized coordinate $\phi(x)$, where x is a spacetime four-vector. In this way we treat $\vec{x}$ and t on equal footing, as required for a relativistic approach. The Lagrangian $L\big(\{q_j\},\{\dot{q}_j\},t\big)$ is replaced by a Lagrangian density $\mathcal{L}(\phi(x),\partial_\mu\phi)$, which depends on the generalized cooordinate $\phi(x)$ and the corresponding four-velocity $\partial_\mu\phi(x)$. The fact that the derivates with respect to time and space should be combined into a four-velocity $\partial_\mu\phi(x)$ is needed for a proper relativistic treatment, as we will see later on. In practice we only work with Lagrangian densities, so we usually refer to $\mathcal{L}$ in a sloppy way as 'the Lagrangian'.

Now that we have a Lagrangian, we need to formulate Hamilton's variation principle for continuous systems:

$$\delta S \;=\; \delta\Big(\int_{t_1}^{t_2}\mathrm{d}t\int \mathrm{d}\vec{x}\;\mathcal{L}(\phi,\partial_\mu\phi)\Big) \;\equiv\; \delta\Big(\int_{x_1}^{x_2}\mathrm{d}^4x\;\mathcal{L}(\phi,\partial_\mu\phi)\Big) \;=\; 0$$

$$\text{for }\; \phi(x)\to\phi(x)+\delta\phi(x) \;\text{ such that }\; \delta\phi(x) \xrightarrow{|\vec{x}\,|\to\infty} 0 \;\text{ and }\; \forall_{\vec{x}}\;\delta\phi(t_1,\vec{x}\,) = \delta\phi(t_2,\vec{x}\,) = 0 \; .$$

This means that the system evolves between two field configurations that are kept fixed at the temporal and spatial boundaries of the four-dimensional integration region. The latter requirement follows from the fact that we will consider systems with finite properties only.

From this variation principle it follows that

$$\begin{aligned}\delta S &= \int_{x_1}^{x_2} \mathrm{d}^4x \left\{ \frac{\partial\mathcal{L}}{\partial\phi}\,\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\delta(\partial_\mu\phi) \right\} \\ &= \int_{x_1}^{x_2} \mathrm{d}^4x\; \partial_\mu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\delta\phi \Big) + \int_{x_1}^{x_2} \mathrm{d}^4x \left\{ \frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \Big) \right\} \delta\phi \;=\; 0\end{aligned}$$

for all allowed variations $\delta\phi$. According to Gauss' divergence theorem, the first integral in the final expression vanishes since it gives rise to an integral over the boundary of the four-dimensional integration region. The final result is the so-called Euler–Lagrange equation for a stationary action:

$$\partial_\mu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \Big) \;=\; \frac{\partial\mathcal{L}}{\partial\phi}\;.$$

We get the same equation for each extra field occuring in $\mathcal{L}$.

An immediate consequence of the variation principle is that the equation of motion (Euler–Lagrange equation) does not change if we add a ϕ-dependent four-divergence to the Lagrangian: $\mathcal{L} \to \mathcal{L} + \partial_\mu G^\mu$. The reason is that this extra term adds a boundary contribution to S. Such a boundary contribution remains unaffected by a field variation with fixed boundaries. Note that we have not considered the possibility of having terms in the Lagrangian that couple $\phi(t,\vec{x}\,)$ to $\phi(t,\vec{y}\,)$. This follows from the locality requirement that we have to impose on viable quantum field theories. As a result, only $\phi(x)$ and $\partial_\mu\phi(x)$ occur.

Just like the Lagrangian, the Hamiltonian H in the discrete case becomes an integral of the Hamiltonian density $\mathcal{H}$ in the continuous case:

$$H\big(\{q_j\},\{p_j\}\big) \;\equiv\; \sum_j p_j\dot{q}_j - L \quad\longrightarrow\quad \int \mathrm{d}\vec{x}\;\mathcal{H}(\phi,\vec{\nabla}\phi,\pi) \;\equiv\; \int \mathrm{d}\vec{x}\,\left\{ \pi\,\frac{\partial\phi}{\partial t} - \mathcal{L} \right\}\;,$$

with the conjugate momenta for both cases defined as

$$p_j \;\equiv\; \frac{\partial L}{\partial\dot{q}_j} \quad\longrightarrow\quad \pi \;\equiv\; \frac{\partial\mathcal{L}}{\partial(\partial\phi/\partial t)}\;.$$

Note the preferred treatment of t with respect to $\vec{x}$ in the definition of $\mathcal{H}$: $\partial\phi/\partial t$ occurs in the definition of π. That means that t and $\vec{x}$ are not treated on equal footing in the Hamiltonian formalism, making the Hamiltonian formalism less suitable for dealing with relativistic field theories than the Lagrangian formalism. We will need to know H, though, for performing the quantization of the classical theory.

Example: consider the following Lagrangian containing a set of fields labeled by $a \in \mathbb{N}$

$$\mathcal{L}\big(\{\phi_a\},\{\partial_\mu\phi_a\}\big) \;=\; \frac{1}{2}\dot{\phi}_a^2 - \frac{1}{2}(\vec{\nabla}\phi_a)^2 - \frac{1}{2}m^2\phi_a^2 \;=\; \frac{1}{2}(\partial_\mu\phi_a)(\partial^\mu\phi_a) - \frac{1}{2}m^2\phi_a^2\;.$$

A summation convention is implied here, so $\phi_a^2 = \sum_a \phi_a^2$. Note that according to the standard convention in the book $\partial_\mu \phi_a = (\partial_0 \phi_a, \vec{\nabla}\phi_a)$ and $\partial^\mu \phi_a = (\partial_0 \phi_a, -\vec{\nabla}\phi_a)$. Using Einstein's standard summation convention for repeated Minkowski indices, the Euler–Lagrange equations then read

$$\partial_\mu(\partial^\mu \phi_a) + m^2 \phi_a = (\partial_0^2 - \vec{\nabla}^2 + m^2)\phi_a \equiv (\Box + m^2)\phi_a = 0 ,$$

i.e. all fields ϕ_a satisfy the familiar Klein-Gordon equation. The conjugate momenta and the Hamiltonian density are given by

$$\pi_a = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_a} = \dot{\phi}_a \qquad \text{and} \qquad \mathcal{H} = \dot{\phi}_a \pi_a - \mathcal{L} = \frac{1}{2}\pi_a^2 + \frac{1}{2}(\vec{\nabla}\phi_a)^2 + \frac{1}{2}m^2\phi_a^2 .$$

The first (kinetic) term in the Hamiltonian density corresponds to the energy cost of "moving" in time, the second (elastic) term to the energy cost of "shearing" in space, and the third (mass) term is the energy cost of having the field around at all. Note that in deriving this Hamiltonian we sum over all fields in the term $\dot{\phi}_a \pi_a$. This makes sense, since all fields ϕ_a are independent.

> 2b *Question: apart from being local, what requirements do we have to impose on the Lagrangian density of a relativistic quantum field theory?*

Relativity principle: the guiding principle will be the relativity principle, which states that in each inertial frame the physics should be the same. One option is to use a passive transformation to go from one inertial frame to the other. In that case we have to find a relativistic wave equation that keeps its form under Lorentz transformations: $Df(x) = 0 \;\Rightarrow\; D'f'(x') = 0$, where D is a differential operator and f a field. The prime indicates Lorentz-transformed objects. Alternatively, we can physically transform all fields and demand the relativistic wave equation to be invariant. This is called an active transformation. To phrase it differently, if a field satisfies the equation of motion, then the same should hold for the Lorentz-transformed field:

$$Df(x) = 0 \quad \Rightarrow \quad Df'(x) = 0 .$$

This is automatically guaranteed if the associated Lagrangian density $\mathcal{L}$ is a Lorentz scalar field, since the action S will in that case be Lorentz invariant and therefore an extremum of the action will indeed yield another extremum upon Lorentz transformation. Similar arguments hold for constant translations $x' = x + x_0$, where x_0 is a constant four-vector.

Proof: in order to prove that the action is Lorentz invariant if $\mathcal{L}$ is a Lorentz scalar field, we first give the official definition of a Lorentz scalar field. Consider to this end the Lorentz transformation $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$, with Λ a continuous Lorentz transformation tensor (describing rotations and boosts). Then $\phi(x) \in \mathbb{R}$ is called a Lorentz scalar field if it transforms as $\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x)$ under the Lorentz transformation, i.e. the transformed field evaluated at the transformed spacetime point gives the same value as the original field in the spacetime point prior to the Lorentz transformation. The Jacobian of this transformation is 1, since $\det\Lambda = 1$ for a continuous Lorentz transformation. Therefore, for a Lorentz scalar Lagrangian density $\mathcal{L}$

$$\mathcal{L}(x) \xrightarrow{\text{scalar}} \mathcal{L}'(x) = \mathcal{L}(\Lambda^{-1}x) \equiv \mathcal{L}(y) \quad \Rightarrow$$

$$S = \int \mathrm{d}^4x\, \mathcal{L}(x) \to S' = \int \mathrm{d}^4x\, \mathcal{L}'(x) = \int \mathrm{d}^4x\, \mathcal{L}(y) \xlongequal{x=\Lambda y\,,\,\text{Jacobian}=1} \int \mathrm{d}^4y\, \mathcal{L}(y) = S\ .$$

Note, though, that the endpoints t_1 and t_2 of the temporal integration interval will change under boosts.

1.2.1 Noether's theorem for continuous symmetries

> 3 *As a next ingredient for setting up quantum field theory we will try to identify conserved currents and "charges" that are present in the theory. These conserved charges are instrumental in quantizing the theory and finding its particle interpretation.*

Consider a field $\phi(x)$ that satisfies the Euler–Lagrange equation of $\mathcal{L}(\phi, \partial_\mu\phi)$ and apply the infinitesimal continuous transformation

$$\phi(x) \to \phi'(x) = \phi(x) + \alpha\,\Delta\phi(x)\ , \quad \text{with } \alpha \text{ independent of } x \text{ and infinitesimal}\ .$$

We speak of a symmetry under this transformation if $\mathcal{L}(x)$ changes by a four-divergence: $\mathcal{L}(x) \to \mathcal{L}(x) + \alpha\,\partial_\mu G^\mu(x)$, since that implies that the equation of motion is left invariant (cf. the remark on page 5). In that case

$$\begin{aligned}\alpha\,\partial_\mu G^\mu &= \alpha\,\Delta\mathcal{L} = \alpha\Big(\frac{\partial\mathcal{L}}{\partial\phi}\,\Delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\Delta(\partial_\mu\phi)\Big)\\ &= \alpha\,\partial_\mu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\Delta\phi\Big) + \alpha\left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Big)\right]\Delta\phi\ .\end{aligned}$$

The second term is zero if $\phi(x)$ is a solution to the Euler–Lagrange equation. In that case we are left with

$$\partial_\mu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\Delta\phi - G^\mu\Big) \equiv \partial_\mu j^\mu = 0\ ,$$

i.e. j^μ is a conserved current when expressed in terms of solutions to the Euler–Lagrange equations. This is trivially extended to cases with more fields and automatically leads to

Noether's theorem: for each continuous symmetry there is a conserved current.

This theorem has two important consequences:

- The "charge" $Q(t) = \int \mathrm{d}\vec{x}\; j^0(x)$ is conserved globally if $\vec{j}(x)$ vanishes sufficiently fast for $|\vec{x}| \to \infty$. Proof: if $\vec{j}(x)$ vanishes sufficiently fast for $|\vec{x}| \to \infty$ we have

$$\frac{\mathrm{d}Q(t)}{\mathrm{d}t} \;=\; \int \mathrm{d}\vec{x}\, \frac{\partial j^0}{\partial t} \;\overset{\partial_\mu j^\mu = 0}{=\!=\!=}\; -\int \mathrm{d}\vec{x}\; \vec{\nabla}\cdot\vec{j} \;\overset{\text{Gauss}}{=\!=\!=}\; -\int \mathrm{d}\vec{s}\cdot\vec{j} \;=\; 0\;.$$

- More importantly this charge conservation also holds locally!
Proof: following the previous case

$$\frac{\mathrm{d}}{\mathrm{d}t}\,Q_V(t) \;\equiv\; \frac{\mathrm{d}}{\mathrm{d}t}\int\limits_V \mathrm{d}\vec{x}\; j^0(x) \;=\; -\int\limits_V \mathrm{d}\vec{x}\; \vec{\nabla}\cdot\vec{j} \;\overset{\text{Gauss}}{=\!=\!=}\; -\int\limits_{S(V)} \mathrm{d}\vec{s}\cdot\vec{j}\;.$$

  In other words: any charge leaving the closed volume V must be accounted for by an explicit flow of the current $\vec{j}$ through the surface $S(V)$ of V.

Translation symmetry: from imposing the relativity principle we know that $\mathcal{L}(x)$ should be a Lorentz scalar, so under an infinitesimal translation

$$x^\nu \;\to\; x'^\nu \;=\; x^\nu - \alpha^\nu \quad \text{where } \alpha^\nu \text{ is a constant infinitesimal four-vector}$$

we have

$$\mathcal{L}(x) \;\to\; \mathcal{L}'(x) \;=\; \mathcal{L}(\overbrace{x+\alpha}^{\text{inverse}}) \;\approx\; \mathcal{L}(x) + \alpha^\nu\partial_\nu\mathcal{L}(x) \;=\; \mathcal{L}(x) + \alpha_\nu\partial_\mu\big(g^{\mu\nu}\mathcal{L}(x)\big)\;.$$

The last term is a total four-divergence, so relativistic field theories have translation symmetry with $\big[G^\mu(x)\big]^\nu = g^{\mu\nu}\mathcal{L}(x)$ for all four independent translations labeled by ν. Now suppose that $\mathcal{L}$ depends on an arbitrary collection of fields $f_a(x)$ that transform as

$$f_a(x) \;\to\; f_a(x+\alpha) \;\approx\; f_a(x) + \alpha_\nu\partial^\nu f_a(x) \;\equiv\; f_a(x) + \alpha_\nu\big[\Delta f_a(x)\big]^\nu\;,$$

which is valid for all components of viable quantum fields. For $f_a(x)$ satisfying the Euler–Lagrange equations, this results in four conserved currents:

$$T^{\mu\nu} \;\equiv\; \Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu f_a)}\Big)\partial^\nu f_a - g^{\mu\nu}\mathcal{L} \qquad (\nu = 0,\cdots,3)\;,$$

and hence four conserved charges:

$$\int \mathrm{d}\vec{x}\; T^{00} \;=\; \int \mathrm{d}\vec{x}\left[\frac{\partial\mathcal{L}}{\partial\dot{f}_a}\,\dot{f}_a - \mathcal{L}\right] \;=\; \int \mathrm{d}\vec{x}\left[\pi_a\dot{f}_a - \mathcal{L}\right] \;=\; \int \mathrm{d}\vec{x}\; \mathcal{H} \;=\; H\;,$$

$$\int \mathrm{d}\vec{x}\; T^{0j} \;=\; -\int \mathrm{d}\vec{x}\left[\frac{\partial\mathcal{L}}{\partial\dot{f}_a}\,\nabla^j f_a - 0\right] \;=\; -\int \mathrm{d}\vec{x}\; \pi_a\nabla^j f_a \;\equiv\; P^j\;.$$

Summation over a is again implied. The quantity $T^{\mu\nu}$ is called the stress-energy tensor or energy-momentum tensor, H is the physical energy carried by the fields f_a, and P^j is the j^{th} component of the physical momentum carried by the fields f_a. We will see later that what we just did does not just hold for scalar fields, but also for any component of a vector, spinor, ... field.

3a *The field energy H will play a crucial role in the quantization of free field theories, since it will feature in the quantum mechanical requirement of having an energy spectrum that is bounded from below. On top of that it determines the quantum mechanical time evolution. The field momentum will help us in determining the particle interpretation of free quantum field theories.*

Intermezzo 1: the energy-momentum tensor in cosmology

In general relativity a curved-spacetime version of the energy-momentum tensor $\Theta^{\mu\nu}$ features, which is symmetric under the interchange of μ and ν. In the modified Einstein equation including cosmological constant:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} \; = \; -\,8\pi G\Theta_{\mu\nu} \qquad\qquad (G = \text{Newton's constant})\ ,$$

this symmetrized energy-momentum tensor describes matter and energy in the universe, whereas the Ricci tensor $R_{\mu\nu}$, scalar curvature $R = g^{\rho\sigma}R_{\rho\sigma}$ and cosmological constant Λ describe the "structure" of spacetime for an empty space (i.e. for the vacuum).

The flat-spacetime version of the energy-momentum tensor $T^{\mu\nu}$ that we have just derived is in general not guaranteed to be symmetric under the interchange of μ and ν. However, this can be arranged by adding an appropriate extra term $\partial_\rho K^{\rho\mu\nu}$ with $K^{\rho\mu\nu} = -\,K^{\mu\rho\nu}$ such that $\Theta^{\mu\nu} = T^{\mu\nu} + \partial_\rho K^{\rho\mu\nu} = \Theta^{\nu\mu}$ and $\partial_\mu\Theta^{\mu\nu} = \partial_\mu T^{\mu\nu} + \partial_\mu\partial_\rho K^{\rho\mu\nu} = \partial_\mu\partial_\rho K^{\rho\mu\nu} = 0$. For an explicit example, see Ex. 2.1 in the textbook by Peskin & Schroeder.

By bringing the cosmological-constant term to the right-hand side of the modified Einstein equation, it can be viewed as representing the energy-momentum tensor of empty space itself (taking into account such effects as vacuum energy and vacuum pressure). Such a cosmological-constant term therefore constitutes the vacuum contribution to the curvature. In view of its proportionality to the metric tensor, the cosmological-constant term is the same for all inertial observers in the flat-spacetime case, which is compatible with the notion that in that case the vacuum should not have a preferred frame. So, the presence of a cosmological-constant term does not conflict with any first-principle requirements!

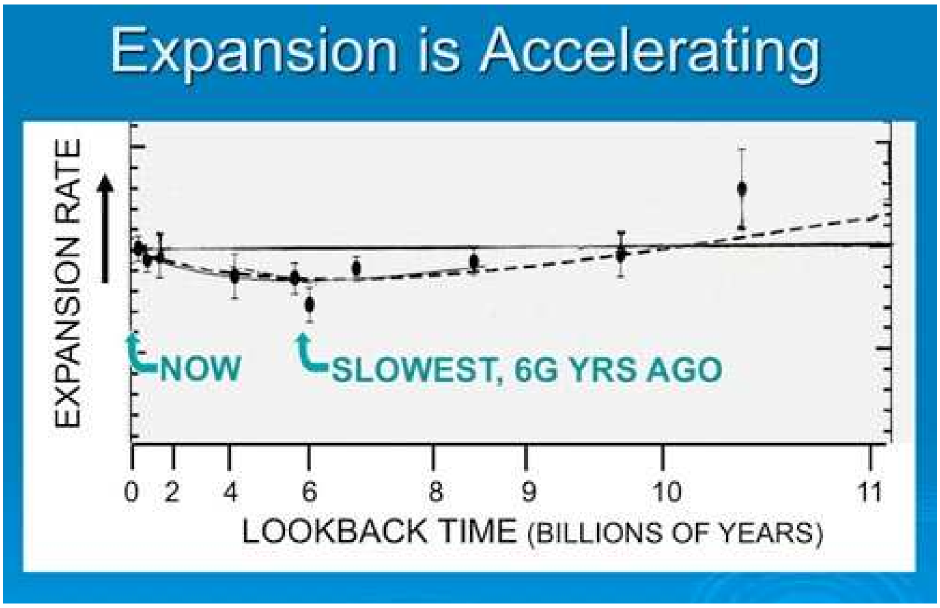
For a positive cosmological constant ($\Lambda > 0$) the energy density of the vacuum is positive and the associated pressure is negative, resulting in an accelerated expansion of empty space as seems to be supported by experiment (see next page). Such a vacuum energy is usually referred to as dark energy. We will see shortly that field quantization can actually provide a source of dark energy.

The reason why the pressure is negative follows from the simple fact that energy is released if the volume of space expands, whereas a positive "pressure on space" would require work to be exerted during the expansion. For a positive cosmological constant the vacuum represents an unlimited energy reservoir, which is tapped when the universe inflates.

Accelerated expansion of the universe

If all of the energy in the universe would be in the form of matter, radiation and gravitational waves, the rate of expansion of the universe would decrease continuously after the Big Bang due to gravity. However, if empty space itself would also carry a positive energy density, which could be viewed as "the energy cost of having space", then this is not necessarily true anymore. Such a dark-energy density would have a repelling effect. Moreover, if this density would be constant it would not be affected by the expansion of the universe, whereas the density of matter decreases as the universe expands. This would imply that the universe could undergo a transition from being matter/radiation dominated at early stages to being dark-energy dominated at later stages, resulting in a reacceleration of the universe from a certain moment onwards. Precisely this scenario seems to be borne out by experiment (see the figure below and the lecture course "Gravity and the Cosmos").

Saul Perlmutter, Brian Schmidt, Adam Riess (2011 Nobel Prize in Physics)



The expansion rate of the universe at different times can be inferred from the redshift of far away objects, provided that we can determine in a reliable way how much distance the light has travelled before reaching us. To this end supernova 1a explosions are used as standard candles. Since these explosions produce as much light as an entire galaxy at peak luminosity, they can be used as beacons to look into the distant past. Another crucial feature of supernova 1a explosions is that they have a well-defined mechanism: a white dwarf accretes matter from a companion star until it reaches a critical mass at which a runaway carbon fusion is triggered that sets off the explosion. These supernova 1a explosions produce a distinctive luminosity spectrum, which makes them identifiable. The distance travelled by the light then follows from the observed peak luminosity, by comparing it to the known peak luminosity at the time of emission.

Symmetry under rotations and boosts (continuous Lorentz transformations): under an infinitesimal continuous Lorentz transformation

$$x^\rho \;\to\; x'^\rho \;=\; \Lambda^\rho{}_\sigma x^\sigma \;\approx\; x^\rho + \omega^\rho{}_\sigma x^\sigma \;,$$

where $\omega^{\rho\sigma} = -\,\omega^{\sigma\rho} \in \mathbb{R}$ is an infinitesimal tensor with six independent components. The Lagrangian is a Lorentz scalar, so

$$\begin{aligned}
\mathcal{L}(x) \;\to\; \mathcal{L}(\Lambda^{-1}x) \;&\approx\; \mathcal{L}(x-\omega x) \;\approx\; \mathcal{L}(x) - \omega^\rho{}_\sigma x^\sigma \partial_\rho \mathcal{L}(x) \\
&\stackrel{\omega^\rho{}_\rho=0}{=\!=\!=}\; \mathcal{L}(x) - \omega^\rho{}_\sigma \partial_\rho x^\sigma \mathcal{L}(x) \;=\; \mathcal{L}(x) - \omega_{\rho\sigma}\partial_\mu\big(g^{\mu\rho}x^\sigma \mathcal{L}(x)\big) \\
&\stackrel{\omega_{\rho\sigma}=-\,\omega_{\sigma\rho}}{=\!=\!=\!=\!=}\; \mathcal{L}(x) - \frac{1}{2}\omega_{\rho\sigma}\partial_\mu\big([g^{\mu\rho}x^\sigma - g^{\mu\sigma}x^\rho]\,\mathcal{L}(x)\big) \;.
\end{aligned}$$

Since $\mathcal{L}(x)$ changes by a total four-divergence, relativistic field theories have a symmetry under continuous Lorentz transformations with $\big[G^\mu(x)\big]^{\rho\sigma} = -\,[g^{\mu\rho}x^\sigma - g^{\mu\sigma}x^\rho]\,\mathcal{L}(x)$ for all six independent components of $\omega_{\rho\sigma}$.

Now consider a Lagrangian for a scalar field $\phi(x)$. Such a field transforms as

$$\phi(x) \;\to\; \phi'(x) \;=\; \phi(\Lambda^{-1}x) \;\approx\; \phi(x) - \frac{1}{2}\,\omega_{\rho\sigma}\,[x^\sigma\partial^\rho - x^\rho\partial^\sigma]\,\phi(x) \;\equiv\; \phi(x) + \frac{1}{2}\,\omega_{\rho\sigma}\,\big[\Delta\phi(x)\big]^{\rho\sigma} \;.$$

This results in six conserved currents, one for each independent component of $\omega_{\rho\sigma}$:

$$J^{\mu\rho\sigma}(x) \;=\; \Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Big)[x^\rho\partial^\sigma - x^\sigma\partial^\rho]\,\phi(x) \;+\; [g^{\mu\rho}x^\sigma - g^{\mu\sigma}x^\rho]\,\mathcal{L}(x) \;=\; T^{\mu\sigma}(x)\,x^\rho - T^{\mu\rho}(x)\,x^\sigma \;,$$

and hence six conserved "charges":

- Rotations ($\rho,\sigma = i,j$): $J^k \equiv \frac{1}{2}\epsilon^{ijk}\int\!\mathrm{d}\vec{x}\,\big[T^{0j}(x)x^i - T^{0i}(x)x^j\big]$, with summation over the spatial indices i and j implied. This is the k^{th} component of the physical angular momentum carried by the field $\phi(x)$.
- Boosts ($\rho,\sigma = 0,i$): $K^i \;\equiv\; \int\!\mathrm{d}\vec{x}\,\big[T^{0i}(x)x^0 - T^{00}(x)x^i\big] \;=\; x^0P^i - \int\!\mathrm{d}\vec{x}\,x^i\,T^{00}(x)$. Conservation of these three "charges" implies that $\dfrac{\mathrm{d}}{\mathrm{d}t}\big(x^0P^i - \int\!\mathrm{d}\vec{x}\,x^i\,T^{00}(x)\big) = P^i - \dfrac{\mathrm{d}}{\mathrm{d}t}\int\!\mathrm{d}\vec{x}\;x^i\,T^{00}(x) = 0$. Since $\int\!\mathrm{d}\vec{x}\;T^{00}(x) = H$, this equation can be interpreted as saying that the "centre-of-energy" of the field travels at constant velocity, in analogy with the movement of the centre-of-mass of a free classical system.

3a *The angular momentum of a field depends on the type of field and will thus be useful after quantization. It will help us to determine the intrinsic spin of the particles described by the free quantum field theory that corresponds to a given wave equation. As a result of the relativity principle, each type of wave equation will give rise to a specific particle spin.*

Abelian internal symmetry ("global $U(1)$ gauge symmetry"): an internal symmetry involves a transformation of the fields that acts in the same way at every spacetime point, whereas abelian implies multiplying all fields by a constant phase factor. Consider a complex scalar field $\phi(x)$ that satisfies the Euler–Lagrange equations of the Lagrangian

$$\mathcal{L}(\phi, \phi^*, \partial_\mu\phi, \partial_\mu\phi^*) \;=\; (\partial_\mu\phi)(\partial^\mu\phi^*) - m^2\phi\phi^*.$$

This Lagrangian is invariant under the continuous transformation $\phi \to e^{i\alpha}\phi$, $\phi^* \to e^{-i\alpha}\phi^*$, where $\alpha \in \mathbb{R}$ is a constant. This implies that under the infinitesimal version of this transformation, i.e.

$$\phi \;\to\; \phi + \alpha(i\phi) \;\equiv\; \phi + \alpha\Delta\phi \quad \text{and} \quad \phi^* \;\to\; \phi^* + \alpha(-i\phi^*) \;\equiv\; \phi^* + \alpha\Delta\phi^* \;,$$

we get $\Delta\mathcal{L} = 0 \;\Rightarrow\; G^\mu = 0$. As a result the current

$$j^\mu \;=\; i\phi\partial^\mu\phi^* - i\phi^*\partial^\mu\phi \;=\; i\big[(\partial^\mu\phi^*)\phi - \phi^*\partial^\mu\phi\big]$$

is conserved. For an extended example of a gauge symmetry see Ex. 3.

> 3b *We will see later that the conserved charge arising from currents of this type have the interpretation of electric charge or particle number. The associated $U(1)$ gauge symmetry will feature prominently in a symmetry-based description of electromagnetic interactions.*

Symmetries versus unobservable quantities: the above-given symmetries are in fact all related to quantities that are fundamentally unobservable.

> 3 *The abelian internal symmetry is linked to the unobservability of the absolute phase of a QM wave function. Translation and rotation symmetry are the result of the unobservability of the absolute position and direction in spacetime. Symmetry under boosts is related to the unobservability of the absolute velocity of a chosen reference frame.*

1.3 The free Klein-Gordon theory (real case, § 2.3 in the book)

We start our tour of the relativistic quantum-field-theory world with the simplest example: the quantum field theory for real scalar fields that satisfy the free Klein-Gordon (KG) equation. The classical Lagrangian for a real scalar field $\phi(x)$ that satisfies the free KG equation is given by

$$\mathcal{L} \;=\; \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 \quad \xrightarrow{\text{Euler-Lagrange}} \quad (\Box + m^2)\phi(x) \;=\; 0 \quad , \quad \pi \;=\; \frac{\partial\mathcal{L}}{\partial\dot{\phi}} \;=\; \dot{\phi} \;.$$

The corresponding time-independent Hamiltonian reads (cf. page 6)

$$H \;=\; \int \mathrm{d}\vec{x}\,\Big[\,\frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2\,\Big] \;.$$

4a Question: how should we quantize such a classical field theory?

1) Canonical quantization: in principle we could approach this in the same way as in the case of the quantization of Newtonian mechanics: the dynamical coordinates and associated conjugate momenta become operators that satisfy canonical commutation relations. In the Schrödinger picture this reads

- Discrete quantum mechanics: $\big[\hat{q}_j, \hat{p}_k\big] = i\delta_{jk}\hat{1} \quad , \quad \big[\hat{q}_j, \hat{q}_k\big] = \big[\hat{p}_j, \hat{p}_k\big] = 0\,.$

- Continuous quantum field theories: $\big[\hat{\phi}_j(\vec{x}\,), \hat{\pi}_k(\vec{y}\,)\big] = i\delta_{jk}\,\delta(\vec{x}-\vec{y}\,)\hat{1}\,,$

$$\big[\hat{\phi}_j(\vec{x}\,), \hat{\phi}_k(\vec{y}\,)\big] = \big[\hat{\pi}_j(\vec{x}\,), \hat{\pi}_k(\vec{y}\,)\big] = 0\,.$$

Subsequently, the fully covariant (time-dependent) versions of these commutation relations can be obtained by switching to the Heisenberg picture. This type of quantization procedure is called canonical quantization.

2) Quantizing an infinite number of linear harmonic oscillators: in a general quantum field theory the spectrum of $\hat{H}$ is hard to find, since it involves an infinite number of degrees of freedom that in general do not evolve independently. However, in the case of free theories each degree of freedom does evolve independently. The reason behind this is that the corresponding equations of motion are linear wave equations, with all quantum fields as well as their individual components satisfying the KG equation. This latter requirement is needed in order to implement particle-wave duality in the right way by giving rise to the correct relation between energy and momentum for the free particles described by the theory. Consider now such a field component $f(\vec{x},t) \in \mathbb{R}$ with $(\Box + m^2)f(\vec{x},t) = 0$. In order to decouple the degrees of freedom we use the momentum representation (Fourier decomposition)

$$f(\vec{x},t) \;\equiv\; \int \frac{\mathrm{d}\vec{p}}{(2\pi)^3}\, e^{i\vec{p}\cdot\vec{x}} g(\vec{p},t)\ ,$$

so that the KG equation changes into

$$\Big(\frac{\partial^2}{\partial t^2} + (\vec{p}^{\,2} + m^2)\Big)\, g(\vec{p},t) \;=\; 0$$

for each Fourier-mode $\vec{p}$. This means that $g(\vec{p},t)$ solves the equation of motion of a harmonic oscillator vibrating at a frequency $\omega_{\vec{p}} \equiv \sqrt{\vec{p}^{\,2}+m^2}$. The most general solution to the KG equation will therefore be a linear superposition of simple harmonic oscillators, each with a different amplitude and frequency. So, in order to quantize $f(\vec{x},t)$, one simply has to quantize the infinite number of oscillators in terms of raising (creation) and lowering (annihilation) operators. The associated harmonic energy quanta are interpreted as particles.

Next it will be proven that both procedures are actually equivalent.

Comparing both procedures: let's first recall how the quantization of a linear harmonic oscillator goes. Consider to this end the corresponding Hamilton operator

$$\hat{H} \;=\; \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 \;\equiv\; \frac{1}{2}\hat{P}^2 + \frac{1}{2}\omega^2\hat{Q}^2 \qquad \text{with} \qquad \big[\hat{Q},\hat{P}\big] \;=\; i\hat{1}\ ,$$

using $\hat{P} = \hat{p}/\sqrt{m}$ and $\hat{Q} = \hat{x}\sqrt{m}$. Next we introduce a lowering operator $\hat{a}$ and raising operator $\hat{a}^\dagger$ according to

$$\hat{Q} \;\equiv\; \frac{\hat{a}+\hat{a}^\dagger}{\sqrt{2\omega}} \quad , \quad \hat{P} \;\equiv\; -\,i\omega\,\frac{\hat{a}-\hat{a}^\dagger}{\sqrt{2\omega}}\ .$$

From this the fundamental bosonic commutation relation $\big[\hat{a},\hat{a}^\dagger\big] = \hat{1}$ follows and

$$\hat{H} \;=\; \frac{1}{2}\,\omega\,(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger) \;=\; \omega\,(\hat{a}^\dagger\hat{a} + \tfrac{1}{2}\hat{1}) \;\equiv\; \omega\,(\hat{n} + \tfrac{1}{2}\hat{1})\ ,$$

where $\hat{n} = \hat{a}^\dagger\hat{a}$ can be interpreted as a counting operator. Using $\big[\hat{H},\hat{a}^\dagger\big] = \omega\,\hat{a}^\dagger$, the energy eigenvalues E_n and eigenfunctions $|n\rangle$ of this Hamilton operator can be obtained:

$$E_n \;=\; (n+\tfrac{1}{2})\omega \quad , \quad |n\rangle \;\equiv\; \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}\,|0\rangle \qquad\quad (n = 0, 1, \cdots)\ .$$

Based on this we use the following ansatz for the quantized KG field and its conjugate momentum in terms of a continuous set of oscillator modes labeled by $\vec{p}$:

$$\hat{\phi}(\vec{x}\,) \;=\; \int\frac{\mathrm{d}\vec{p}}{(2\pi)^3}\,\frac{\hat{a}_{\vec{p}} + \hat{a}^\dagger_{-\vec{p}}}{\sqrt{2\omega_{\vec{p}}}}\,e^{i\vec{p}\cdot\vec{x}} \;=\; \int\frac{\mathrm{d}\vec{p}}{(2\pi)^3}\,\frac{1}{\sqrt{2\omega_{\vec{p}}}}\Big(\hat{a}_{\vec{p}}\,e^{i\vec{p}\cdot\vec{x}} + \hat{a}^\dagger_{\vec{p}}\,e^{-i\vec{p}\cdot\vec{x}}\Big) \;=\; \hat{\phi}^\dagger(\vec{x}\,)\ ,$$

$$\hat{\pi}(\vec{x}\,) \;=\; -\,i\int\frac{\mathrm{d}\vec{p}}{(2\pi)^3}\,\omega_{\vec{p}}\,\frac{\hat{a}_{\vec{p}} - \hat{a}^\dagger_{-\vec{p}}}{\sqrt{2\omega_{\vec{p}}}}\,e^{i\vec{p}\cdot\vec{x}} \;=\; -\,i\int\frac{\mathrm{d}\vec{p}}{(2\pi)^3}\,\sqrt{\frac{\omega_{\vec{p}}}{2}}\,\Big(\hat{a}_{\vec{p}}\,e^{i\vec{p}\cdot\vec{x}} - \hat{a}^\dagger_{\vec{p}}\,e^{-i\vec{p}\cdot\vec{x}}\Big) \;=\; \hat{\pi}^\dagger(\vec{x}\,)\ ,$$

with $\big[\hat{a}_{\vec{p}},\hat{a}^\dagger_{\vec{p}\,'}\big] = (2\pi)^3\delta(\vec{p}-\vec{p}\,')\hat{1}$ and all other commutators 0.

Let's now see whether we have succeeded in properly quantizing and decoupling the free real KG theory. From the fundamental bosonic commutation relations for creation and annihilation operators it follows that

$$\begin{aligned}\big[\hat{\phi}(\vec{x}\,),\hat{\pi}(\vec{y}\,)\big] \;&=\; -\,\frac{i}{2}\int\frac{\mathrm{d}\vec{p}\,\mathrm{d}\vec{p}\,'}{(2\pi)^6}\sqrt{\frac{\omega_{\vec{p}\,'}}{\omega_{\vec{p}}}}\;e^{i(\vec{p}\cdot\vec{x}+\vec{p}\,'\cdot\vec{y})}\big[\hat{a}_{\vec{p}} \,+\, \hat{a}^\dagger_{-\vec{p}}\,,\hat{a}_{\vec{p}\,'} \,-\, \hat{a}^\dagger_{-\vec{p}\,'}\big]\\ &=\; \frac{i}{(2\pi)^3}\,\hat{1}\int\mathrm{d}\vec{p}\;e^{i\vec{p}\cdot(\vec{x}-\vec{y}\,)} \;=\; i\delta(\vec{x}-\vec{y}\,)\hat{1}\ ,\end{aligned}$$

in agreement with canonical quantization.

Energy spectrum and zero-point energy: the Hamilton operator of the free real KG theory now reads

$$
\begin{aligned}
\hat{H} &= \int \mathrm{d}\vec{x}\left[\frac{1}{2}\,\hat{\pi}^2 + \frac{1}{2}\,(\vec{\nabla}\hat{\phi})^2 + \frac{1}{2}\,m^2\hat{\phi}^2\right] \\
&= \int \mathrm{d}\vec{x} \int \frac{\mathrm{d}\vec{p}\,\mathrm{d}\vec{p}\,'}{(2\pi)^6}\,\frac{e^{i\vec{x}\cdot(\vec{p}+\vec{p}\,')}}{4\sqrt{\omega_{\vec{p}}\,\omega_{\vec{p}\,'}}}\Big[-\ \omega_{\vec{p}}\,\omega_{\vec{p}\,'}\,(\hat{a}_{\vec{p}} - \hat{a}^{\dagger}_{-\vec{p}})(\hat{a}_{\vec{p}\,'} - \hat{a}^{\dagger}_{-\vec{p}\,'}) \\
&\qquad\qquad + (m^2 - \vec{p}\cdot\vec{p}\,')(\hat{a}_{\vec{p}} + \hat{a}^{\dagger}_{-\vec{p}})(\hat{a}_{\vec{p}\,'} + \hat{a}^{\dagger}_{-\vec{p}\,'})\Big] \\
&\overset{\vec{x}\text{ integral}}{=\!=} \int \frac{d\vec{p}}{(2\pi)^3}\,\frac{1}{2}\,\omega_{\vec{p}}\,(\hat{a}_{\vec{p}}\,\hat{a}^{\dagger}_{\vec{p}} + \hat{a}^{\dagger}_{-\vec{p}}\,\hat{a}_{-\vec{p}}) \\
&\overset{\vec{p}\to-\vec{p}\text{ in 2nd term}}{=\!=} \int \frac{d\vec{p}}{(2\pi)^3}\,\omega_{\vec{p}}\,\hat{a}^{\dagger}_{\vec{p}}\,\hat{a}_{\vec{p}} + \int \frac{d\vec{p}}{(2\pi)^3}\,\frac{1}{2}\,\omega_{\vec{p}}\,(2\pi)^3\delta(\vec{0}\,)\hat{1}\ ,
\end{aligned}
$$

which is indeed nicely decoupled and properly time-independent. The last term in the final expression is called the zero-point energy. It is a consequence of the uncertainty principle and represents the ground-state energy in the absence of any oscillator quanta.

(*4b*) *Question: have we obtained an energy spectrum that is bounded from below?*

From the decoupled form of the Hamilton operator of the free real KG theory we can read off that

- the energy spectrum is indeed bounded from below by the zero-point energy;
- only positive-energy quanta feature in the Hamilton operator;
- the zero-point energy is infinite:
  - We have $(2\pi)^3\delta(\vec{0}\,) = \int \mathrm{d}\vec{x}\; e^{i\vec{x}\cdot\vec{p}}\Big|_{\vec{p}=\vec{0}} = \lim\limits_{L\to\infty} \int_{-L}^{L} \mathrm{d}\vec{x}\; e^{i\vec{x}\cdot\vec{p}}\Big|_{\vec{p}=\vec{0}} = \lim\limits_{L\to\infty} \int_{-L}^{L} \mathrm{d}\vec{x} = V.$ This is an infinity originating from the fact that space is infinite. Such a long-distance infinity is often referred to as an infra-red (IR) divergence, since it is related to $\vec{p} = \vec{0}$.
  - The zero-point energy density $\displaystyle\int \frac{\mathrm{d}\vec{p}}{(2\pi)^3}\,\frac{1}{2}\,\omega_{\vec{p}}$ is still infinite, originating from the $|\vec{p}\,| \to \infty$ limit of the integrand. This type of infinity is called ultra-violet (UV) divergence, being related to short distances/high frequencies. It is the consequence of our unrealistic assumption that the theory is valid up to arbitrarily high energies. As we will see later, the $\vec{p}$-integral should be cut off at a value where the theory breaks down or a more fancy technique should be used to quantify the UV infinity if we do not want to introduce a new energy scale.

4d *The zero-point energy is inessential for the particle interpretation, but it is measurable in bounded sytems through the Casimir effect (as is explained in the bachelor course "Kwantummechanica 3") and it has explicit cosmological implications in view of the fact that it contributes to the cosmological constant.*

About 68% of the energy density in the universe bears the characteristics of a cosmological constant with energy scale $10^{-3}\,\text{eV}$, which is surprisingly small. With the Planck mass $M_{\text{pl}} = \mathcal{O}(10^{28}\,\text{eV})$ being the natural scale of gravity, where ordinary quantum field theory most likely breaks down, we would expect the energy scale belonging to the cosmological constant to be $\mathcal{O}(M_{\text{pl}})$ if it has a gravitational origin. One of the big questions in present-day high-energy physics therefore reads *"Why is the cosmological constant so small?"*.

The art of covering up: normal ordering.
In most textbooks all issues related to properties of the vacuum of the theory are simply circumvented by removing vacuum energies, charges, etc..[1] This is done by applying normal ordering, i.e. bringing all creation operators to the front:

$$\hat{a}^\dagger \hat{a} \;\to\; N(\hat{a}^\dagger \hat{a}) = \hat{a}^\dagger \hat{a} \quad , \quad \hat{a}\hat{a}^\dagger \;\to\; N(\hat{a}\hat{a}^\dagger) = \hat{a}^\dagger \hat{a} \quad \Rightarrow \quad N(\hat{H}) = \int \frac{\mathrm{d}\vec{p}}{(2\pi)^3}\, \omega_{\vec{p}}\, \hat{a}^\dagger_{\vec{p}} \hat{a}_{\vec{p}} \;.$$

After quantization the momentum carried by the KG field (cf. page 8) becomes

$$\begin{aligned}\hat{\vec{P}} &= -\int \mathrm{d}\vec{x}\; \hat{\pi}(\vec{x}\,)\vec{\nabla}\hat{\phi}(\vec{x}\,) = \int \frac{\mathrm{d}\vec{p}}{(2\pi)^3}\, \frac{i}{2}\,(-i\vec{p}\,)(\hat{a}_{\vec{p}} - \hat{a}^\dagger_{-\vec{p}})(\hat{a}_{-\vec{p}} + \hat{a}^\dagger_{\vec{p}}) \\ &= -\frac{1}{2}\int \frac{\mathrm{d}\vec{p}}{(2\pi)^3}\, \vec{p}\,(\hat{a}^\dagger_{-\vec{p}}\hat{a}_{-\vec{p}} - \hat{a}_{\vec{p}}\hat{a}^\dagger_{\vec{p}} + \hat{a}^\dagger_{-\vec{p}}\hat{a}^\dagger_{\vec{p}} - \hat{a}_{\vec{p}}\hat{a}_{-\vec{p}}) = \int \frac{\mathrm{d}\vec{p}}{(2\pi)^3}\, \vec{p}\,\hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{p}} \;,\end{aligned}$$

where in the last step we have taken $\vec{p} \to -\vec{p}$ in the first term and we have used that $\vec{p}\,\hat{a}_{\vec{p}}\hat{a}_{-\vec{p}}$, $\vec{p}\,\hat{a}^\dagger_{-\vec{p}}\hat{a}^\dagger_{\vec{p}}$ and $\vec{p}\,(2\pi)^3\delta(\vec{0}\,)$ are all odd under $\vec{p} \to -\vec{p}$ whereas the integration is even. This time there is no zero-point contribution and as such there is no need for normal ordering, which is consistent with the fact that quantum fluctuations should have no preferred direction.

Particle interpretation of the free real KG theory:

4d *In the next step we determine the particle interpretation of the theory, mostly by simply reading it off from $N(\hat{H})$ and $\hat{\vec{P}}$.*

- Vacuum (ground state): $|0\rangle$ such that $\langle 0|0\rangle = 1$ and $\hat{a}_{\vec{p}}|0\rangle = 0$ for all $\vec{p}$. Then $N(\hat{H})|0\rangle = 0$ and $\hat{\vec{P}}|0\rangle = \vec{0}$, i.e. the vacuum "has" energy $E = 0$ and momentum $\vec{P} = \vec{0}$.

[1] The tacit assumption here is that some underlying (high-scale) physics takes care of this

- Excited states: obtained as $(\text{constant}) * \hat{a}^\dagger_{\vec{p}}\hat{a}^\dagger_{\vec{q}}\cdots|0\rangle$. Then $E = \omega_{\vec{p}} + \omega_{\vec{q}} + \cdots$ and $\vec{P} = \vec{p} + \vec{q} + \cdots$, which follows from $\big[\hat{H}, \hat{a}^\dagger_{\vec{p}}\big] = \omega_{\vec{p}}\,\hat{a}^\dagger_{\vec{p}}$ and $\big[\hat{\vec{P}}, \hat{a}^\dagger_{\vec{p}}\big] = \vec{p}\,\hat{a}^\dagger_{\vec{p}}$.
  - For higher excitations $\hat{a}^\dagger_{\vec{p}}$ is replaced by $(\hat{a}^\dagger_{\vec{p}})^n/\sqrt{n!}$.
  - The excitations are interpreted as particles.
  - In view of the bosonic commutation relations for the associated creation and annihilation operators these particles are bosons.
  - In fact the particles are spin-0 bosons. This follows from $\big[\hat{J}^k, \hat{a}^\dagger_{\vec{0}}\big] = 0$ for $k = 1, 2, 3$. Bearing in mind that a zero-momentum particle does not give rise to an orbital angular momentum, this indeed implies that the particles in the real KG theory also carry no intrinsic angular momentum.

    Proof: the quantized version of the angular momentum derived on page 11 yields

$$\big[\hat{J}^k, \hat{a}^\dagger_{\vec{0}}\big] = \Big[\epsilon^{ijk}\int \mathrm{d}\vec{x}\ \hat{\pi}(\vec{x})\,\nabla^i\hat{\phi}(\vec{x})\,x^j, \hat{a}^\dagger_{\vec{0}}\Big]$$

$$= \frac{1}{2}\epsilon^{ijk}\int \mathrm{d}\vec{x}\,x^j\int\frac{\mathrm{d}\vec{p}\,\mathrm{d}\vec{p}\,'}{(2\pi)^6}\,p^i\sqrt{\frac{\omega_{\vec{p}\,'}}{\omega_{\vec{p}}}}\,e^{i\vec{x}\cdot(\vec{p}+\vec{p}\,')}\big[\,(\hat{a}_{\vec{p}\,'} - \hat{a}^\dagger_{-\vec{p}\,'})(\hat{a}_{\vec{p}} + \hat{a}^\dagger_{-\vec{p}}), \hat{a}^\dagger_{\vec{0}}\,\big]$$

$$= \frac{1}{2}\epsilon^{ijk}\int \mathrm{d}\vec{x}\,x^j\int\frac{\mathrm{d}\vec{p}\,\mathrm{d}\vec{p}\,'}{(2\pi)^3}\,p^i\sqrt{\frac{\omega_{\vec{p}\,'}}{\omega_{\vec{p}}}}\,e^{i\vec{x}\cdot(\vec{p}+\vec{p}\,')}\Big(\delta(\vec{p}\,')\,[\hat{a}_{\vec{p}} + \hat{a}^\dagger_{-\vec{p}}] + \delta(\vec{p})\,[\hat{a}_{\vec{p}\,'} - \hat{a}^\dagger_{-\vec{p}\,'}]\Big)\ .$$

    The second term in the last expression vanishes trivially. The first term vanishes as well since $i \neq j$ and consequently the x^i integral will be proportional to $\delta(p^i)$.
  - An example of such a particle is the π^0 pion.

Normalization of states and completeness relation: note that we did not specify yet what normalization factor to use in the definition of the 1-particle states. Unlike what is done in non-relativistic quantum mechanics, where the normalization factor is usually taken to be 1, we will use a relativistically motivated normalization of the 1-particle states:

$$|\vec{p}\,\rangle \equiv \sqrt{2\omega_{\vec{p}}}\;\hat{a}^\dagger_{\vec{p}}\,|0\rangle \quad\Rightarrow\quad \langle\vec{p}\,|\vec{q}\,\rangle = 2\sqrt{\omega_{\vec{p}}\,\omega_{\vec{q}}}\,\langle 0|\hat{a}_{\vec{p}}\,\hat{a}^\dagger_{\vec{q}}\,|0\rangle = 2\sqrt{\omega_{\vec{p}}\,\omega_{\vec{q}}}\,\langle 0|\big[\hat{a}_{\vec{p}}, \hat{a}^\dagger_{\vec{q}}\big]|0\rangle$$

$$= 2\omega_{\vec{p}}\,(2\pi)^3\,\delta(\vec{p} - \vec{q}\,)\ .$$

The latter expression is invariant under continuous Lorentz transformations.

Proof: in order to prove this statement we first derive the important integration identity

$$\int\frac{\mathrm{d}\vec{p}}{(2\pi)^3}\,\frac{1}{2\omega_{\vec{p}}} = \int\frac{\mathrm{d}^4p}{(2\pi)^3}\,\delta(p^2 - m^2)\,\Theta(p^0)\qquad(\text{Lorentz invariant integration measure}),\quad(1)$$

with Θ the Heaviside step function. We get this identity by using that

$$\delta\big(h(x)\big) = \sum_j\frac{\delta(x - x_j)}{|h'(x_j)|}\qquad\text{for}\ \ h(x_j) = 0\ \ \text{and}\ \ h'(x_j) \neq 0\ ,$$

which leads to

$$\delta(p^2 - m^2) \;=\; \delta\big(p_0^2 - [\vec{p}^{\,2} + m^2]\big) \;=\; \delta(p_0^2 - \omega_{\vec{p}}^2) \;=\; \frac{\delta(p_0 - \omega_{\vec{p}})}{2\omega_{\vec{p}}} + \frac{\delta(p_0 + \omega_{\vec{p}})}{2\omega_{\vec{p}}}\,.$$

Since p^0 cannot change sign for $p^2 > 0$, the right-hand-side of equation (1) only contains Lorentz invariant objects. As a result, the expression on the left-hand-side is Lorentz invariant as well and the same goes for $\langle \vec{p}\,|\vec{q}\,\rangle$, since

$$\int \frac{\mathrm{d}\vec{p}}{(2\pi)^3}\,\frac{\langle \vec{p}\,|\vec{q}\,\rangle}{2\omega_{\vec{p}}} \;=\; \int \mathrm{d}\vec{p}\;\delta(\vec{p} - \vec{q}\,) \;=\; 1.$$

The 1-particle completeness relation is then given by

$$\int \frac{\mathrm{d}\vec{p}}{(2\pi)^3}\,\frac{1}{2\omega_{\vec{p}}}\;|\vec{p}\,\rangle\langle\vec{p}\,| \;=\; \hat{1}\,\Big|_{\text{1-particle subspace}}$$

since

$$\underset{\vec{q}}{\forall}\quad \int \frac{\mathrm{d}\vec{p}}{(2\pi)^3}\,\frac{1}{2\omega_{\vec{p}}}\;|\vec{p}\,\rangle\langle\vec{p}\,|\vec{q}\,\rangle \;=\; \int \frac{\mathrm{d}\vec{p}}{(2\pi)^3}\,\frac{2\omega_{\vec{p}}(2\pi)^3\,\delta(\vec{p} - \vec{q}\,)}{2\omega_{\vec{p}}}\;|\vec{p}\,\rangle \;=\; |\vec{q}\,\rangle\,.$$

Finally we may ask the question what state is actually created by $\hat{\phi}(\vec{x}\,) = \hat{\phi}^\dagger(\vec{x}\,)$. Letting this operator act on the vacuum one obtains

$$\hat{\phi}(\vec{x}\,)|0\rangle \;=\; \int \frac{\mathrm{d}\vec{p}}{(2\pi)^3}\,\frac{e^{i\vec{p}\cdot\vec{x}}}{\sqrt{2\omega_{\vec{p}}}}\,(\hat{a}_{\vec{p}} + \hat{a}^\dagger_{-\vec{p}})\,|0\rangle \;\underset{\overline{}}{\xrightarrow{\vec{p}\to-\vec{p}}}\; \int \frac{\mathrm{d}\vec{p}}{(2\pi)^3}\,\frac{e^{-i\vec{p}\cdot\vec{x}}}{2\omega_{\vec{p}}}\,|\vec{p}\,\rangle\,.$$

From this it can be concluded that a particle is created "at position $\vec{x}$", since

$$\langle\vec{q}\,|\hat{\phi}(\vec{x}\,)|0\rangle \;=\; \int \frac{\mathrm{d}\vec{p}}{(2\pi)^3}\,\frac{e^{-i\vec{p}\cdot\vec{x}}}{2\omega_{\vec{p}}}\,\langle\vec{q}\,|\vec{p}\,\rangle \;=\; e^{-i\vec{q}\cdot\vec{x}}$$

is indeed identical to $\langle\vec{q}\,|\vec{x}\,\rangle$ in non-relativistic quantum mechanics.

Point to ponder: you might wonder now whether this contradicts the earlier statement that there is no local single-particle concept in Quantum Field Theory. To check this we consider the overlap between two such position states:

$$\langle 0|\hat{\phi}(\vec{x}\,)\hat{\phi}(\vec{y}\,)|0\rangle \;\propto\; e^{-m|\vec{x}-\vec{y}|} \qquad \text{for large enough } |\vec{x}-\vec{y}\,|\,,$$

as determined on page 27 of the textbook by Peskin & Schroeder. In the non-relativistic limit, which effectively corresponds to the limit $m\to\infty$, the overlap vanishes for $\vec{x}\neq\vec{y}$ and $\hat{\phi}(\vec{x}\,)|0\rangle$ makes sense as a local particle state at position $\vec{x}$. For finite masses, though, $\hat{\phi}(\vec{x}\,)|0\rangle$ is always an extended object with the Compton wavelength $\lambda_c = 1/m$ governing its effective range. This length scale represents the inherent minimum uncertainty on the particle's position, just as we predicted earlier. This also tells us that in Quantum Field Theory a truly local measurement of a single particle at position $\vec{x}$ actually does not exist!

1.4 Switching on the time dependence (§ 2.4 in the book)

4c *Next we add the time dependence by switching to the Heisenberg picture, which makes all operators time dependent according to* $\hat{O} \to \hat{O}(t) \equiv e^{i\hat{H}t}\hat{O}e^{-i\hat{H}t}$ *as expected from the fact that* $\hat{H}$ *is the generator of time translations. This implies that the canonical (equal-time) commutation relations have the same form as in the Schrödinger picture:* $\big[\hat{\phi}(\vec{x},t), \hat{\pi}(\vec{y},t)\big] = i\delta(\vec{x}-\vec{y})\hat{1}$, *with all other commutators being 0.*

Short derivation of $\hat{\phi}(x) = \hat{\phi}(\vec{x},t)$: we have $\big[\hat{H}, \hat{a}_{\vec{p}}\big] = -\,\omega_{\vec{p}}\,\hat{a}_{\vec{p}} \Rightarrow \hat{H}\hat{a}_{\vec{p}} = \hat{a}_{\vec{p}}\,(\hat{H}-\omega_{\vec{p}})$ and $\hat{H}^n\hat{a}_{\vec{p}} = \hat{a}_{\vec{p}}\,(\hat{H}-\omega_{\vec{p}})^n$. That means that $e^{i\hat{H}t}\hat{a}_{\vec{p}}\,e^{-i\hat{H}t} = \hat{a}_{\vec{p}}\,e^{i(\hat{H}-\omega_{\vec{p}})t}e^{-i\hat{H}t} = \hat{a}_{\vec{p}}\,e^{-i\omega_{\vec{p}}t}$ and $e^{i\hat{H}t}\hat{a}^\dagger_{\vec{p}}\,e^{-i\hat{H}t} = \hat{a}^\dagger_{\vec{p}}\,e^{i\omega_{\vec{p}}t}$. Applied to $\hat{\phi}(x)$ this yields:

$$\hat{\phi}(x) \;=\; \int\frac{\mathrm{d}\vec{p}}{(2\pi)^3}\,\frac{1}{\sqrt{2\omega_{\vec{p}}}}\,\big(\hat{a}_{\vec{p}}\,e^{-ip\cdot x} + \hat{a}^\dagger_{\vec{p}}\,e^{ip\cdot x}\big)\Big|_{p_0=\omega_{\vec{p}}} \qquad \text{and} \qquad \hat{\pi}(x) \;=\; \frac{\partial}{\partial t}\hat{\phi}(x)\;,$$

where the first term corresponds to the positive frequency modes and the second term to the negative frequency modes. This reflects particle-wave duality, with each frequency mode corresponding to the creation/annihilation of fundamental quanta of the theory. Analogously:

$$\big[\hat{\vec{P}}, \hat{a}_{\vec{p}}\big] \;=\; -\,\vec{p}\,\hat{a}_{\vec{p}} \quad\Rightarrow\quad e^{-i\hat{\vec{P}}\cdot\vec{x}}\hat{a}_{\vec{p}}\,e^{i\hat{\vec{P}}\cdot\vec{x}} \;=\; \hat{a}_{\vec{p}}\,e^{i\vec{p}\cdot\vec{x}}\;.$$

Combining both identities yields

$$\hat{\phi}(x) \;\overset{[\hat{H},\hat{\vec{P}}\,]\,=\,0}{=\!=\!=\!=}\; e^{i(\hat{H}t\,-\,\hat{\vec{P}}\cdot\vec{x})}\hat{\phi}(0)\,e^{-i(\hat{H}t\,-\,\hat{\vec{P}}\cdot\vec{x})} \;=\; e^{i\hat{P}\cdot x}\hat{\phi}(0)\,e^{-i\hat{P}\cdot x}\;.$$

This reflects the fact that the quantized conserved Noether charges are the generators of the corresponding continuous transformations, which in this case implies that $\hat{P}^\mu$ is the generator of spacetime translations.

Next we invoke the following relativistic requirement.

4c *Causality: a measurement performed at one spacetime point* y *can only affect a measurement at another spacetime point* x *whose separation from the first point is timelike or lightlike, i.e.* $(x-y)^2 \geq 0$.

This latter requirement means that in such cases a particle can physically travel the corresponding spatial distance within the corresponding time period, since $(x-y)^2 \geq 0$ corresponds to a spacetime separation inside or on the lightcone $|\vec{x}-\vec{y}\,| = |x^0-y^0|$. In the coordinate representation any observable involving scalar particles can be written in terms of KG fields. So, if $\big[\hat{\phi}(x), \hat{\phi}(y)\big] = 0$ for $(x-y)^2 < 0$, then the measurements do not influence each other for spacelike separations (i.e. outside the lightcone) and causality is preserved.

For the real KG field we find

$$
\begin{aligned}
\big[\hat\phi(x),\hat\phi(y)\big] &= \int\frac{\mathrm{d}\vec p\,\mathrm{d}\vec q}{(2\pi)^6}\,\frac{1}{2\sqrt{\omega_{\vec p}\,\omega_{\vec q}}}\,\big[\hat a_{\vec p}\,e^{-ip\cdot x}+\hat a^\dagger_{\vec p}\,e^{ip\cdot x},\ \hat a_{\vec q}\,e^{-iq\cdot y}+\hat a^\dagger_{\vec q}\,e^{iq\cdot y}\big]\Big|_{p_0=\omega_{\vec p},\,q_0=\omega_{\vec q}}\\
&= \int\frac{\mathrm{d}\vec p}{(2\pi)^3}\,\frac{1}{2\omega_{\vec p}}\,\big(e^{-ip\cdot(x-y)}-e^{ip\cdot(x-y)}\big)\,\hat 1\,\Big|_{p_0=\omega_{\vec p}} \;\equiv\; D(x-y)\,\hat 1 - D(y-x)\,\hat 1\ .
\end{aligned}
$$

The function $D(x)$ has the following properties:

1. In the previous expression each of the terms on the left-hand-side of the second line is Lorentz invariant according to equation (1). As a result, the function $D(x)$ is Lorentz invariant as well and hence $D(x)=D(\Lambda x)\equiv D(x')$.

2. $D(x)=D(-x)$ if $x_0=0$. This follows directly by taking $\vec p\to-\vec p$ in the integration.

Bearing in mind that for $(x-y)^2<0$ there exists a Lorentz transformation Λ such that $x_0'-y_0'=0$, we can derive from these two properties that

$$
\begin{aligned}
0 \;\overset{\text{property 2}}{=\!=\!=}\;& D(x'-y')-D(y'-x') \;=\; D\big(\Lambda(x-y)\big)-D\big(\Lambda(y-x)\big)\\
\overset{\text{property 1}}{=\!=\!=}\;& D(x-y)-D(y-x) \qquad\quad \text{if}\ \ (x-y)^2<0\ .
\end{aligned}
$$

This automatically implies that causality is preserved in the real KG theory because propagation from y to x, given by $\langle 0|\hat\phi(x)\hat\phi(y)|0\rangle = D(x-y)$, is indistinguishable from propagation from x to y, given by $\langle 0|\hat\phi(y)\hat\phi(x)|0\rangle = D(y-x)$, if $(x-y)^2<0$. This sounds weird, but in the spacelike regime we cannot think of propagation as particle movement. There is no Lorentz invariant way to order events, since if we have in one frame that $x_0-y_0>0$ a Lorentz transformation can yield another frame where $x_0-y_0<0$.

> 4 *In fact, quantizing using canonical quantization conditions was already sufficient for properly implementing causality. In spite of its non-covariant form, there is no preferred treatment of time by quantizing in the canonical way!*

Proof: the proof of this statement exploits the fact that $\big[\hat\phi(x),\hat\phi(y)\big]$ is Lorentz invariant, as well as the fact that for $(x-y)^2<0$ there exists a Lorentz transformation Λ such that $x_0'-y_0'=0$. Then we can readily obtain the causality requirement

$$
\big[\hat\phi(x),\hat\phi(y)\big] \;\overset{\text{Lor. inv.}}{=\!=\!=}\; \big[\hat\phi(\vec x\,',t'),\hat\phi(\vec y\,',t')\big] \;=\; e^{i\hat H t'}\big[\hat\phi(\vec x\,'),\hat\phi(\vec y\,')\big]\,e^{-i\hat H t'} \;=\; 0
$$

for $(x-y)^2<0$ as a direct consequence of canonical quantization.

1.5 Quantization of the free complex Klein-Gordon theory

The Lagrangian for a complex scalar field $\phi(x)$ satisfying the free KG equation is given by

$$\mathcal{L} \,=\, (\partial_\mu \phi)(\partial^\mu \phi^*) - m^2 \phi\phi^*,$$

which contains twice as many degrees of freedom as the Lagrangian of the real KG theory. This can be seen explicitly by writing $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$ with $\phi_{1,2} \in \mathbb{R}$ (see Ex. 4). Then the Lagrangian becomes

$$\mathcal{L} \,=\, \frac{1}{2}(\partial_\mu \phi_1)(\partial^\mu \phi_1) \,-\, \frac{1}{2}m^2\phi_1^2 \,+\, \frac{1}{2}(\partial_\mu \phi_2)(\partial^\mu \phi_2) \,-\, \frac{1}{2}m^2\phi_2^2 \,.$$

Now we can either treat $\phi_{1,2}$ or ϕ, ϕ^* as independent degrees of freedom. The quantization goes exactly as before, with $\frac{1}{\sqrt{2}}(\hat{a}_{1,\vec{p}} + i\,\hat{a}_{2,\vec{p}}) \equiv \hat{a}_{\vec{p}}$ and $\frac{1}{\sqrt{2}}(\hat{a}^\dagger_{1,\vec{p}} + i\,\hat{a}^\dagger_{2,\vec{p}}) \equiv \hat{b}^\dagger_{\vec{p}} \neq \hat{a}^\dagger_{\vec{p}}$. Hence:

$$\hat{\phi}(x) \,=\, \int \frac{\mathrm{d}\vec{p}}{(2\pi)^3}\,\frac{1}{\sqrt{2\omega_{\vec{p}}}}\,\big(\hat{a}_{\vec{p}}\,e^{-ip\cdot x} + \hat{b}^\dagger_{\vec{p}}\,e^{ip\cdot x}\big)\Big|_{p_0=\omega_{\vec{p}}} \,,$$

where the first term corresponds to particles and the second to so-called antiparticles. The associated commutators are given by:

$$\big[\hat{a}_{\vec{p}}, \hat{a}^\dagger_{\vec{q}}\big] \,=\, \big[\hat{b}_{\vec{p}}, \hat{b}^\dagger_{\vec{q}}\big] \,=\, (2\pi)^3\delta(\vec{p}-\vec{q})\,\hat{1}, \quad \text{with all other commutators being } 0 \,.$$

From these commutation relations we can derive that causality is conserved in the complex Klein-Gordon theory as well:

$$\big[\hat{\phi}(x), \hat{\phi}(y)\big] \,=\, \big[\hat{\phi}^\dagger(x), \hat{\phi}^\dagger(y)\big] \,=\, 0 \,,$$

$$\big[\hat{\phi}(x), \hat{\phi}^\dagger(y)\big] \,=\, D(x-y)\,\hat{1} - D(y-x)\,\hat{1} \,\overset{\text{see before}}{=\!=\!=}\, 0 \quad \text{if} \quad (x-y)^2 < 0 \,.$$

Note that $D(x-y)$ originates from particle propagation, whereas $D(y-x)$ originates from antiparticle propagation. This brings us to the following important conclusion:

> 4c *the correct causal structure of the complex Klein-Gordon theory hinges on the combined treatment of particles and antiparticles, since particle propagation from y to x, $\langle 0|\hat{\phi}(x)\hat{\phi}^\dagger(y)|0\rangle = D(x-y)$, is indistinguishable from antiparticle propagation from x to y, $\langle 0|\hat{\phi}^\dagger(y)\hat{\phi}(x)|0\rangle = D(y-x)$, if $(x-y)^2 < 0$.*

Particle interpretation: as before we can derive the particle interpretation by looking at the energy, momentum and "charge" operators (see Ex. 4 for a critical discussion). After quantization these operators read:

$$\hat{H} \,=\, \int \frac{\mathrm{d}\vec{p}}{(2\pi)^3}\,\omega_{\vec{p}}\,(\hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{p}} + \hat{b}^\dagger_{\vec{p}}\hat{b}_{\vec{p}}) \,+\, \text{zero-point energy} \,,$$

$$\hat{\vec{P}} \,=\, \int \frac{\mathrm{d}\vec{p}}{(2\pi)^3}\,\vec{p}\,(\hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{p}} + \hat{b}^\dagger_{\vec{p}}\hat{b}_{\vec{p}}) \,,$$

$$\hat{Q} \,=\, \int \frac{\mathrm{d}\vec{p}}{(2\pi)^3}\,(-\,\hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{p}} + \hat{b}^\dagger_{\vec{p}}\hat{b}_{\vec{p}}) \,=\, -\,\hat{N}_{particles} \,+\, \hat{N}_{antiparticles} \,.$$

The zero-point term for the charge operator has to vanish to guarantee Lorentz-invariant vacuum properties (see Ex. 4), so normal ordering is a physical requirement in that case! This charge operator is the generator of $U(1)$ phase transformations:

$$\big[\hat{Q}, \hat{\phi}(x)\big] \;=\; \hat{\phi}(x) \qquad \Rightarrow \qquad e^{i\theta\hat{Q}}\,\hat{\phi}(x)\,e^{-i\theta\hat{Q}} \;=\; e^{i\theta}\hat{\phi}(x) \qquad \text{for } \theta \in \mathbb{R} \text{ constant} .$$

Since the aforementioned conserved quantities only contain number operators after quantization, we have

$$\big[\hat{H}, \hat{\vec{P}}\big] \;=\; \big[\hat{H}, \hat{Q}\big] \;=\; \big[\hat{H}, \hat{N}_{particles}\big] \;=\; \big[\hat{H}, \hat{N}_{antiparticles}\big] \;=\; 0 .$$

4d *In free KG theories (in fact in all free theories) energy, momentum, number of particles and number of antiparticles are all conserved. In interacting theories the number of particles and the number of antiparticles are no longer separately conserved, but their difference quite often is.*

Now we can read off the particle interpretation of the complex KG theory: it resembles the one for the real KG theory, with the difference being that for every particle state there should now also be an antiparticle state with opposite "charge" quantum numbers and the same 4-momentum quantum numbers. An example of such a scalar particle–antiparticle combination is given by the $\pi^{\mp}$ pions. The case $\hat{\phi} = \hat{\phi}^\dagger$ is special in the sense that particle and antiparticle states coincide, so all "charges" should be 0.

Lorentz transformations and $\hat{\phi}(x)$: as before $\hat{\phi}(x) = e^{i\hat{P}\cdot x}\hat{\phi}(0)\,e^{-i\hat{P}\cdot x}$, but what about Lorentz transformations? We know that $|\vec{p}\rangle = \sqrt{2\omega_{\vec{p}}}\,\hat{a}^\dagger_{\vec{p}}|0\rangle$ and that a similar expression holds for antiparticle states, so we can use this to define the unitary operator that implements (active) Lorentz transformations in the Hilbert space of quantum states:

$$|\overrightarrow{\Lambda p}\rangle \equiv \hat{U}(\Lambda)|\vec{p}\rangle \;\Rightarrow\; \sqrt{2\omega_{\overrightarrow{\Lambda p}}}\,\hat{a}^\dagger_{\overrightarrow{\Lambda p}}|0\rangle \;=\; \sqrt{2\omega_{\vec{p}}}\,\hat{U}(\Lambda)\,\hat{a}^\dagger_{\vec{p}}|0\rangle \;\overset{\hat{U}(\Lambda)|0\rangle\equiv|0\rangle}{=\!=\!=\!=}\; \sqrt{2\omega_{\vec{p}}}\,\hat{U}(\Lambda)\,\hat{a}^\dagger_{\vec{p}}\,\hat{U}^{-1}(\Lambda)|0\rangle$$

$$\Rightarrow \text{ define: } \hat{U}(\Lambda)\,\hat{a}^\dagger_{\vec{p}}\,\hat{U}^{-1}(\Lambda) \;=\; \sqrt{\frac{\omega_{\overrightarrow{\Lambda p}}}{\omega_{\vec{p}}}}\;\hat{a}^\dagger_{\overrightarrow{\Lambda p}} ,$$

with a similar expression for $\hat{b}^\dagger_{\vec{p}}$. As a result:

$$\hat{U}(\Lambda)\hat{\phi}(x)\hat{U}^{-1}(\Lambda) \;=\; \int \frac{\mathrm{d}\vec{p}}{(2\pi)^3}\,\frac{1}{2\omega_{\vec{p}}}\,\sqrt{2\omega_{\overrightarrow{\Lambda p}}}\,\big(\hat{a}_{\overrightarrow{\Lambda p}}\,e^{-ip\cdot x} + \hat{b}^\dagger_{\overrightarrow{\Lambda p}}\,e^{ip\cdot x}\big)$$

$$\overset{p'=\Lambda p}{=\!=\!=} \int \frac{\mathrm{d}\vec{p}\,'}{(2\pi)^3}\,\frac{1}{2\omega_{\vec{p}\,'}}\,\sqrt{2\omega_{\vec{p}\,'}}\,\big(\hat{a}_{\vec{p}\,'}e^{-ip'\cdot\Lambda x} + \hat{b}^\dagger_{\vec{p}\,'}e^{ip'\cdot\Lambda x}\big) \;=\; \hat{\phi}(\Lambda x) ,$$

where the second line is obtained by using that $\int \mathrm{d}\vec{p}/(2\omega_{\vec{p}})$ and $e^{\pm ip\cdot x}$ are all Lorentz invariant. This implies that the transformed field creates/destroys antiparticles/particles at the spacetime point Λx.

1.6 Inversion of the Klein-Gordon equation (§ 2.4 in the book)

4e *For certain physical applications it is important to know the inverse of the KG equation, for instance for deriving scattering amplitudes or for solving systems that involve a KG field being coupled to a classical source.*

Since a solution to $(\Box + m^2)\phi_0(x) = 0$ exists, the inversion of the differential operator $(\Box + m^2)$ does not exist formally, so it has to be defined. Once we have defined this inverse $(\Box + m^2)^{-1}$ properly an appropriate solution to the equation $(\Box + m^2)\phi = j$ is given by $\phi = (\Box + m^2)^{-1} j + \phi_0$, given that $\phi = \phi_0$ in the absence of the source j.

Green's function: let's try to find the so-called Green's function $G(x-y)$, which is the inverse KG operator $(\Box + m^2)^{-1}$ written in coordinate space. By convention this Green's function is required to satisfy $(\Box_x + m^2)G(x-y) \equiv -i\,\delta^{(4)}(x-y) = -i\,\delta(x^0-y^0)\,\delta(\vec{x}-\vec{y})$, where the right-hand-side represents (up to the conventional factor $-i$) the unit operator in coordinate space. In momentum space this becomes

$$G(x-y) \;\equiv\; \int \frac{\mathrm{d}^4p}{(2\pi)^4}\,\tilde{G}(p)\,e^{-ip\cdot(x-y)} \quad \text{and} \quad \delta^4(x-y) \;=\; \int \frac{\mathrm{d}^4p}{(2\pi)^4}\,e^{-ip\cdot(x-y)}\ ,$$

so that

$$(-p^2 + m^2)\tilde{G}(p) \;=\; -i \quad \Rightarrow \quad \tilde{G}(p) \;=\; \frac{i}{p^2 - m^2}\ .$$

The problem with defining the inverse of the KG operator is apparent now: it resides in the fact that $p^2 - m^2 = p_0^2 - (\vec{p}^{\,2} + m^2) = p_0^2 - \omega_{\vec{p}}^2 = 0$ for the physical (anti)particles of the KG theory. In these so-called on-mass-shell (or short: on-shell) situations with $p^2 = m^2$ the Fourier coefficient of the Green's function blows up, thereby leading to an ill-defined Fourier integral. That brings us to the key question that we have to address if we want to define a proper Green's function:

how should we go around the poles of $(p^2 - m^2)^{-1} = (p_0 - \omega_{\vec{p}})^{-1}(p_0 + \omega_{\vec{p}})^{-1}$ while performing the Fourier integral?

There are several options for this, reflecting the fact that the Green's function cannot be defined uniquely. We mention here two useful possible definitions.

1) The retarded Green's function: for taking into account influences that lie in the past it is useful to shift the poles into the lower-half of the complex plane by an infinitesimal amount $-i\epsilon$ (see figure 2), where the infinitesimal constant $\epsilon \in \mathbb{R}^+$ should be taken to 0 at the end of the calculation.

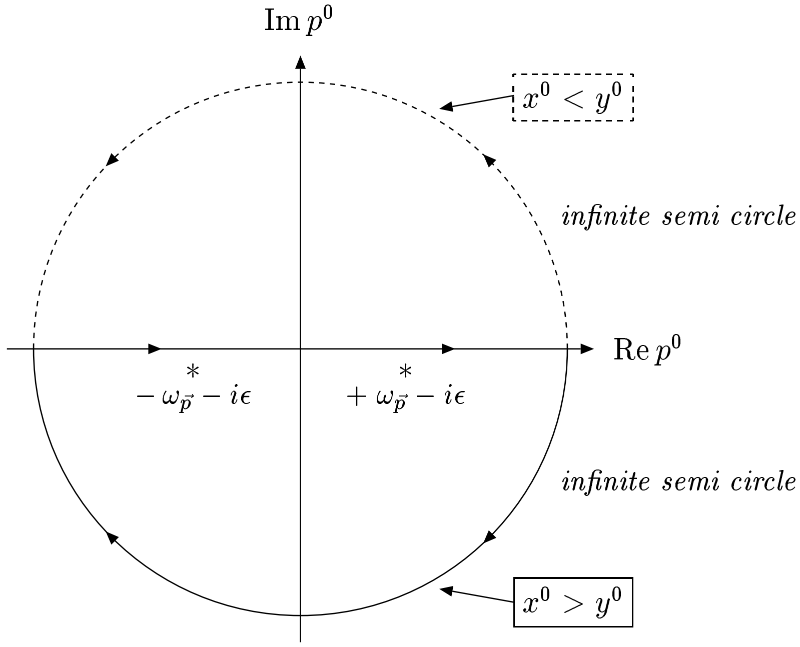


Figure 2: Complex poles and closed integration contours for the retarded Green's function.

Using the complex integration contours as indicated in figure 2, the Fourier integration yields

$$\begin{aligned} D_R(x-y) &= \Theta(x^0-y^0)(-2\pi i)\int\frac{\mathrm{d}\vec{p}}{(2\pi)^4}\left\{\frac{ie^{-ip\cdot(x-y)}}{2\omega_{\vec{p}}}\bigg|_{p_0=\omega_{\vec{p}}}+\frac{ie^{-ip\cdot(x-y)}}{-2\omega_{\vec{p}}}\bigg|_{p_0=-\omega_{\vec{p}}}\right\} \\ &\underline{\underline{\vec{p}\to-\vec{p}\text{ in 2nd term}}}\;\Theta(x^0-y^0)\int\frac{\mathrm{d}\vec{p}}{(2\pi)^3}\frac{1}{2\omega_{\vec{p}}}\left(e^{-ip\cdot(x-y)}-e^{ip\cdot(x-y)}\right)\Big|_{p_0=\omega_{\vec{p}}}\ , \end{aligned}$$

which means that

$$D_R(x-y) \;=\; \Theta(x^0-y^0)\big(D(x-y)-D(y-x)\big) \;=\; \Theta(x^0-y^0)\langle 0|\big[\hat{\phi}(x),\hat{\phi}^\dagger(y)\big]|0\rangle\ .$$

Application: consider a real KG field coupled to an external classical source $j(x)$ that is switched on during a finite time interval. Then

$$(\Box+m^2)\hat{\phi}(x) \;=\; j(x)\in\mathbb{R}\ ,$$

which would correspond to an extra term $+j(x)\phi(x)$ in the Lagrangian (resembling a forced oscillator). Before $j(x)$ is turned on we have

$$\hat{\phi}(x) \;=\; \hat{\phi}_0(x) \;=\; \int\frac{\mathrm{d}\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{p}}}}\left(\hat{a}_{\vec{p}}\,e^{-ip\cdot x}+\hat{a}^\dagger_{\vec{p}}\,e^{ip\cdot x}\right)\Big|_{p_0=\omega_{\vec{p}}}\ ,$$

with $\phi_0(x)$ a solution to the free KG equation $(\Box+m^2)\phi_0(x)=0$. After $j(x)$ is turned on we have

$$\hat{\phi}(x) \;=\; \hat{\phi}_0(x) \,+\, i\!\int\! \mathrm{d}^4y\, D_R(x-y)\,j(y)$$

$$=\; \hat{\phi}_0(x) \,+\, i\!\int\! \mathrm{d}^4y \int\! \frac{\mathrm{d}\vec{p}}{(2\pi)^3}\,\frac{j(y)}{2\omega_{\vec{p}}}\,\Theta(x^0-y^0)\left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\right)\Big|_{p_0=\omega_{\vec{p}}} \;.$$

If x^0 is smaller than the switch-on time of j then $\Theta(x^0-y^0)j(y) = 0$ and only $\hat{\phi}_0(x)$ remains, in agreement with the initial condition we started out with. If x^0 is larger than the switch-off time of j, then $\Theta(x^0-y^0)j(y) = j(y)$. Using that $\int \mathrm{d}^4y\, e^{ip\cdot y} j(y) \equiv \tilde{j}(p)$ and $\int \mathrm{d}^4y\, e^{-ip\cdot y} j(y) \overset{j\in\mathbb{R}}{=\!=} \tilde{j}^{\,*}(p)$ we find in that case that

$$\hat{\phi}(x) \;=\; \int\! \frac{\mathrm{d}\vec{p}}{(2\pi)^3}\,\frac{1}{\sqrt{2\omega_{\vec{p}}}}\left\{\Big(\hat{a}_{\vec{p}} + i\,\frac{\tilde{j}(p)}{\sqrt{2\omega_{\vec{p}}}}\Big)e^{-ip\cdot x} \,+\, \Big(\hat{a}^\dagger_{\vec{p}} - i\,\frac{\tilde{j}^{\,*}(p)}{\sqrt{2\omega_{\vec{p}}}}\Big)e^{ip\cdot x}\right\}\Bigg|_{p_0=\omega_{\vec{p}}}$$

$$\equiv\; \int\! \frac{\mathrm{d}\vec{p}}{(2\pi)^3}\,\frac{1}{\sqrt{2\omega_{\vec{p}}}}\left(\hat{\alpha}_{\vec{p}}\, e^{-ip\cdot x} + \hat{\alpha}^\dagger_{\vec{p}}\, e^{ip\cdot x}\right)\Big|_{p_0=\omega_{\vec{p}}}$$

and

$$N(\hat{H}) \;=\; \int\! \frac{\mathrm{d}\vec{p}}{(2\pi)^3}\,\omega_{\vec{p}}\,\hat{\alpha}^\dagger_{\vec{p}}\hat{\alpha}_{\vec{p}} \;,$$

with N denoting normal ordering. The operator $\hat{\alpha}_{\vec{p}}$ is a quasi-particle annihilation operator, satisfying

$$\hat{\alpha}_{\vec{p}}\,|0\rangle \;=\; i\,\frac{\tilde{j}(p)}{\sqrt{2\omega_{\vec{p}}}}\,|0\rangle \;\equiv\; \lambda_{\vec{p}}\,|0\rangle \;.$$

So, the free-particle vacuum state $|0\rangle$ is now a quasi-particle coherent state. Its energy has changed by an amount

$$\Delta E_0 \;=\; \langle 0|N(\hat{H})|0\rangle \;=\; \int\! \frac{\mathrm{d}\vec{p}}{(2\pi)^3}\,\frac{1}{2}\,|\tilde{j}(p)|^2 \;,$$

corresponding to $\langle 0|\int \frac{\mathrm{d}\vec{p}}{(2\pi)^3}\,\hat{\alpha}^\dagger_{\vec{p}}\hat{\alpha}_{\vec{p}}\,|0\rangle = \int \frac{\mathrm{d}\vec{p}}{(2\pi)^3}\,|\lambda_{\vec{p}}|^2 = \int \frac{\mathrm{d}\vec{p}}{(2\pi)^3}\,\frac{|\tilde{j}(p)|^2/2}{\omega_{\vec{p}}}$ quasi-particles.

> *The particle interpretation has changed as a result of the influence of the external source! This example shows that particles and quasi-particles are derived quantities and that the retarded Green's functions are handy tools for dealing with external influences that are switched on during a finite amount of time.*

2) Feynman propagator: an alternative way of shifting the poles is given in figure 3. As will be worked out in Ex. 5, this pole configuration is equivalent with replacing $(p^2-m^2)^{-1}$ by $(p^2-m^2+i\epsilon)^{-1}$, where again the infinitesimal constant $\epsilon \in \mathbb{R}^+$ should be taken to 0 at the end of the calculation.

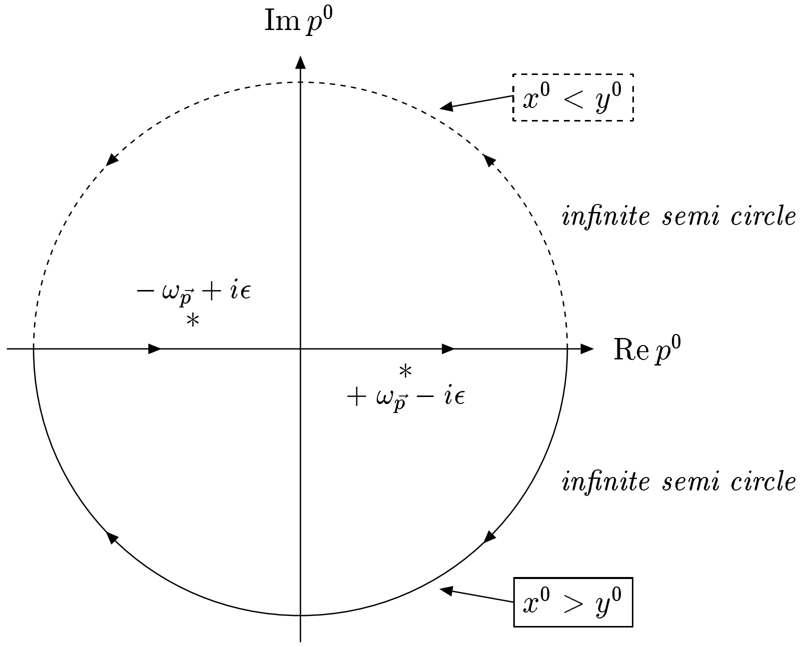


Figure 3: Complex poles and closed integration contours for the Feynman propagator.

Using the complex integration contours as indicated in figure 3, the Fourier integration yields

$$D_F(x-y) \;=\; \begin{cases} -2\pi i \displaystyle\int \frac{\mathrm{d}\vec{p}}{(2\pi)^4}\, \frac{ie^{-ip\cdot(x-y)}}{2\omega_{\vec{p}}}\Bigg|_{p_0=\omega_{\vec{p}}} & \text{if } \ x^0 > y^0 \\ +2\pi i \displaystyle\int \frac{\mathrm{d}\vec{p}}{(2\pi)^4}\, \frac{ie^{-ip\cdot(x-y)}}{-2\omega_{\vec{p}}}\Bigg|_{p_0=-\omega_{\vec{p}}} & \text{if } \ x^0 < y^0 \end{cases} ,$$

which means that

$$\begin{aligned} D_F(x-y) \;&=\; \Theta(x^0-y^0)D(x-y) \,+\, \Theta(y^0-x^0)D(y-x) \\ &=\; \Theta(x^0-y^0)\,\langle 0|\hat{\phi}(x)\hat{\phi}^\dagger(y)|0\rangle \,+\, \Theta(y^0-x^0)\,\langle 0|\hat{\phi}^\dagger(y)\hat{\phi}(x)|0\rangle \\ &\equiv\; \langle 0|T(\hat{\phi}(x)\hat{\phi}^\dagger(y))|0\rangle \ . \end{aligned}$$

This is the definition of time ordering: the operator at the latest time is put in front. The Feynman propagator $D_F(x-y)$ is the time-ordered propagation amplitude.

4e *The Feynman propagator will feature prominently in the derivation of scattering amplitudes in perturbation theory!*

2 Interacting scalar fields and Feynman diagrams

The next eight lectures cover large parts of Chapters 4 and 7 as well as a few aspects of Chapter 10 of Peskin & Schroeder.

> 5 *The task that we set ourselves is to investigate the consequences of adding interactions that couple different Fourier modes and, as such, the associated particles. This will be quite a bit more complicated than the free theories that we have encountered in the previous chapter, where the relevant quantities were diagonal (i.e. decoupled) in the momentum representation and particle numbers were conserved explicitly. Even worse, up to now nobody has been able to solve general interacting field theories. Therefore we will focus on weakly coupled field theories, which can be investigated by means of perturbation theory.*

Causality dictates us to add local terms only, i.e. $\hat{\mathcal{L}}_{int}(x)$ and not $\hat{\mathcal{L}}_{int}(x,y)$. In order to investigate what is meant by "weak interactions", the following interacting real scalar theory is considered:

$$\mathcal{L} \,=\, \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) \,-\, \frac{1}{2}m^2\phi^2 \,+\, \mathcal{L}_{int} \qquad \text{with} \qquad \mathcal{L}_{int} \,=\, -\sum_{n\geq 3}\frac{\lambda_n}{n!}\,\phi^n \quad (\phi\in\mathbb{R})\ ,$$

where $\lambda_n \in \mathbb{R}$ is called a coupling constant. Note that $\mathcal{L}_{int} = -\,\mathcal{H}_{int}$, since it contains no derivatives. The corresponding Euler-Lagrange equation is not a simple linear (wave) equation anymore:

$$\partial_\mu(\partial^\mu\phi) \,+\, m^2\phi \,+\, \sum_{n\geq 3}\frac{\lambda_n}{(n-1)!}\,\phi^{n-1} \,=\, 0 \quad \Rightarrow \quad (\Box + m^2)\phi \,=\, -\sum_{n\geq 3}\frac{\lambda_n}{(n-1)!}\,\phi^{n-1}\ .$$

Since $\pi_\phi = \partial_0\phi$ is unaffected by the interaction, the quantum mechanical basis

$$\big[\hat{\phi}(\vec{x}\,), \hat{\pi}_\phi(\vec{y}\,)\big] \,=\, i\delta(\vec{x}-\vec{y}\,)\hat{1} \quad \text{and all other commutators being 0}$$

is the same as in the free KG case. Hence, $\hat{\phi}(\vec{x}\,)$ and $\hat{\pi}_\phi(\vec{x}\,)$ can be given the same Fourier-decomposed form as before (cf. page 14). However, since the non-linear $\hat{\phi}^{n-1}$ term contains for example $(\hat{a}^\dagger)^{n-1}$, the number of particles is not conserved anymore as a result of the interaction. Consequently, also the particle interpretation, which can be obtained from the Hamilton operator, will be different.

2.1 When are interaction terms small? (§ 4.1 in the book)

> 5a *To answer this question we have to perform a dimensional analysis: the action $S = \int d^4x\,\mathcal{L}$ is dimensionless, so $\mathcal{L}$ must have dimension (mass)^4, or short "dimension 4". The shorthand notation for this is $[\mathcal{L}] = 4$.*

Kinetic term: the kinetic term has the form $(\partial_\mu \phi)(\partial^\mu \phi)$. Since $[\partial_\mu] = 1$, that means that $[\phi] = 1$, which is consistent with the dimension of the mass term $\propto m^2\phi^2$.

Interaction terms: since $[\phi^n] = n$, the coupling constants have a dimension $[\lambda_n] = 4-n$. So, λ_n is not dimensionless, except when $n = 4$. Three cases can be distinguished:

1. Coupling constants with positive mass dimension. Take λ_3 as an example. Using the dimension of the field, we can see that $[\lambda_3] = +1$. In a process at energy scale E the coupling constant λ_3 will enter in the dimensionless combination λ_3/E. The ϕ^3 interaction can therefore be considered weak at high energies $(E \gg \lambda_3)$ and strong at small energies $(E \ll \lambda_3)$. For the latter reason such interactions are called relevant.

2. Dimensionless coupling constants. For our real scalar theory, the only dimensionless coupling constant is λ_4 since $[\lambda_4] = 0$. The ϕ^4 interaction can be considered weak if the coupling constant is small $(\lambda_4 \ll 1)$. Such interactions are called marginal, since they are equally important at all energy scales.

3. Coupling constants with negative mass dimension. For the coupling constants with $n \geq 5$ we have $[\lambda_{n \geq 5}] = 4 - n < 0$. In a process at energy scale E the coupling constants $\lambda_{n \geq 5}$ will enter in the dimensionless combinations $\lambda_n E^{n-4}$. The $\phi^{n \geq 5}$ interactions can therefore be considered weak at low energies and strong at high energies. Because of this suppressed influence on low-energy physics, such interactions are called irrelevant. Such interactions have their origin in underlying physics that takes place at higher energy scales.

   5a *Complication: it is impossible to avoid high energies in quantum field theory, because of the occurrence of integrals over all momenta at higher orders in perturbation theory. We have in fact already encountered an example of this in § 1.3 while discussing the zero-point energy and its infinities.*

2.2 Renormalizable versus non-renormalizable theories

Renormalizable theories: a renormalizable theory has the marked property that it is not sensitive to our lack of knowledge about high-scale physics. It therefore

- keeps its predictive power at all energy scales in spite of the occurrence of high-energy effects in the quantum corrections;

- can be used to make precise theoretical predictions that can be confronted with experiment;

- does not involve coupling constants with negative mass dimension.

Guided by our quest for the ultimate "theory of everything", the prevalent view in high-energy physics used to be that any sensible theory that describes nature should be renormalizable. However, this requirement is based on the unrealistic assumption that any theory that attempts to describe aspects of nature has to be valid up to arbitrarily large energies. It is much more likely that at some energy scale new physics will kick in, causing the original theory to be incomplete.

Non-renormalizable theories

> 5b *In situations where our present theoretical knowledge proves insufficient or where we prefer to describe the physics up to a minimum length scale, another class of theories is particularly useful. These mostly non-renormalizable theories are obtained by parametrizing our ignorance (scenario 1 discussed below) or by "integrating out" known/anticipated physics at small length scales (scenario 2 discussed below).*

Non-renormalizable theories, scenario 1: unknown new physics.
Suppose that we are starting to observe experimental deviations from our favourite model of the world, caused by some unknown high-scale physics. If we only have access to this high-scale physics through low-energy data (see the Fermi-model example below), we sometimes have to content ourselves with an incomplete model that describes the physics as seen through blurry glasses. In that case we only know the physics up to a certain energy scale μ (i.e. down to a length scale $1/\mu$) with higher energy scales (i.e. smaller length scales) being integrated out. This will in general result in a non-renormalizable effective theory that describes nature up to the energy scale μ and a Lagrangian that will parametrize our lack of knowledge about the physics that takes place at higher energy scales. Such effective theories

- have limited predictive power, since the physics at high energy scales $E \gg \mu$ is not described properly;
- can contain interactions with coupling constants of negative mass dimension, which would formally lead to uncontrolled UV infinities at higher orders in perturbation theory as a result of integrals over all momenta (if we would assume the theory to be correct at all energy scales, ... which would be incorrect);
- can nevertheless be used to make reliable predictions at $\mathcal{O}(\mu)$ energies provided that the unknown high-scale physics resides at an energy scale $\Lambda_{\text{NP}} \gg \mu$;
- may reveal at which energy scale the unknown high-scale physics must emerge.

Non-renormalizable theories, scenario 2: known/anticipated new physics. The moment we (think to) know the underlying physics model that is responsible for the observed low-energy phenomena, we can explicitly integrate out the high-energy degrees of freedom from the model. This results in the same type of effective Lagrangian, but this time the underlying physics model has left its fingerprints on the coupling constants. For instance, if the energy/mass scale of the underlying physics resides at $\Lambda_{\rm NP}$, then this scale will act as a natural scaling factor in the couplings. This procedure of explicitly linking the coupling constants of the effective theory to the parameters of the underlying physics model is called matching.

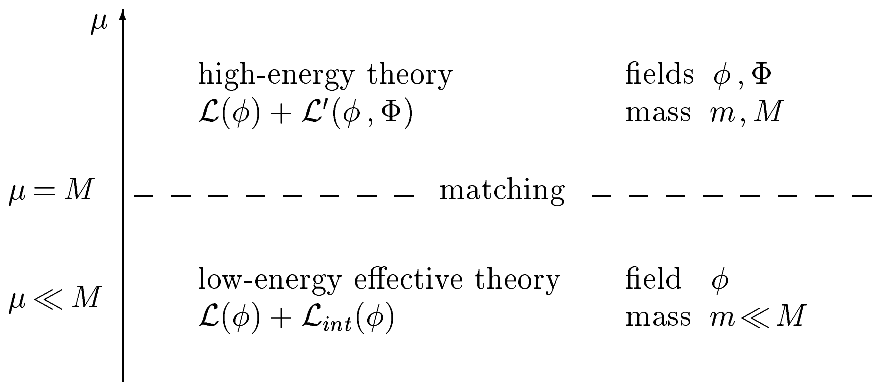
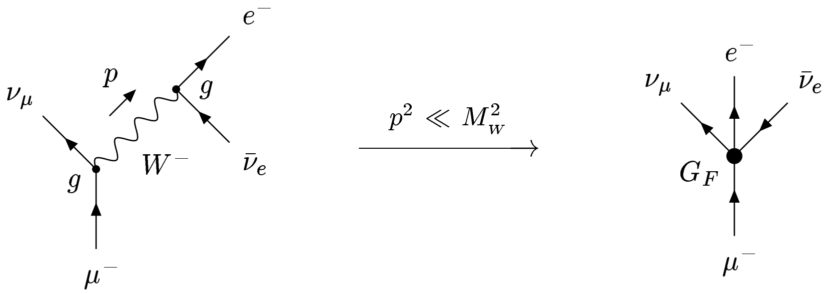


Figure 4: Schematic display of a low-energy effective theory containing a light field ϕ with mass m, originating from a high-energy theory that also includes a heavy field Φ with mass M.

Example: the Fermi-model of weak interactions. This probably sounds rather abstract, so let's have a closer look at the above-given statements by considering an explicit example. The so-called Fermi-model of weak interactions has in fact started out along the lines just described. In this example the role of $\Lambda_{\rm NP}$ is played by the mass M_W of the W boson. As will be explained in courses covering the Standard Model, decay processes like $\mu^- \to \nu_\mu e^- \bar{\nu}_e$ (muon decay) proceed through the exchange of a W boson with a mass of about 80 GeV between the particles. The associated decay amplitude contains a factor $1/(p^2 - M_W^2)$, originating from the propagator of the W-boson (cf. page 25), and two factors of g, corresponding to the coupling constant of the weak interactions. However, at the typical energy scale of the decay process, i.e. $E = \mathcal{O}(m_\mu = 0.1\,\text{GeV})$, the momentum carried by the W boson is much smaller than its mass M_W. In that case, the propagator factor is perceived as having a constant value:

$$\frac{g^2}{p^2 - M_W^2} \quad \xrightarrow{\;p^2 \,\ll\, M_W^2\;} \quad -\frac{g^2}{M_W^2} + \mathcal{O}(p^2/M_W^4)\ .$$

In terms of a diagrammatic representation of the physics that goes on in the decay process (see later) this corresponds to



On the basis of such "low-energy" decay processes the existence of (effective) 4-particle interactions was postulated (Fermi, 1932), with the corresponding dimensionful effective coupling constant (Fermi-coupling) being small in view of the absorbed $1/M_W^2$ suppression factor. This explains the name "weak interactions", which simply refers to the fact that these interactions were perceived as weak at low energies. At $p^2 = \mathcal{O}(M_W^2)$ the weak-interaction physics underlying the W-boson exchange will reveal itself and the weak interactions will no longer be weak.

> 5b *This is of course all hindsight, since in 1932 the correct model for the weak interactions did not exist yet. In fact, the above argument can be reversed. The low-energy Fermi-coupling was measured to be of* $\mathcal{O}(10^{-5}\, GeV^{-2}) \approx \mathcal{O}(\Lambda_{NP}^{-2})$*, which correctly signals that the physics underlying the weak interactions must reveal itself at an energy scale of* $\mathcal{O}(100\, GeV)$.

Planck scale: applying the same reasoning to the even smaller gravitational constant, i.e. $G = \mathcal{O}(10^{-38}\,\mathrm{GeV}^{-2})$, we would predict that gravity becomes strong at an energy scale of $\mathcal{O}(10^{19}\,\mathrm{GeV})$, which is commonly referred to as the Planck scale Λ_{P}.

Generic properties of effective field theories: the philosophy behind effective field theories is mostly a pragmatic one. If you want to describe certain physical phenomena quantitatively, it is an overkill to use a physics model that also gives details about experimentally inaccessible phenomena (like strong gravitational effects). In that case it is more practical to make use of a simpler, effective description that captures the most important physics of the system without giving unnecessary detail. Additional (small) effects resulting from the more fundamental theory can be taken into account by adding them as small perturbations (like relativistic corrections in non-relativistic quantum mechanics).

Consider for instance a fundamental theory with dimensionless coupling constants that describes the world at $\mathcal{O}(\Lambda_{\mathrm{NP}})$ energies. Assume, for argument's sake, that this theory contains a real scalar field ϕ that describes light particles with mass $m \ll \Lambda_{\mathrm{NP}}$ and another real scalar field Φ that describes much heavier particles with mass $M = \mathcal{O}(\Lambda_{\mathrm{NP}})$.

The laws of physics at $E \ll \Lambda_{\rm NP}$ are best formulated in terms of the light scalar field with interactions that are produced by the fundamental high-energy theory. After all, the heavy particles cannot be produced directly at these energies and therefore it is more practical to remove them from the description (i.e. integrate them out). This results in an effective Lagrangian as given before with effective couplings $\lambda_n = g_n/\Lambda_{\rm NP}^{n-4}$, where g_n is a dimensionless coupling constant governed by the high-energy theory. So, the impact of the $\phi^{n\geq 5}$ terms on physics at $E \ll \Lambda_{\rm NP}$ is suppressed by factors $(E/\Lambda_{\rm NP})^{n-4}$.

- The interactions that are most likely to affect low-energy experiments are the renormalizable ϕ^3 and ϕ^4 terms. That is why at sufficiently low energies effective theories only contain renormalizable interactions.
- The other interactions are suppressed at low energies and can therefore either be ignored or incorporated as small perturbations. This aspect makes it possible to include formally non-renormalizable interactions in the theory without spoiling its predictive power at low energies. At high energies this is not true anymore, but there the full glory of the underlying high-energy theory should be taken into account.
- Since the impact of the ϕ^n terms is extremely small for larger n, it is in general very tough to figure out the entire high-energy theory from low-energy data alone!

Remark: the physics at different length/energy scales can be related through the so-called renormalization group (see later). In particular in condensed-matter physics this renormalization group is a powerful analyzing tool, since different condensed-matter phenomena are quite often governed by different characteristic length scales. As we will see later, also in high-energy physics the renormalization group will prove very handy. The main difference between the field-theoretical treatments of both branches of physics resides in the absence of a smallest length scale in high-energy physics, whereas the atomic scale provides a natural cutoff in condensed-matter physics.

2.3 Perturbation theory (§ 4.2 in the book)

5c *Our ultimate aim is to calculate scattering cross sections and decay rates, from which information can be obtained on the fundamental particles that exist in nature and their mutual interactions. The following two models will be used in the remainder of this chapter:*

1. ϕ^4-theory: $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$ with $\phi \in \mathbb{R}$. This model contains the type of quartic interaction with dimensionless coupling constant that also features in the Higgs model.

2. Scalar Yukawa theory: $\mathcal{L} = (\partial_\mu \psi^*)(\partial^\mu \psi) + \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - M^2 \psi^* \psi - \frac{1}{2} m^2 \phi^2 - g \psi^* \psi \phi$ with $\phi \in \mathbb{R}$ and $\psi \in \mathbb{C}$. This is a toy model that resembles the Yukawa theory for the interaction between fermions and scalars, which will be discussed at a later stage. Apart from spin aspects these two theories differ in the dimension of the coupling constant, being $+1$ for the scalar Yukawa theory and 0 for the true Yukawa theory.

Non-relativistic quantum mechanics: in non-relativistic quantum mechanics scattering reactions are characterized by

- asymptotic free (non-interacting) situations at $t \to \mp\infty$, involving free particles in beam, target and detector (due to negligible wave-function overlap);
- a collision stage around $t = 0$ when the colliding particles interact/vanish and new particles may be produced.

Quantum field theory: we would like to use the same reasoning in quantum field theory, assuming the initial and final states of the reaction to be free-particle states. In that case the initial and final states of the reaction would be eigenstates of the Hamilton operator of the free Klein-Gordon theory, which are therefore also eigenstates of the particle and antiparticle number operator. In the end we will have to correct for two aspects that are not taken into account properly in this way (see later):

- bound states may form;
- more importantly, a particle well-separated from the other particles in the reaction is nevertheless not alone in quantum field theory, being surrounded by a cloud of virtual particles. It is not possible to switch off interactions in quantum field theory, so we have to correct for this later.

The Heisenberg picture: let's ignore these issues for the moment and try to develop a calculational toolbox based on the asymptotic free situations at $t \to \pm\infty$. As mentioned on page 27, we start out with the same quantum mechanical basis as in the free theory, so the Schrödinger picture field $\hat{\phi}(\vec{x}\,)$ can be given the same Fourier-decomposed form as before. The fact that we are dealing with an interacting theory manifests itself through the time-independent Hamilton operator, which is used in the Heisenberg picture and which is needed for determining the particle interpretation:

$$\hat{H} \;=\; \hat{H}_0 + \hat{H}_{int} \;=\; \hat{H}_0 + \int \mathrm{d}\vec{x}\; \hat{\mathcal{H}}_{int}(\vec{x}\,) \;=\; \hat{H}_0 - \int \mathrm{d}\vec{x}\; \hat{\mathcal{L}}_{int}(\vec{x}\,)\;.$$

The interaction Hamiltonian H_{int} is assumed to be weak compared to the Hamiltonian H_0 of the free theory. In the last step we have used that there are no derivatives in the interaction, so $\mathcal{H}_{int} = -\,\mathcal{L}_{int}$. This leads to Heisenberg fields

$$\hat{\phi}(x) \;\equiv\; \hat{\phi}(t, \vec{x}\,) \;=\; e^{i\hat{H}t}\hat{\phi}(\vec{x}\,)e^{-i\hat{H}t}\;,$$

where $e^{\pm i\hat{H}t}$ introduces extra creation/annihilation operators as a result of the presence of $\hat{H}_{int}$ and therefore changes the particle content and interpretation of the creation and annihilation operators. The ground state of the interacting theory will be denoted by $|\Omega\rangle$, which in general does not coincide with the vacuum state of the free theory (see the example on page 25). For this state we have $\hat{H}|\Omega\rangle = E_0|\Omega\rangle$, with E_0 the lowest energy level.

The interaction picture: the asymptotic free situation can be described by the free-particle Hamilton operator $\hat{H}_0$, so the corresponding time-dependent fields are given by

$$\hat{\phi}_{I}(x) \;=\; e^{i\hat{H}_0 t}\hat{\phi}(\vec{x}\,)\,e^{-i\hat{H}_0 t}$$

and are called interaction-picture fields. This is actually the situation we have encountered in the previous chapter, i.e. $\hat{\phi}_{I}(x) = \hat{\phi}_{\rm free}(x)$. The creation and annihilation operators have the same meaning as in the free theory, so the ground state is in this case the stable vacuum $|0\rangle$ of the free theory, with $N(\hat{H}_0)|0\rangle = 0$ after normal ordering.

Switching between pictures: there is an operator that allows you to switch between interaction picture and Heisenberg picture:

$$\hat{\phi}(x) \;=\; e^{i\hat{H}t}\hat{\phi}(\vec{x}\,)e^{-i\hat{H}t} \;=\; e^{i\hat{H}t}e^{-i\hat{H}_0 t}\hat{\phi}_{I}(x)e^{i\hat{H}_0 t}e^{-i\hat{H}t} \;\equiv\; \hat{U}^{-1}(t,0)\hat{\phi}_{I}(x)\hat{U}(t,0)\;.$$

The operator $\hat{U}(t,0)$ satisfies the differential equation

$$i\frac{\partial}{\partial t}\hat{U}(t,0) \;=\; e^{i\hat{H}_0 t}(\hat{H}-\hat{H}_0)e^{-i\hat{H}t} \;=\; e^{i\hat{H}_0 t}\hat{H}_{int}e^{-i\hat{H}_0 t}e^{i\hat{H}_0 t}e^{-i\hat{H}t} \;\overset{P.\&S.}{\equiv}\; \hat{H}_{I}(t)\hat{U}(t,0)\;,$$

with boundary condition $\hat{U}(0,0) = \hat{1}$ and with $\hat{H}_{I}(t)$ only referring to the interaction term (according to the definition in the textbook of Peskin & Schroeder).

6 *This constitutes a natural starting point for a perturbative expansion:*

$$\begin{aligned}\hat{U}(t\geq 0,0) \;&=\; \hat{1} \,+\, (-i)\int_0^t \mathrm{d}t_1\,\hat{H}_{I}(t_1)\hat{U}(t_1,0)\\ &=\; \hat{1} \,+\, (-i)\int_0^t \mathrm{d}t_1\,\hat{H}_{I}(t_1) \,+\, (-i)^2\int_0^t \mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2\,\hat{H}_{I}(t_1)\hat{H}_{I}(t_2) \,+\,\cdots\;,\end{aligned}$$

where the product $\hat{H}_{I}(t_1)\hat{H}_{I}(t_2)$ in the last term is ordered in time. In Ex. 6 it will be derived that

$$\hat{U}(t,0) \;=\; T\Big(e^{-i\int_0^t dt'\,\hat{H}_{I}(t')}\Big) \;\equiv\; \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_0^t \mathrm{d}t_1\cdots\int_0^t \mathrm{d}t_n\,T\big(\hat{H}_{I}(t_1)\cdots\hat{H}_{I}(t_n)\big)\;,$$

which can be truncated at the required perturbative order. Such an object is called a time-ordered exponential. For now we will define time ordering according to

$$T\big(\hat{O}_1(t_1)\hat{O}_2(t_2)\big) \;=\; \begin{cases}\hat{O}_1(t_1)\hat{O}_2(t_2) & t_1 > t_2\\[2ex] \hat{O}_2(t_2)\hat{O}_1(t_1) & t_2 > t_1\end{cases}\;.$$

Later on we will have to extend the definition of time ordering to fermionic operator fields. Since $\hat{H}_I(t')$ consists of interaction-picture fields only, we have succeeded in rewriting $\hat{\phi}(x)$ in terms of free fields through $\hat{\phi}(x) = \hat{U}^{-1}(t,0)\hat{\phi}_I(x)\hat{U}(t,0)$.

The definition of $\hat{U}$ can be extended to arbitrary reference points:

$$\hat{U}(t,t_1) \;\equiv\; e^{i\hat{H}_0 t}e^{-i\hat{H}(t-t_1)}e^{-i\hat{H}_0 t_1} \;=\; \hat{U}(t,0)\hat{U}^{-1}(t_1,0)\ .$$

This operator still satisfies the differential equation $i\dfrac{\partial}{\partial t}\hat{U}(t,t_1) = \hat{H}_I(t)\hat{U}(t,t_1)$, but with boundary condition $\hat{U}(t_1,t_1) = \hat{1}$. The same procedure as before yields:

$$\hat{U}(t,t_1) \;=\; T\Big(e^{-i\int_{t_1}^{t} dt'\,\hat{H}_I(t')}\Big) \qquad\qquad (t \geq t_1)\ .$$

This operator has the following properties that follow trivially from the above-given definition of $\hat{U}(t,t_1)$:

$$\hat{U}(t_1,t_2)\hat{U}(t_2,t_3) \;=\; \hat{U}(t_1,t_3) \qquad \text{and} \qquad \hat{U}(t_1,t_3)\hat{U}^{-1}(t_2,t_3) \;=\; \hat{U}(t_1,t_2)\ .$$

Note that we have not used that $\hat{H}_0$ and $\hat{H}$ are hermitian, by sticking to $\hat{U}^{-1}$ instead of writing $\hat{U}^{\dagger}$. So, $\hat{U}(t,t_1)$ can be generalized to non-hermitian $\hat{H}_I(t)$ or complex-valued time trajectories, as is used in some of the textbooks on quantum field theory.

2.4 Wick's theorem (§ 4.3 in the book)

The scattering amplitude for going from a free-particle initial state $|i\rangle$ to a free-particle final state $|f\rangle$ now takes the form

$$\lim_{t_\pm\to\pm\infty}\langle f|\hat{U}(t_+,t_-)|i\rangle \;\equiv\; \langle f|\hat{S}|i\rangle \;\equiv\; \langle f|(\hat{1}+i\hat{T})|i\rangle\ .$$

In this expression the matrix $\langle f|\hat{S}|i\rangle$ is called the S-matrix (scattering matrix), the unit operator occurring on the right-hand-side corresponds to the case where no scattering takes place, and $\hat{T}$ is the transition operator that describes actual scattering.

6a Question: what should be done to calculate such an S-matrix element at lowest order in perturbation theory?

The clumsy way of calculating *S*-matrix elements: let's consider the scalar Yukawa theory, where $\hat{H}_{int} = g\int \mathrm{d}\vec{x}\ \hat{\psi}^{\dagger}(\vec{x})\hat{\psi}(\vec{x})\hat{\phi}(\vec{x})$. Remember that ψ is a complex Klein-Gordon field, i.e. $\hat{\psi}^{\dagger}\neq\hat{\psi}$, whereas ϕ is a real Klein-Gordon field, i.e. $\hat{\phi}^{\dagger}=\hat{\phi}$. Then we have:

$$\lim_{t_\pm\to\pm\infty}\hat{U}(t_+,t_-) \;=\; T\Big(e^{-i\int_{-\infty}^{\infty} dt'\,\hat{H}_I(t')}\Big) \;=\; \hat{1} \;-\; ig\int \mathrm{d}^4x\ \hat{\psi}_I^{\dagger}(x)\hat{\psi}_I(x)\hat{\phi}_I(x) \;+\; \mathcal{O}(g^2)\ .$$

Consider the following decay process within the scalar Yukawa theory:

$$\phi(\vec{p}) \;\to\; \psi(\vec{q}_1) + \bar{\psi}(\vec{q}_2)\ ,$$

where $\phi(\vec{p})$ denotes a ϕ-particle with mass m and momentum $\vec{p}$, whereas $\psi(\vec{q}_1)$ and $\bar{\psi}(\vec{q}_2)$ denote a ψ-particle and a ψ-antiparticle with mass M and momenta $\vec{q}_1$ and $\vec{q}_2$ respectively. The ingredients for the calculation are:

$$|i\rangle \;=\; \sqrt{2\omega_{\vec{p}}}\,\hat{a}^{\dagger}_{\vec{p}}\,|0\rangle\ ,$$

$$\langle f| \;=\; \sqrt{2\omega_{\vec{q}_1}2\omega_{\vec{q}_2}}\;\langle 0|\,\hat{c}_{\vec{q}_2}\,\hat{b}_{\vec{q}_1}\ ,$$

$$\hat{\phi}_I(x) \;=\; \int\frac{\mathrm{d}\vec{k}}{(2\pi)^3}\,\frac{1}{\sqrt{2\omega_{\vec{k}}}}\,\Big(\hat{a}_{\vec{k}}\,e^{-ik\cdot x} + \hat{a}^{\dagger}_{\vec{k}}\,e^{ik\cdot x}\Big)\bigg|_{k_0=\omega_{\vec{k}}=\sqrt{\vec{k}^2+m^2}}\ ,$$

$$\hat{\psi}_I(x) \;=\; \int\frac{\mathrm{d}\vec{k}_1}{(2\pi)^3}\,\frac{1}{\sqrt{2\omega_{\vec{k}_1}}}\,\Big(\hat{b}_{\vec{k}_1}e^{-ik_1\cdot x} + \hat{c}^{\dagger}_{\vec{k}_1}e^{ik_1\cdot x}\Big)\bigg|_{k_{1_0}=\omega_{\vec{k}_1}=\sqrt{\vec{k}_1^2+M^2}}\ ,$$

$$\hat{\psi}^{\dagger}_I(x) \;=\; \int\frac{\mathrm{d}\vec{k}_2}{(2\pi)^3}\,\frac{1}{\sqrt{2\omega_{\vec{k}_2}}}\,\Big(\hat{c}_{\vec{k}_2}e^{-ik_2\cdot x} + \hat{b}^{\dagger}_{\vec{k}_2}e^{ik_2\cdot x}\Big)\bigg|_{k_{2_0}=\omega_{\vec{k}_2}=\sqrt{\vec{k}_2^2+M^2}}\ .$$

Using that $\langle f|i\rangle = 0$ we get

$$\langle f|\hat{S}|i\rangle \;=\; \sqrt{8\omega_{\vec{p}}\,\omega_{\vec{q}_1}\omega_{\vec{q}_2}}\,\langle 0|\hat{c}_{\vec{q}_2}\hat{b}_{\vec{q}_1}\Big(-ig\int\mathrm{d}^4x\;\hat{\psi}^{\dagger}_I(x)\hat{\psi}_I(x)\hat{\phi}_I(x)\Big)\hat{a}^{\dagger}_{\vec{p}}\,|0\rangle\ .$$

Since the $\hat{a}$-, $\hat{b}$- and $\hat{c}$-operators mutually commute, the $\hat{a}^{\dagger}_{\vec{k}}$ term in $\hat{\phi}_I$ can be commuted to the left and will annihilate the vacuum. Similarly $\hat{b}_{\vec{k}_1}$ in $\hat{\psi}_I$ and $\hat{c}_{\vec{k}_2}$ in $\hat{\psi}^{\dagger}_I$ can be commuted to the right and will annihilate the vacuum there, bearing in mind that the vacuum expectation value of an operator that involves an odd number of $\hat{c}$-operators vanishes trivially. In other words, only the $\hat{a}_{\vec{k}}$ term in $\hat{\phi}_I$, the $\hat{c}^{\dagger}_{\vec{k}_1}$ term in $\hat{\psi}_I$ and the $\hat{b}^{\dagger}_{\vec{k}_2}$ term in $\hat{\psi}^{\dagger}_I$ will contribute:

$$\langle f|\hat{S}|i\rangle \;=\; -ig\int\mathrm{d}^4x\iiint\frac{\mathrm{d}\vec{k}\,d\vec{k}_1\,\mathrm{d}\vec{k}_2}{(2\pi)^9}\,\Big(\frac{\omega_{\vec{p}}\,\omega_{\vec{q}_1}\omega_{\vec{q}_2}}{\omega_{\vec{k}}\,\omega_{\vec{k}_1}\omega_{\vec{k}_2}}\Big)^{\frac{1}{2}}e^{i(k_1+k_2-k)\cdot x}\,\langle 0|\hat{c}_{\vec{q}_2}\hat{b}_{\vec{q}_1}\big(\hat{b}^{\dagger}_{\vec{k}_2}\,\hat{c}^{\dagger}_{\vec{k}_1}\,\hat{a}_{\vec{k}}\big)\,\hat{a}^{\dagger}_{\vec{p}}\,|0\rangle\ .$$

We know that

$$\hat{a}_{\vec{k}}\,\hat{a}^{\dagger}_{\vec{p}}\,|0\rangle \;=\; \big[\hat{a}_{\vec{k}}\,,\hat{a}^{\dagger}_{\vec{p}}\big]\,|0\rangle \;=\; (2\pi)^3\delta(\vec{k}-\vec{p}\,)\,|0\rangle\ ,$$

and similarly that

$$\langle 0|\,\hat{b}_{\vec{q}_1}\hat{b}^{\dagger}_{\vec{k}_2} \;=\; \langle 0|\,(2\pi)^3\delta(\vec{k}_2-\vec{q}_1) \qquad\text{and}\qquad \langle 0|\,\hat{c}_{\vec{q}_2}\hat{c}^{\dagger}_{\vec{k}_1} \;=\; \langle 0|\,(2\pi)^3\delta(\vec{k}_1-\vec{q}_2)\ .$$

This leads to the following result for the lowest-order decay amplitude:

$$\langle f|\hat{S}|i\rangle \;=\; -\,ig\int\mathrm{d}^4x\;e^{i(q_2+q_1-p)\cdot x}\,\langle 0|0\rangle \;=\; -\,ig\,(2\pi)^4\delta^{(4)}(q_1+q_2-p)\ ,$$

with g the strength of the interaction that is responsible for the decay. The δ-function ensures that energy and momentum are conserved in the decay. In the reference frame of the decaying particle we have: $p = (m, \vec{0}) \Rightarrow \vec{q}_1 + \vec{q}_2 = \vec{0}$, $\omega_{\vec{q}_1} + \omega_{\vec{q}_2} = m$ with $\omega_{\vec{q}_j} = \sqrt{\vec{q}_j^{\,2} + M^2} \geq M$ for $j = 1, 2$. So, the decay is only possible if $m \geq 2M$.

The smart way of calculating S-matrix elements:

> 6b *the trick will be to bring all creation operators to the left and all annihilation operators to the right, with the vacuum state doing the rest. In other words, in order to calculate S-matrix elements we need a way to rewrite time-ordered fields in normal-ordered form ... as will be provided by Wick's theorem!*

Step 1: consider a real Klein-Gordon field

$$\hat{\phi}_I(x) \;=\; \int \frac{\mathrm{d}\vec{p}}{(2\pi)^3} \, \frac{e^{-ip\cdot x}}{\sqrt{2\omega_{\vec{p}}}} \, \hat{a}_{\vec{p}} \;+ \int \frac{\mathrm{d}\vec{p}}{(2\pi)^3} \, \frac{e^{ip\cdot x}}{\sqrt{2\omega_{\vec{p}}}} \, \hat{a}^{\dagger}_{\vec{p}} \;\equiv\; \hat{\phi}^{+}_I(x) + \hat{\phi}^{-}_I(x) \; ,$$

where the first term corresponds to the positive-frequency part and the second term to the negative-frequency part. The $\hat{\phi}^{+}_I$ and $\hat{\phi}^{-}_I$ fields have the following useful property:

$$\hat{\phi}^{+}_I(x)\,|0\rangle \;=\; 0 \qquad \text{and} \qquad \langle 0|\,\hat{\phi}^{-}_I(x) \;=\; 0 \; .$$

Since $\hat{\phi}^{+}_I$ only contains annihilation operators, the fields $\hat{\phi}^{+}_I(x)$ and $\hat{\phi}^{+}_I(y)$ commute. Similarly, $\hat{\phi}^{-}_I$ only contains creation operators, so the fields $\hat{\phi}^{-}_I(x)$ and $\hat{\phi}^{-}_I(y)$ commute as well. As a result

$$\begin{aligned}
x^0 > y^0: \quad T\big(\hat{\phi}_I(x)\hat{\phi}_I(y)\big) \;&=\; \big(\hat{\phi}^{+}_I(x) + \hat{\phi}^{-}_I(x)\big)\big(\hat{\phi}^{+}_I(y) + \hat{\phi}^{-}_I(y)\big) \\
&=\; N\big(\hat{\phi}_I(x)\hat{\phi}_I(y)\big) \;+\; \big[\hat{\phi}^{+}_I(x), \hat{\phi}^{-}_I(y)\big] \;=\; N\big(\hat{\phi}_I(x)\hat{\phi}_I(y)\big) \;+\; D(x-y)\,\hat{1} \; , \\
x^0 < y^0: \quad T\big(\hat{\phi}_I(x)\hat{\phi}_I(y)\big) \;&=\; \big(\hat{\phi}^{+}_I(y) + \hat{\phi}^{-}_I(y)\big)\big(\hat{\phi}^{+}_I(x) + \hat{\phi}^{-}_I(x)\big) \\
&=\; N\big(\hat{\phi}_I(y)\hat{\phi}_I(x)\big) \;+\; \big[\hat{\phi}^{+}_I(y), \hat{\phi}^{-}_I(x)\big] \;=\; N\big(\hat{\phi}_I(x)\hat{\phi}_I(y)\big) \;+\; D(y-x)\,\hat{1} \; .
\end{aligned}$$

Now we define a so-called <u>contraction</u>:

$$\overline{\hat{\phi}_I(x)\hat{\phi}_I(y)} \;=\; \underline{\hat{\phi}_I(x)\hat{\phi}_I(y)} \;\equiv\; \begin{cases} \big[\hat{\phi}^{+}_I(x), \hat{\phi}^{-}_I(y)\big] \;=\; D(x-y)\,\hat{1} & \text{if } x^0 > y^0 \\[2ex] \big[\hat{\phi}^{+}_I(y), \hat{\phi}^{-}_I(x)\big] \;=\; D(y-x)\,\hat{1} & \text{if } x^0 < y^0 \end{cases} \;=\; D_F(x-y)\,\hat{1} \; ,$$

with $D_F(x-y)$ the Feynman propagator of the free Klein-Gordon theory. With this definition, the time-ordered expression can be rewritten as

$$T\big(\hat{\phi}_I(x)\hat{\phi}_I(y)\big) \;=\; N\big(\hat{\phi}_I(x)\hat{\phi}_I(y)\big) \;+\; \overline{\hat{\phi}_I(x)\hat{\phi}_I(y)} \; .$$

As a consequence of normal ordering we get, as expected, that

$$\langle 0|T\big(\hat{\phi}_I(x)\hat{\phi}_I(y)\big)|0\rangle \;=\; 0 \;+\; D_F(x-y) \; .$$

Step 2, Wick's theorem: let's for the moment skip the annoying subscript I and use the shorthand notation $\hat{\phi}_j \equiv \hat{\phi}_I(x_j)$ for $j = 1, \cdots, n$. Wick's theorem then states:

$$T(\hat{\phi}_1 \cdots \hat{\phi}_n) \;=\; N(\hat{\phi}_1 \cdots \hat{\phi}_n + \text{all possible contractions}) \; .$$

For example:

$$\begin{aligned} T(\hat{\phi}_1\hat{\phi}_2\hat{\phi}_3\hat{\phi}_4) = N\Big(& \hat{\phi}_1\hat{\phi}_2\hat{\phi}_3\hat{\phi}_4 + \wick{\c{\hat{\phi}}_1\c{\hat{\phi}}_2}\hat{\phi}_3\hat{\phi}_4 + \wick{\c{\hat{\phi}}_1\hat{\phi}_2\c{\hat{\phi}}_3}\hat{\phi}_4 + \wick{\c{\hat{\phi}}_1\hat{\phi}_2\hat{\phi}_3\c{\hat{\phi}}_4} + \hat{\phi}_1\wick{\c{\hat{\phi}}_2\c{\hat{\phi}}_3}\hat{\phi}_4 + \hat{\phi}_1\wick{\c{\hat{\phi}}_2\hat{\phi}_3\c{\hat{\phi}}_4} \\ & + \hat{\phi}_1\hat{\phi}_2\wick{\c{\hat{\phi}}_3\c{\hat{\phi}}_4} + \wick{\c1{\hat{\phi}}_1\c1{\hat{\phi}}_2\c2{\hat{\phi}}_3\c2{\hat{\phi}}_4} + \wick{\c1{\hat{\phi}}_1\c2{\hat{\phi}}_2\c1{\hat{\phi}}_3\c2{\hat{\phi}}_4} + \wick{\c1{\hat{\phi}}_1\c2{\hat{\phi}}_2\c2{\hat{\phi}}_3\c1{\hat{\phi}}_4}\Big) \; , \end{aligned}$$

with $N(\wick{\c{\hat{\phi}}_1\hat{\phi}_2\c{\hat{\phi}}_3}\hat{\phi}_4) \;\equiv\; D_F(x_1 - x_3)N(\hat{\phi}_2\hat{\phi}_4)$.

The decomposition stated in Wick's theorem has the following important consequence:

> *leftover (uncontracted) normal-ordered terms vanish upon taking the vacuum expectation value!*

For example:

$$\begin{aligned} \langle 0|T(\hat{\phi}_1\hat{\phi}_2\hat{\phi}_3\hat{\phi}_4)|0\rangle \;\;&=\;\; D_F(x_1 - x_2)D_F(x_3 - x_4) \;+\; D_F(x_1 - x_3)D_F(x_2 - x_4) \\ &\quad + \; D_F(x_1 - x_4)D_F(x_2 - x_3) \; . \end{aligned}$$

6b *Feynman propagators thus play a central role in the resulting expressions.*

Proof of Wick's theorem: assume that the theorem is correct for all $n \leq m-1$, knowing that it is okay for $n = 1, 2$. For convenience we take $x_1^0 \geq x_2^0 \geq \cdots \geq x_m^0$, bearing in mind that the order of the scalar fields is irrelevant for time ordering and normal ordering. Then

$$\begin{aligned} T(\hat{\phi}_1 \cdots \hat{\phi}_m) \;&=\; \hat{\phi}_1\hat{\phi}_2 \cdots \hat{\phi}_m \;=\; \hat{\phi}_1 T(\hat{\phi}_2 \cdots \hat{\phi}_m) \\ &\overset{\text{by assumption}}{=\!=\!=\!=\!=} \;\; \hat{\phi}_1 N(\hat{\phi}_2 \cdots \hat{\phi}_m \;+\; \text{all possible contractions of } \hat{\phi}_2 \cdots \hat{\phi}_m) \\ &=\; (\hat{\phi}_1^+ + \hat{\phi}_1^-)N(\cdots) \;=\; \hat{\phi}_1^+ N(\cdots) \;+\; N(\hat{\phi}_1^- \cdots) \; , \end{aligned}$$

where in the last step we have used that $\hat{\phi}_1^-$ contains creation operators only and therefore already is in the right position. In contrast, $\hat{\phi}_1^+$ contains annihilation operators only and should be placed after all other fields. To get it in normal-ordered form, we need to commute it past all other fields:

$$\hat{\phi}_1^+ N(\cdots) \;=\; N(\cdots)\hat{\phi}_1^+ \;+\; \text{corrections for all uncontracted } \hat{\phi}_{j>1}^- \; .$$

For instance:

$$
\begin{aligned}
\hat{\phi}_1^+ N(\hat{\phi}_2 \cdots \hat{\phi}_m) \;&=\; N\big([\hat{\phi}_1^+, \hat{\phi}_2^-] \hat{\phi}_3 \cdots \hat{\phi}_m \;+\; \hat{\phi}_2 [\hat{\phi}_1^+, \hat{\phi}_3^-] \hat{\phi}_4 \cdots \hat{\phi}_m \;+\; \cdots \\
&\qquad\qquad +\; \hat{\phi}_2 \cdots \hat{\phi}_{m-1} [\hat{\phi}_1^+, \hat{\phi}_m^-] \big) \;+\; N(\hat{\phi}_2 \cdots \hat{\phi}_m)\, \hat{\phi}_1^+ \\
&\xlongequal{x_1^0 \geq x_{j>1}^0}\; N\big(\hat{\phi}_1^+ \hat{\phi}_2 \cdots \hat{\phi}_m \;+\; \overbracket{\hat{\phi}_1 \hat{\phi}_2} \hat{\phi}_3 \cdots \hat{\phi}_m \;+\; \overbracket{\hat{\phi}_1 \hat{\phi}_2 \hat{\phi}_3} \hat{\phi}_4 \cdots \hat{\phi}_m \;+\; \cdots \big) \, ,
\end{aligned}
$$

where we have used that $N(\hat{\phi}_2 \cdots \hat{\phi}_m)\, \hat{\phi}_1^+ = N(\hat{\phi}_1^+ \hat{\phi}_2 \cdots \hat{\phi}_m)$. Consequently

$$
\hat{\phi}_1 N(\hat{\phi}_2 \cdots \hat{\phi}_m) \;=\; N(\hat{\phi}_1 \hat{\phi}_2 \cdots \hat{\phi}_m \;+\; \text{all single contractions of } \hat{\phi}_1 \text{ with another } \hat{\phi}_j) \, .
$$

The other (contracted) terms can be worked out in an analogous way, completing the inductive proof of Wick's theorem.

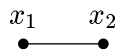
2.5 Diagrammatic notation: Feynman diagrams (§ 4.4 in the book)

In order to study the implications of Wick's theorem we will focus here on the interacting scalar ϕ^4-theory, with the scalar Yukawa theory being worked out in the exercises.

> 6c *For calculating amplitudes it will prove handy to introduce a diagrammatic notation, called Feynman diagrams, for time-ordered vacuum expectation values of interaction-picture fields.*

Propagator : we start with a diagrammatic notation for contractions

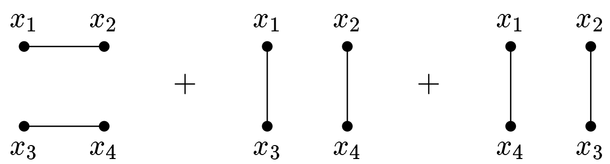
$$
\langle 0 | T\big(\hat{\phi}_I(x_1) \hat{\phi}_I(x_2)\big) |0\rangle \;=\; \overbracket{\hat{\phi}_I(x_1) \hat{\phi}_I(x_2)} \;=\; D_F(x_1 - x_2) \;\equiv
$$



,

where the solid line represents the contraction (propagator) and the dots at the end of the line represent the so-called external points in position space. From this it follows, for example, that

$$
\langle 0 | T\big(\hat{\phi}_I(x_1) \cdots \hat{\phi}_I(x_4)\big) |0\rangle \quad =
$$



.

Later on we will need more complicated vacuum expectation values of the form

$$
\lim_{t_\pm \to \pm\infty} \langle 0 | T\Big(\hat{\phi}_I(x_1) \cdots \hat{\phi}_I(x_n)\, e^{-i \int_{t_-}^{t_+} dt\, \hat{H}_I(t)} \Big) |0\rangle \, ,
$$

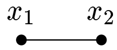
so let's further develop the diagrammatic notation. We again start with the case $n = 2$:

$$
\langle 0 | T\Big(\hat{\phi}_I(x_1) \hat{\phi}_I(x_2)\, e^{-i \int d^4x\, \hat{\mathcal{H}}_I(x)} \Big) |0\rangle \;\xlongequal{\text{Taylor}}\; \langle 0 | T\Big(\hat{\phi}_I(x_1) \hat{\phi}_I(x_2) \Big[\hat{1} - i \int \mathrm{d}^4x\, \hat{\mathcal{H}}_I(x) + \cdots \Big] \Big) |0\rangle .
$$

We can now calculate this quantity up to the required perturbative order.

Lowest order:

$$\langle 0|T\big(\hat{\phi}_I(x_1)\hat{\phi}_I(x_2)\big)|0\rangle \;=\; D_F(x_1-x_2) \;=\;$$



.

First order in $\boldsymbol{\lambda}$:

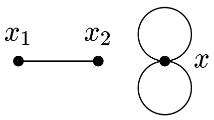
$$\langle 0|T\Big(\hat{\phi}_I(x_1)\hat{\phi}_I(x_2)\Big[-i\int \mathrm{d}^4x\,\frac{\lambda}{4!}\,\hat{\phi}_I^4(x)\Big]\Big)|0\rangle$$

$$\stackrel{\text{Wick}}{=\!=}\; 3\Big(\frac{-i\lambda}{4!}\Big)\int \mathrm{d}^4x\,\langle 0|\hat{\phi}_I(x_1)\hat{\phi}_I(x_2)\hat{\phi}_I(x)\hat{\phi}_I(x)\hat{\phi}_I(x)\hat{\phi}_I(x)|0\rangle$$

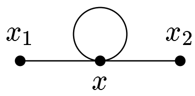
$$+\;12\Big(\frac{-i\lambda}{4!}\Big)\int \mathrm{d}^4x\,\langle 0|\hat{\phi}_I(x_1)\hat{\phi}_I(x_2)\hat{\phi}_I(x)\hat{\phi}_I(x)\hat{\phi}_I(x)\hat{\phi}_I(x)|0\rangle$$

$$=\quad -\frac{i\lambda}{8}\,D_F(x_1-x_2)\int \mathrm{d}^4x\,D_F^2(x-x)\;-\;\frac{i\lambda}{2}\int \mathrm{d}^4x\,D_F(x_1-x)D_F(x_2-x)D_F(x-x)$$

$=$



$+$



.

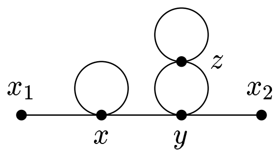
$\boxed{\text{Vertex}}$: the spacetime point x that is integrated over is called an <u>internal point</u> or <u>vertex</u>. To such a vertex we assign the analytic expression $-i\lambda\int d^4x$, which is the amplitude for emission and/or absorption of particles at the spacetime point x, summed over all points where this can occur. Also notice that we encounter for the first time pieces of diagram that involve <u>closed loops</u>.

An example of a higher-order term involving three powers in $\boldsymbol{\lambda}$:

$$\mathcal{P}\,\frac{1}{3!}\Big(\frac{-i\lambda}{4!}\Big)^3\langle 0|\hat{\phi}_I(x_1)\hat{\phi}_I(x_2)\int \mathrm{d}^4x\,\hat{\phi}_I\hat{\phi}_I\hat{\phi}_I\hat{\phi}_I\int \mathrm{d}^4y\,\hat{\phi}_I\hat{\phi}_I\hat{\phi}_I\hat{\phi}_I\int \mathrm{d}^4z\,\hat{\phi}_I\hat{\phi}_I\hat{\phi}_I\hat{\phi}_I|0\rangle$$

$$=\quad \frac{i\lambda^3}{8}\int \mathrm{d}^4x\int \mathrm{d}^4y\int \mathrm{d}^4z\,D_F(x_1-x)D_F(x-x)D_F(x-y)D_F(x_2-y)D_F^2(y-z)D_F(z-z)$$

$=$



.

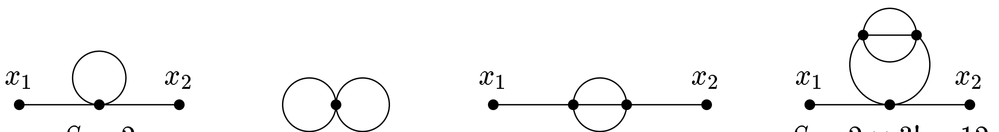
Here $\int \mathrm{d}^4x\;\hat{\phi}_I\hat{\phi}_I\hat{\phi}_I\hat{\phi}_I$ is a shorthand notation for $\int \mathrm{d}^4x\;\hat{\phi}_I(x)\hat{\phi}_I(x)\hat{\phi}_I(x)\hat{\phi}_I(x)$. The factor $\frac{1}{3!}\big(\frac{-i\lambda}{4!}\big)^3$ follows directly from the expansion of $e^{-i\int d^4x\,\hat{\mathcal{H}}_I(x)}$, whereas the factor $\mathcal{P}$ represents the number of times the contractions can be permuted without changing the contribution. This permutation factor is a product of the following terms:

- 3! from permuting x, y and z;
- 4×3 from the x contractions;
- 4×3 from the y contractions;
- 4×3 from the z contractions;

From the $\mathcal{O}(\lambda)$ and $\mathcal{O}(\lambda^3)$ examples we see that the factor $1/n!$ is cancelled by the $n!$ permutation factor from interchanging vertices, and that the factors $1/4!$ are largely compensated by the number of ways the contractions can be placed into $\hat{\phi}_I\hat{\phi}_I\hat{\phi}_I\hat{\phi}_I$.

$\boxed{\text{Symmetry factor}}$: we end up with a leftover factor $1/S$, with S the symmetry factor that represents the number of ways in which diagram components can be interchanged such that exactly the same diagram is obtained.

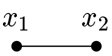
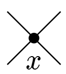
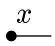
Examples:



$S = 2$ $\quad$ $S = 2^3 = 8$ $\quad$ $S = 3! = 6$ $\quad$ $S = 2 \times 3! = 12$.

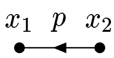
> 6c *The expression $\langle 0|T\big(\hat{\phi}_I(x_1)\hat{\phi}_I(x_2)\, e^{-i\int d^4x\,\hat{\mathcal{H}}_I(x)}\big)|0\rangle$ can now be represented by the sum of all possible Feynman diagrams with two external points, where a Feynman diagram is a collection (drawing) of propagators, vertices and external points. The rules for associating analytic expressions with specific pieces of diagrams are called the Feynman rules of the theory.*

Feynman rules for the scalar ϕ^4-theory in position space:

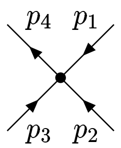
1. For each propagator  insert $D_F(x_1 - x_2)$.
2. For each vertex  insert $(-i\lambda)\int \mathrm{d}^4x$.
3. For each external point  insert 1.
4. Divide by the symmetry factor.

> 6c *Given a specific diagram, the complete analytic expression is obtained by multiplying the above-given analytic expressions for the specific pieces of the diagram.*

Switching to momentum space: usually it is more convenient to work in momentum space, rather than position space. First we consider the Feynman propagator:

$$D_F(x_1 - x_2) \;=\; \int \frac{\mathrm{d}^4p}{(2\pi)^4}\,\frac{i}{p^2 - m^2 + i\epsilon}\, e^{-ip\cdot(x_1 - x_2)} \;=\;$$  ,

where the sign (direction) of p is arbitrary since $D_F(x_1 - x_2) = D_F(x_2 - x_1)$ for a scalar field. In other words, we can assign a four-momentum p and complex factor $i/(p^2 - m^2 + i\epsilon)$ to each propagator, indicating the direction of the momentum flow by an arrow. This arrow has no deeper meaning than that in ϕ^4-theory, but in the scalar Yukawa theory it will be needed to distinguish particles from antiparticles. Using this momentum-flow convention a vertex corresponds to the following Fourier integral:

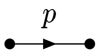
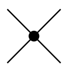
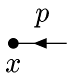


$$\rightarrow \qquad \int \mathrm{d}^4 z \, e^{-i(p_1+p_2+p_3-p_4)\cdot z} \; = \; (2\pi)^4 \delta^{(4)}(p_1 + p_2 + p_3 - p_4) \; .$$

On the left-hand-side of this equation the integral follows from the Feynman rule for the vertex and the exponential factor is caused by the momentum-space expressions for the Feynman propagators.

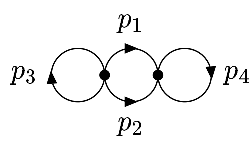
> 6c *In momentum space we hence obtain four-dimensional δ-functions that represent energy-momentum conservation at each vertex. These δ-functions can be used to perform some of the integrals that originate from the Feynman propagators.*

Feynman rules for the scalar ϕ^4-theory in momentum space:

1. For each propagator  insert $i/(p^2 - m^2 + i\epsilon)$.
2. For each vertex  insert $-i\lambda$.
3. For each external point  insert $e^{-ip\cdot x}$.
4. Impose momentum conservation at each vertex.
5. Integrate over each undetermined momentum p_j: $\int \frac{d^4 p_j}{(2\pi)^4}$.
6. Divide by the symmetry factor.

Vacuum bubbles: the pieces of diagram that are disconnected from the external points are called vacuum bubbles. For example:

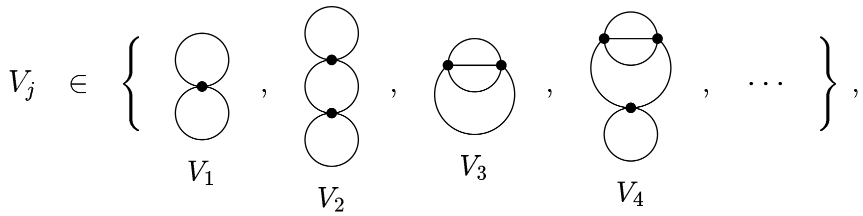
$$\int \mathrm{d}^4 x \, \hat{\phi}_I(x)\hat{\phi}_I(x)\hat{\phi}_I(x)\hat{\phi}_I(x) \int \mathrm{d}^4 y \, \hat{\phi}_I(y)\hat{\phi}_I(y)\hat{\phi}_I(y)\hat{\phi}_I(y) \quad = $$



The corresponding diagram will give rise to two energy-momentum δ-functions

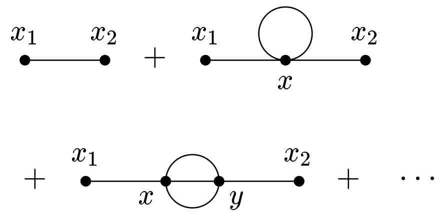
$$(2\pi)^4 \delta^{(4)}(p_1 + p_2)(2\pi)^4 \delta^{(4)}(p_1 + p_2) \; .$$

Upon inserting the first δ-function, the last δ-function will yield $\delta^{(4)}(0)$. This represents the infinite spacetime volume factor that originates from the fact that this vacuum bubble can occur at any spacetime point! We have in fact already encountered an example of such an IR divergence in § 1.3 while discussing the infinities of the zero-point energy. Let's now label the possible vacuum bubbles by



then the following identity holds:

$$\langle 0|T\Big(\hat{\phi}_I(x_1)\hat{\phi}_I(x_2)\, e^{-i\int d^4x\, \hat{\mathcal{H}}_I(x)}\Big)|0\rangle \;=\; e^{\sum_j V_j}\Bigg($$



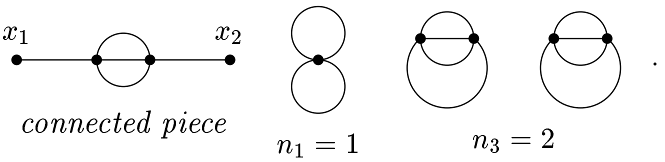
$$\Bigg) . \qquad (2)$$

The part between parantheses on the right-hand-side is the sum of all connected diagrams, i.e. continuous drawings that connect external points, whereas the exponential factor in front is the vacuum-bubble contribution. This vacuum-bubble contribution involves no external points and is therefore given by

$$e^{\sum_j V_j} \;=\; \langle 0|T\Big(e^{-i\int d^4x\, \hat{\mathcal{H}}_I(x)}\Big)|0\rangle \;.$$

Note: in the ϕ^4-theory each vertex has an even number of lines coming together. So, x_1 and x_2 must be connected to each other. The reason for this is that internal lines of a diagram connect two vertices and therefore count as two lines that are attached to a vertex. As such, a connected piece of diagram involves an even number of external lines and points.

Proof of identity (2): consider a diagram with n_j vacuum bubbles of type V_j and one connected piece without vacuum bubbles, like



From the Feynman rules it follows that

$$\text{analytic expression diagram} \;=\; (\text{analytic expression connected piece}) \times \Big(\prod_j \frac{1}{n_j!}\,(V_j)^{n_j}\Big)\,,$$

where the symmetry factor comes from interchanging the n_j copies of V_j. Hence we find

$$\langle 0|T\Big(\hat{\phi}_I(x_1)\hat{\phi}_I(x_2)\, e^{-i\int d^4x\, \hat{\mathcal{H}}_I(x)}\Big)|0\rangle \;=\; \text{sum of all diagrams}$$

$$= \sum_{\substack{\text{all possible}\\ \text{connected pieces}}} \; \sum_{\text{all } \{n_j\}} (\text{analytic expression connected piece}) \times \Big(\prod_j \frac{1}{n_j!}\,(V_j)^{n_j}\Big)$$

$$= (\text{sum of all connected diagrams}) \times \sum_{\text{all } \{n_j\}} \Big(\prod_j \frac{1}{n_j!}\,(V_j)^{n_j}\Big)$$

$$= (\text{sum of all connected diagrams}) \times \langle 0|T\Big(e^{-i\int d^4x\, \hat{\mathcal{H}}_I(x)}\Big)|0\rangle \;.$$

The only thing left to prove is that the last factor is indeed equal to $e^{\sum_j V_j}$:

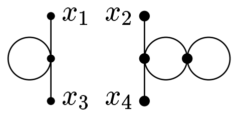
$$e^{\sum_j V_j} \;=\; \prod_j e^{V_j} \;=\; \prod_j \Big(\sum_{n_j} \frac{1}{n_j!}\,(V_j)^{n_j}\Big) \;=\; \Big(\sum_{n_1} \frac{1}{n_1!}\,(V_1)^{n_1}\Big)\Big(\sum_{n_2} \frac{1}{n_2!}\,(V_2)^{n_2}\Big)\cdots$$

$$= \sum_{\text{all } \{n_j\}} \Big(\prod_j \frac{1}{n_j!}\,(V_j)^{n_j}\Big) \;.$$

We can generalize the above-given separation between connected diagrams and vacuum bubbles to

$$\langle 0|T\Big(\hat{\phi}_I(x_1)\cdots\hat{\phi}_I(x_n)\, e^{-i\int d^4x\, \hat{\mathcal{H}}_I(x)}\Big)|0\rangle$$

$$= \langle 0|T\Big(e^{-i\int d^4x\, \hat{\mathcal{H}}_I(x)}\Big)|0\rangle \times (\text{sum of all connected diagrams with } n \text{ external points}) \;.$$

For $4, 6, \cdots$ external points this generalized sum will contain diagrams like 

that do not have all external points connected to each other.

Remark: we will see later that the sum $\sum_j V_j = \log\Big(\langle 0|T\Big(e^{-i\int d^4x\, \hat{\mathcal{H}}_I(x)}\Big)|0\rangle\Big)$ of all vacuum bubbles is actually related to the difference in the ground-state zero-point energies of the interacting theory and the free theory.

2.6 Scattering amplitudes (§ 4.6 in the book)

7 *At this point you might wonder what such time-ordered vacuum expectation values of interaction-picture fields have to do with amplitudes for decay processes or scattering reactions.*

In order to calculate scattering cross sections and decay rates we will have to work out plane-wave amplitudes of the form ${}_{out}\langle \vec{p}_1\vec{p}_2\cdots|\vec{k}_A\vec{k}_B\rangle_{in}$. Here $|\vec{k}_A\vec{k}_B\rangle_{in}$ is the so-called "in-state". In the case of scattering this is a 2-particle momentum state that is constructed in the far past, also referred to as "the initial state". Similarly ${}_{out}\langle \vec{p}_1\vec{p}_2\cdots|$ is the so-called "out-state", which represents the final state particles in the far future, i.e. the particles that will end up in the detectors of the experiment.

> 7 *Since the detectors are in general not able to resolve positions at the level of the de Broglie wavelengths of the particles, it is correct to work with plane-wave states rather than wave packets in order to describe the collision.*

The states ${}_{out}\langle \vec{p}_1\vec{p}_2\cdots|$ and $|\vec{k}_A\vec{k}_B\rangle_{in}$ are plane-wave states in the Heisenberg picture. Normally states are time-independent in the Heisenberg picture. However, the in and out states that we use here are defined as eigenstates of momentum operators that do depend on time. As such, the in-state contains the time stamp $t = t_- \to -\infty$ and the out-state $t = t_+ \to +\infty$. By evolving these states to the eigenstates at $t = 0$, one unique set of Heisenberg-picture plane-wave states is obtained:

$$
{}_{out}\langle \vec{p}_1\vec{p}_2\cdots|\vec{k}_A\vec{k}_B\rangle_{in} \;\equiv\; \langle \vec{p}_1\vec{p}_2\cdots|\hat{S}|\vec{k}_A\vec{k}_B\rangle \;\equiv\; \langle \vec{p}_1\vec{p}_2\cdots|\big(\hat{1}+i\hat{T}\big)|\vec{k}_A\vec{k}_B\rangle \ .
$$

Because of the infinite time interval and the way we normalize the states $\langle \vec{p}_1\vec{p}_2\cdots|$ and $|\vec{k}_A\vec{k}_B\rangle$, these matrix elements are Lorentz invariant. As was mentioned earlier, the matrix element $\langle \vec{p}_1\vec{p}_2\cdots|\hat{S}|\vec{k}_A\vec{k}_B\rangle$ is called the S-matrix element and is naturally split into two parts: a part containing $\hat{1}$, which corresponds to the case where no scattering takes place, and a part containing the transition operator $\hat{T}$, which describes actual scattering. So, the latter part contains all the interesting physics.

The matrix element: the next step is to pull out the anticipated energy-momentum conservation factor according to

$$
\begin{aligned}
\langle \vec{p}_1\vec{p}_2\cdots|i\hat{T}|\vec{k}_A\vec{k}_B\rangle \;&\equiv\; (2\pi)^4\delta^{(4)}\big(k_A+k_B-[p_1+p_2+\cdots]\big)\,i\mathcal{M}(k_A,k_B\to p_1,p_2,\cdots) \\
&\equiv\; (2\pi)^4\delta^{(4)}\Big(\sum_i k_i-\sum_f p_f\Big)\,i\mathcal{M}(\{k_i\}\to\{p_f\}) \ ,
\end{aligned}
$$

where $\mathcal{M}$ is called the invariant matrix element (or short: matrix element).[2] All four-momenta occurring in this expression are on-shell, i.e. $p^2 = m^2$ with m the physical mass of the particle. Therefore it suffices to know the three-momenta of the particles and the reaction state they belong to (i.e. initial or final state) in order to obtain the complete four-momenta. By means of this split-up the interaction details ("dynamics") are separated from the momentum details ("kinematics").

[2] Warning: in some textbooks the factor of i is absorbed into the definition of $\mathcal{M}$

Rewriting things in free-particle language (without proof, for now): as will be shown later, the plane-wave states in the interacting theory can be expressed in terms of free-particle plane-wave states ${}_0\langle \vec{p}_1\vec{p}_2 \cdots |$ and $|\vec{k}_A\vec{k}_B\rangle_0$, resulting in

$$\begin{aligned}
\langle \vec{p}_1\vec{p}_2 \cdots | i\hat{T}\, | \vec{k}_A\vec{k}_B\rangle \;&=\; \lim_{t_\pm \to \pm\infty} \Big({}_0\langle \vec{p}_1\vec{p}_2 \cdots | T\Big(e^{-i\int_{t_-}^{t_+} dt\, \hat{H}_I(t)} \Big) | \vec{k}_A\vec{k}_B\rangle_0 \Big)_{\substack{\text{fully connected}\\ \text{and amputated}}} \times \text{factor} \\
&=\; \Big({}_0\langle \vec{p}_1\vec{p}_2 \cdots | T\Big(e^{-i\int d^4x\, \hat{\mathcal{H}}_I(x)} \Big) | \vec{k}_A\vec{k}_B\rangle_0 \Big)_{\substack{\text{fully connected}\\ \text{and amputated}}} \times \text{factor}\ ,
\end{aligned}$$

where the (not yet specified) factor comes in at loop level. In this way everything has been translated into free-particle language, but some of the ingredients still need to be specified.

> 7 *The actual proof of the above statement will be postponed until § 2.9, since we will need to know a bit more about the properties of loop corrections for that purpose. This proof will be based on the type of time-ordered vacuum expectation values of interaction-picture fields that we have encountered previously.*

In order to get a feeling for the essential ingredients of that proof we will consider an explicit example. Let's have a look at the meaning of "fully connected" and "amputated" by considering the S-matrix element belonging to the $2 \to 2$ process $\phi(k_A)\phi(k_B) \to \phi(p_1)\phi(p_2)$ in the scalar ϕ^4-theory.

The $\mathcal{O}(\lambda^0)$ term:

$$\begin{aligned}
{}_0\langle \vec{p}_1\vec{p}_2 | \vec{k}_A\vec{k}_B\rangle_0 \;&=\; 4\sqrt{\omega_{\vec{p}_1}\omega_{\vec{p}_2}\omega_{\vec{k}_A}\omega_{\vec{k}_B}}\, \langle 0 | \hat{a}_{\vec{p}_1}\hat{a}_{\vec{p}_2}\hat{a}^\dagger_{\vec{k}_A}\hat{a}^\dagger_{\vec{k}_B} | 0\rangle \\
&=\; 4\,\omega_{\vec{k}_A}\omega_{\vec{k}_B}(2\pi)^6 \Big[\delta(\vec{p}_1 - \vec{k}_A)\delta(\vec{p}_2 - \vec{k}_B) \,+\, A \leftrightarrow B \Big] \\
&\underset{=\!=\!=}{\underline{\text{diagrammatically}}}\quad \begin{array}{cc} 1 & 2 \\ \Big| & \Big| \\ A & B \end{array} \;+\; \begin{array}{cc} 1 & 2 \\ \Big| & \Big| \\ B & A \end{array}\ .
\end{aligned}$$

This $\mathcal{O}(\lambda^0)$ term is part of the $\hat{1}$ term in $\hat{S} = \hat{1} + i\hat{T}$, so it does not contribute to the matrix element $\mathcal{M}$.

$\boxed{\text{Arrow of time, Peskin \& Schroeder style}}$: the <u>external lines</u> without external points indicate the incoming particles, which are placed at the bottom of the diagram in the notation of Peskin & Schroeder, and outgoing particles, which are placed at the top of the diagram. In many textbooks these diagrams will be turned by 90° with incoming particles on the left and outgoing ones on the right, i.e. in that case the time-axis points from left to right rather than from bottom to top.

The $\mathcal{O}(\lambda)$ term:

$$
{}_0\langle\vec{p}_1\vec{p}_2|T\Big(-i\int \mathrm{d}^4x\,\frac{\lambda}{4!}\,\hat{\phi}_I^4(x)\Big)|\vec{k}_A\vec{k}_B\rangle_0 \;\xlongequal{\text{Wick}}\; \Big(\frac{-i\lambda}{4!}\Big)\,{}_0\langle\vec{p}_1\vec{p}_2|\int \mathrm{d}^4x\; N\big(\hat{\phi}_I^4(x)\big)|\vec{k}_A\vec{k}_B\rangle_0
$$

$$
+\;\Big(\frac{-i\lambda}{4!}\Big)\,{}_0\langle\vec{p}_1\vec{p}_2|\int \mathrm{d}^4x\; N\Big(6\,\hat{\phi}_I(x)\hat{\phi}_I(x)\hat{\phi}_I(x)\hat{\phi}_I(x)\;+\;3\,\hat{\phi}_I(x)\hat{\phi}_I(x)\hat{\phi}_I(x)\hat{\phi}_I(x)\Big)|\vec{k}_A\vec{k}_B\rangle_0\;.
$$

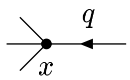
This time terms that are not fully contracted do not vanish automatically:

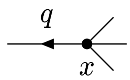
$$
\hat{\phi}_I^+(x)|\vec{k}\rangle_0 \;=\; \int\frac{\mathrm{d}\vec{p}}{(2\pi)^3}\,\frac{1}{\sqrt{2\omega_{\vec{p}}}}\,\hat{a}_{\vec{p}}\,e^{-ip\cdot x}\sqrt{2\omega_{\vec{k}}}\,\hat{a}^\dagger_{\vec{k}}|0\rangle \;=\; e^{-ik\cdot x}|0\rangle\;.
$$

It is now useful to extend the contraction definition with

$$
\hat{\phi}_I(x)|\vec{k}\rangle_0 \;\equiv\; \hat{\phi}_I^+(x)|\vec{k}\rangle_0 \;=\; e^{-ik\cdot x}|0\rangle \qquad\text{and}\qquad {}_0\langle\vec{p}|\hat{\phi}_I(x) \;\equiv\; {}_0\langle\vec{p}|\hat{\phi}_I^-(x) \;=\; \langle 0|e^{ip\cdot x}\;.
$$

This means that we need additional Feynman rules for contractions of field operators with external states:



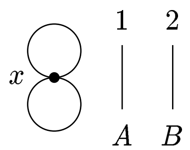
$= \quad e^{-iq\cdot x}$ and


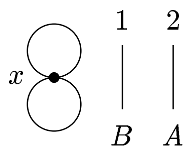
$= \quad e^{iq\cdot x}\,,$

where $e^{-iq\cdot x}$ is the amplitude for finding a particle with four-momentum q at the vertex position x. Diagrammatically the $\mathcal{O}(\lambda)$ terms then consist of the following contributions:

- A term with all $\hat{\phi}_I$'s contracted with each other:

$$
-\frac{i\lambda}{8}\int \mathrm{d}^4x\;{}_0\langle\vec{p}_1\vec{p}_2|\hat{\phi}_I(x)\hat{\phi}_I(x)\hat{\phi}_I(x)\hat{\phi}_I(x)|\vec{k}_A\vec{k}_B\rangle_0 \;=
$$

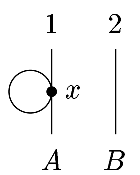


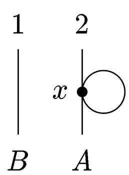
$+$


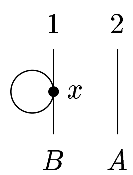
  This is a part of the $\hat{1}$ term in $\hat{S}=\hat{1}+i\hat{T}$, so it does not contribute to the matrix element $\mathcal{M}$.

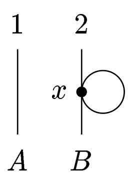
- Terms where some $\hat{\phi}_I$'s are contracted with each other and some with the external states:

$$
-\frac{i\lambda}{2}\int \mathrm{d}^4x\;{}_0\langle\vec{p}_1\vec{p}_2|\hat{\phi}_I(x)\hat{\phi}_I(x)\hat{\phi}_I(x)\hat{\phi}_I(x)|\vec{k}_A\vec{k}_B\rangle_0 \;+\; \text{three similar terms}
$$

$=$


$+$


$+$


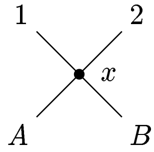
$+$


.

  These terms contribute only if there are as many $\hat{a}$ as $\hat{a}^\dagger$ operators left, so one field should be contracted with an incoming particle state and one with an outgoing particle state. Again this is part of the $\hat{1}$ term in $\hat{S}=\hat{1}+i\hat{T}$, since the integration $\int\mathrm{d}^4x$ yields a momentum-conserving δ-function at each vertex. Again no contribution to the matrix element $\mathcal{M}$ is obtained.

- A term where all $\hat{\phi}_I$'s are contracted with the external states:

$$-\, i\lambda \int \mathrm{d}^4x \; {}_0\langle \vec{p}_1 \vec{p}_2 | \hat{\phi}_I(x) \hat{\phi}_I(x) \hat{\phi}_I(x) \hat{\phi}_I(x) | \vec{k}_A \vec{k}_B \rangle_0 \quad =$$

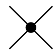


$$= \, -\, i\lambda \int \mathrm{d}^4x \; e^{-i(k_A+k_B-p_1-p_2)\cdot x} \;=\; -\, i\lambda (2\pi)^4 \delta^{(4)}(k_A+k_B-p_1-p_2) \; .$$

This term gives rise to a $-\,\lambda$ contribution to the matrix element $\mathcal{M}$!

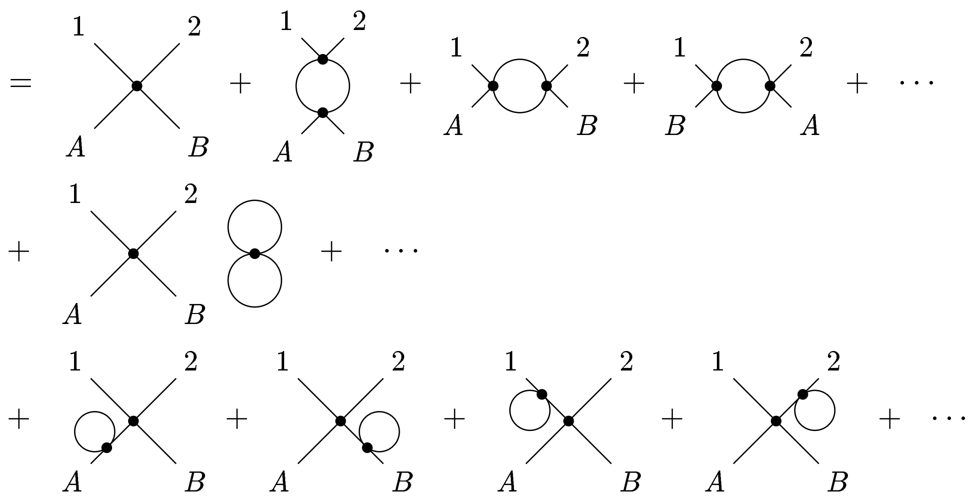
Fully connected diagrams: the discussion above reflects the following general principle.

> 7a *Only fully connected diagrams, in which all lines are connected to each other, contribute to the T-matrix and hence to the matrix element $\mathcal{M}$.*

At lowest non-vanishing order we find $\mathcal{M}(k_A, k_B \to p_1, p_2) = -\,\lambda$ in the scalar ϕ^4-theory, which can be obtained directly from the momentum-space interaction vertex  $= -\, i\lambda$.

Including higher-order terms, while keeping the external lines connected:

$\langle \vec{p}_1 \vec{p}_2 | i\hat{T} | \vec{k}_A \vec{k}_B \rangle \quad =$

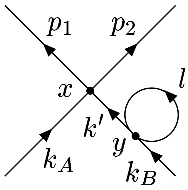


The three sets of diagrams that occur on the separate lines of this expression are now discussed individually.

Set 1: these diagrams contribute to $\mathcal{M}$. Beyond leading order diagrams occur that involve the creation and annihilation of additional "virtual" particles. Such higher-order contributions are called loop corrections.

Set 2: these diagrams involve disconnected vacuum bubbles, which will again exponentiate to an overall phase factor that is irrelevant for physical observables! These graphs take into account the energy shift between the ground state of the free theory and the ground state of the interacting theory with respect to which scattering takes place. So, indeed only fully connected diagrams matter!

Set 3: such diagrams give rise to contributions of the form

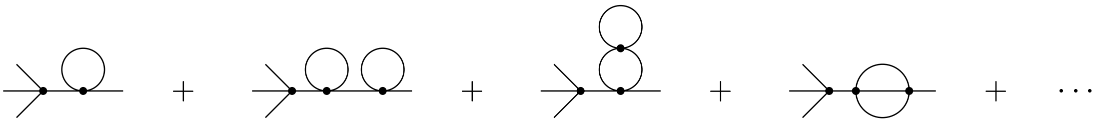


$$= \frac{1}{2}\int \frac{\mathrm{d}^4k'}{(2\pi)^4}\,\frac{i}{k'^2-m^2+i\epsilon}\int\frac{\mathrm{d}^4l}{(2\pi)^4}\,\frac{i}{l^2-m^2+i\epsilon}\,\times$$

$$\times\;(-\,i\lambda)(2\pi)^4\,\delta^{(4)}(k_A+k'-p_1-p_2)(-\,i\lambda)(2\pi)^4\,\delta^{(4)}(k_B-k')$$

$$= \frac{1}{2}(-\,i\lambda)^2(2\pi)^4\,\delta^{(4)}(k_A+k_B-p_1-p_2)\,\frac{i}{k_B^2-m^2+i\epsilon}\int\frac{\mathrm{d}^4l}{(2\pi)^4}\,\frac{i}{l^2-m^2+i\epsilon}\;.$$

This contribution contains two propagators, $D_F(x-y)$ and $D_F(y-y)$, and two δ-functions from the integrals over x and y. It blows up, since external particles are on-shell, i.e. $k_B^2=m^2$. In fact, the diagrams



represent the evolution of $|\vec{p}\rangle_0$ in the free theory into $|\vec{p}\rangle$ in the interacting theory, which causes the complex poles of the propagator to shift away from the free-particle positions at $p^2=m^2$. As we will see later, this evolution will give rise to overall proportionality factors in the T-matrix. All this reflects the fact that a particle is never truly free in quantum field theory, being surrounded by a cloud of virtual particles. In quantum field theory it is simply not possible to switch off interactions.

The amputation procedure: in order to deal with contributions of the latter type, the following procedure is used.

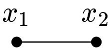
> 7b *Starting at the tip of each external leg, find the last point at which the diagram can be cut by removing a single propagator in such a way that this separates the leg from the rest of the diagram. The amputation procedure tells us to cut the diagram at those points.*

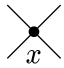
Contributions to the T-matrix are then obtained as

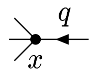
$(2\pi)^4\,\delta^{(4)}\big(k_A+k_B-\sum_f p_f\big)\;i\mathcal{M}\big(k_A,k_B\to\{p_f\}\big)$ = sum of all fully connected amputated Feynman diagrams in position space, multiplied by appropriate proportionality factors at loop level.

The missing details concerning the amputation procedure will be discussed after we have seen some properties of loop corrections.

Position-space Feynman rules for matrix elements in the scalar ϕ^4-theory:

1. For each propagator 

insert $D_F(x_1 - x_2)$.
2. For each vertex 

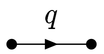
insert $(-i\lambda)\int \mathrm{d}^4x$.
3. For each external line 

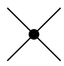
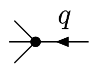
insert $e^{-iq\cdot x}$.
4. Divide by the symmetry factor.

Formulated in momentum space: in order to deal with plane-wave states it is more natural to switch from position space to momentum space. As explained before, in momentum space each interaction vertex gives rise to an energy-momentum δ-function. As we have seen in the example discussed above, one of these δ-functions is the overall energy-momentum δ-function of the T-matrix. Therefore, in momentum space one directly obtains the matrix element as

7c $\boxed{i\mathcal{M}\big(k_A, k_B \to \{p_f\}\big) \;=\; \text{sum of all fully connected amputated Feynman diagrams in momentum space, multiplied by appropriate proportionality factors at loop level.}}$

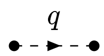
Momentum-space Feynman rules for matrix elements in the scalar ϕ^4-theory:

1. For each propagator 

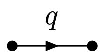
insert $\dfrac{i}{q^2 - m^2 + i\epsilon}$.
2. For each vertex  insert $-i\lambda$.
3. For each external line 

insert 1.
4. Impose momentum conservation at each vertex.
5. Integrate over each undetermined loop momentum l_j: $\displaystyle\int \frac{d^4 l_j}{(2\pi)^4}$.
6. Divide by the symmetry factor.

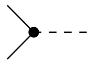
Momentum-space Feynman rules for the scalar Yukawa theory: for completeness we also list here the Feynman rules for the scalar Yukawa theory as derived in the exercises.

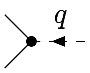
1. For each ϕ-propagator 

insert $\dfrac{i}{q^2 - m^2 + i\epsilon}$.

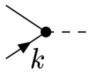
   For each ψ-propagator 

insert $\dfrac{i}{q^2 - M^2 + i\epsilon}$.

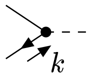
2. For each vertex  insert $-ig$.

3. For each external ϕ-line 

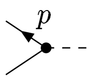
insert 1.

   For each incoming ψ-line 

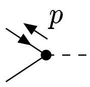
insert 1, originating from $\hat{\psi}$.

   For each incoming $\bar{\psi}$-line 

insert 1, originating from $\hat{\psi}^\dagger$.

   For each outgoing ψ-line 

insert 1, originating from $\hat{\psi}^\dagger$.

   For each outgoing $\bar{\psi}$-line 

insert 1, originating from $\hat{\psi}$.

4. Impose energy-momentum conservation at each vertex.

5. Integrate over each undetermined loop momentum l_j: $\displaystyle\int \frac{d^4 l_j}{(2\pi)^4}$.

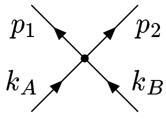
The following observations can be made. First of all, in contrast to the scalar ϕ^4-theory no symmetry factors are needed in the scalar Yukawa theory, since all fields in the interaction are different. Secondly, whereas the arrows on the dashed ϕ-lines have no special meaning, this is not true for the arrows on the solid lines, which correspond to the $\hat{\psi}$ and $\hat{\psi}^\dagger$ fields. This arrow is needed for distinguishing particles (ψ) from antiparticles ($\bar{\psi}$).

7d **Drawing convention:** *draw arrows on the ψ-lines and the $\bar{\psi}$-lines. These arrows represent the direction of particle-number flow: particles flow along the arrow, antiparticles flow against it. In this convention $\hat{\psi}$ corresponds to an arrow flowing into a vertex, whereas $\hat{\psi}^\dagger$ corresponds to an arrow flowing out of a vertex. Since every interaction vertex features both $\hat{\psi}$ and $\hat{\psi}^\dagger$, the arrows link up to form a continuous flow.*

2.7 Non-relativistic limit: forces between particles

7e *We are now in the position to address our first major question: how do forces come about in quantum field theory?*

To answer this question we compare the lowest-order relativistic matrix element for the reaction $\phi(k_A)\phi(k_B) \to \phi(p_1)\phi(p_2)$, i.e.

$i\mathcal{M} =$ 

$= -i\lambda$,

to the non-relativistic amplitude for elastic potential scattering in Born approximation. Since the matrix element is Lorentz invariant, we are free to choose the center-of-mass (CM) frame. In this frame $\vec{k}_A = -\vec{k}_B \equiv \vec{k}$ and $\vec{p}_1 = -\vec{p}_2 \equiv \vec{p}$ with $|\vec{k}\,| = |\vec{p}\,|$ for elastic scattering. The non-relativistic limit amounts to $|\vec{k}\,|$, $|\vec{p}\,| \ll m$, from which it follows that $\omega_{\vec{k}} = \omega_{\vec{p}} \approx m + \mathcal{O}(\vec{k}^{\,2}/m)$. For scattering from states with momenta $\pm\vec{k}$ into states with momenta $\pm\vec{p}$ the comparison then reads:

$$ {}_{\rm NR}\langle \vec{p}\,|V(\hat{\vec{r}}\,)|\vec{k}\,\rangle_{\rm NR} \;=\; \int \mathrm{d}\vec{r}\;V(\vec{r}\,)\,e^{i(\vec{k}-\vec{p})\cdot\vec{r}} \;\equiv\; \int \mathrm{d}\vec{r}\;V(\vec{r}\,)\,e^{i\vec{\Delta}\cdot\vec{r}} \;\approx\; -\,\frac{\mathcal{M}\big(k_A\,,k_B\to p_1\,,p_2\big)/2}{(2m)^2}\;, $$

where the factor $1/2$ multiplying the matrix element originates from having identical particles in the reaction. Furthermore, it has been used that the relativistic and non-relativistic momentum states are related according to

$$ |\vec{p}\,\rangle_0 \;=\; \sqrt{2\omega_{\vec{p}}}\;|\vec{p}\,\rangle_{\rm NR} \;\approx\; \sqrt{2m}\;|\vec{p}\,\rangle_{\rm NR}\;, $$

resulting in a relative factor $(2m)^2$. By inverse Fourier transformation one obtains

$$ V(\vec{r}\,) \;\approx\; \int \frac{\mathrm{d}\vec{\Delta}}{(2\pi)^3}\left(\frac{-\mathcal{M}}{8m^2}\right)e^{-i\vec{\Delta}\cdot\vec{r}} \;\;\underset{\overline{}}{\overset{\mathcal{M}=-\lambda}{=\!=\!=}}\;\; \frac{\lambda}{8m^2}\,\delta(\vec{r}\,) $$

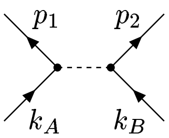
for the interaction potential.

(7e) *The scalar ϕ^4-theory involves a so-called contact interaction $\propto \delta(\vec{r}\,)$,*

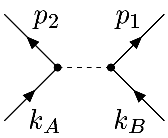
which refers to the fact that the particles interact in one spacetime point at lowest order.

We can repeat this for $\psi(k_A)\psi(k_B) \to \psi(p_1)\psi(p_2)$ scattering in the scalar Yukawa theory. In that case all external on-shell particles have mass M and the lowest-order matrix element reads (see Ex. 9):

$i\mathcal{M}$ =



+



$\equiv\; i\mathcal{M}_1 + i\mathcal{M}_2$

$$ = \; -\,ig^2\left(\frac{1}{(k_A-p_1)^2-m^2+i\epsilon} + \frac{1}{(k_A-p_2)^2-m^2+i\epsilon}\right) $$

$$ \overset{\rm NR}{\approx} \quad ig^2\left(\frac{1}{(\vec{k}-\vec{p}\,)^2+m^2} + \frac{1}{(\vec{k}+\vec{p}\,)^2+m^2}\right)\,, $$

using CM momenta and $k_A^0 - p_1^0 = \sqrt{\vec{k}^{\,2}+M^2} - \sqrt{\vec{p}^{\,2}+M^2} \approx (\vec{k}^{\,2}-\vec{p}^{\,2})/(2M)$. The $+i\epsilon$ terms have been dropped as a result of the fact that the energy components are suppressed.

Note that there are two contributions this time, originating from interchanging the final-state particles (i.e. $\vec{p} \to -\vec{p}$). Using spherical coordinates for the inverse Fourier transform with polar axis along $\vec{r}$ it now follows that

$$
\begin{aligned}
V(\vec{r}) &= -\frac{1}{4M^2}\int \frac{\mathrm{d}\vec{\Delta}}{(2\pi)^3}\,\mathcal{M}_1\, e^{-i\vec{\Delta}\cdot\vec{r}} \;\overset{\Delta\equiv|\vec{\Delta}|}{=\!=\!=}\; -(g/2M)^2\int\limits_{-1}^{1}\frac{\mathrm{d}\cos\theta}{(2\pi)^2}\int\limits_{0}^{\infty}\mathrm{d}\Delta\,\frac{\Delta^2 e^{-i\Delta r\cos\theta}}{\Delta^2+m^2} \\
&= -\frac{(g/2M)^2}{4\pi^2 ir}\int\limits_{0}^{\infty}\mathrm{d}\Delta\;\Delta\,\frac{e^{i\Delta r}-e^{-i\Delta r}}{\Delta^2+m^2} \;=\; -\frac{(g/2M)^2}{4\pi^2 ir}\int\limits_{-\infty}^{\infty}\mathrm{d}\Delta\,\frac{\Delta\, e^{i\Delta r}}{(\Delta+im)(\Delta-im)} \\
&= -\frac{(g/2M)^2}{4\pi^2 ir}\int\limits_{C}\mathrm{d}\Delta\,\frac{\Delta\, e^{i\Delta r}}{(\Delta+im)(\Delta-im)} \;=\; -\frac{(g/2M)^2}{4\pi r}\,e^{-mr}\,,
\end{aligned}
$$

where the integration contour C is given in figure 5.

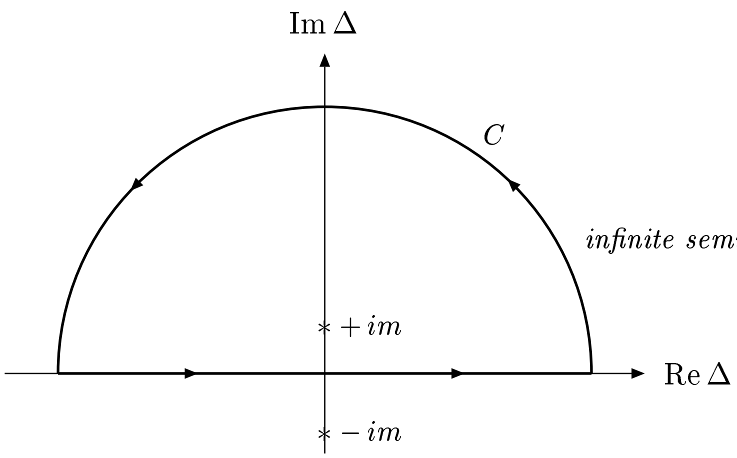


Figure 5: Closed integration contour for the determination of the Yukawa potential.

> 7e *The scalar Yukawa theory involves an attractive Yukawa interaction between the ψ-particles, which dies off exponentially at $1/m$ distances. This length scale (range) is in fact the Compton wavelength of the exchanged virtual ϕ-particles, which mediate the interaction.*

These virtual particles are short-lived off-shell particles, i.e. $p^2 \neq m^2$. In fact, they are too short-lived for their energy to be measured accurately. Hence the name virtual particles. Over $1/m$ distances the energy can fluctuate by $\mathcal{O}(m)$, which is sufficient to create the ϕ-particles. Over larger distances the energy can fluctuate less, resulting in the exponential decrease of the force. If the virtual particles are massless (like the photon) then the Yukawa interaction has an infinite range and changes into the familiar Coulomb potential $\propto 1/r$, which is not decreasing exponentially.

The true Yukawa theory for the interaction between fermions and scalars was used to describe the interactions between nucleons. In that case the mediating particle is a pion. It has a mass of about 140 MeV and therefore an associated characteristic length scale of roughly 1.4 fm, which agrees nicely with the effective range of the nuclear forces.

Forces in quantum field theory: the forces between particles are caused (mediated) by the exchange of virtual particles! Interactions caused by spin-0 force carriers (such as the Yukawa interactions) are universally attractive, just like interactions due to the exchange of spin-2 particles (such as gravity). The exchange of spin-1 particles can result in both attractive and repulsive interactions, as we know from electromagnetism.

The relevant details of this statement are worked out in Ex. 10 and 11. The implications can be seen all around us. Gravity is attractive and gives rise to structure formation in the universe. The force that holds together nucleons inside a nucleus is mediated by the spin-0 pion, giving rise to a strong nuclear force that is attractive and of femtometer range. This nuclear binding force overcomes the repulsive electromagnetic force between the like-charged protons. The proton repulsion influences the nuclear binding-energy properties of heavy nuclei, leading to the observed neutron over proton ratio and nuclear instability of heavy elements as well as the possibility of nuclear fission. The fact that the electromagnetic force can be both repulsive and attractive is responsible for the multi-faceted properties of atoms and the chemistry among molecules. This involves the intricate (quantum-mechanical) interplay between attractive forces that bind electrons to nuclei and the repulsive forces among the electrons and among the nuclei.

Intermezzo 2: flux laws for forces with massless mediators

The previous discussion basically tells us that the interaction potential between particles results from the inverse Fourier transform of the force carrier's propagator. For massless force carriers such as photons (electromagnetism) and gravitons (gravity) this immediately implies a constant flux law for the corresponding force (Gauss' law):

$$-\int\limits_{S(V)} \mathrm{d}\vec{s}\cdot\vec{F}(r) \;=\; \int\limits_{S(V)} \mathrm{d}\vec{s}\cdot\vec{\nabla}V(r) \;\stackrel{m=0}{=\!=\!=}\; -C\int\limits_{S(V)} \mathrm{d}\vec{s}\cdot\vec{\nabla}\int\frac{\mathrm{d}\vec{\Delta}}{(2\pi)^3}\,\frac{e^{-i\vec{\Delta}\cdot\vec{r}}}{\Delta^2}$$

$$\stackrel{\text{Gauss}}{=\!=\!=}\; -C\int\limits_{V}\mathrm{d}\vec{r}\;\vec{\nabla}\cdot\vec{\nabla}\int\frac{\mathrm{d}\vec{\Delta}}{(2\pi)^3}\,\frac{e^{-i\vec{\Delta}\cdot\vec{r}}}{\Delta^2} \;=\; C\int\limits_{V}\mathrm{d}\vec{r}\int\frac{\mathrm{d}\vec{\Delta}}{(2\pi)^3}\,e^{-i\vec{\Delta}\cdot\vec{r}} \;=\; C\int\limits_{V}\mathrm{d}\vec{r}\,\delta(\vec{r}) \;=\; C\,,$$

for a sphere V centered around the origin $\vec{r}=\vec{0}$ of the interaction (CM) and with surface $S(V)$. Since $\mathrm{d}\vec{s}\cdot\vec{F}(r)$ is constant on $S(V)$, we obtain for n spatial dimensions that

$$V^{(n)}(r) \;=\; -\frac{C}{(n-2)\,S_n(1)}\,\frac{1}{r^{n-2}} \qquad\Rightarrow\qquad \vec{F}^{(n)}(r) \;=\; -\frac{C}{S_n(1)}\,\frac{\vec{r}}{r^n} \;=\; -\frac{C}{S_n(1)}\,\frac{\vec{e}_r}{r^{n-1}}$$

for the corresponding interaction potential and force, with $S_n(1)$ the surface area of the unit sphere in n dimensions. For $n = 3$ we obtain $V^{(3)}(r) = -C/(4\pi r)$, which indeed coincides with a massless Yukawa potential with $(g/2M)^2 = C$. The power law for the force simply reflects that at constant force flux the force lines spread (dilute) more rapidly in higher-dimensional spaces.

Application: gravity in compact extra spatial dimensions

An idea to reduce the scale hierarchy between the Standard Model and the energy scale at which gravity becomes strong (Planck scale) is to assume that the graviton can propagate in compact extra spatial dimensions of size R. According to the previous discussion this causes gravity to become stronger at $r < R$ distances due to the different power law:

$$F_{\text{grav}}(r < R) \;=\; \frac{-\,m_1 m_2}{(\Lambda_n\, r)^{n-1}} \quad \xrightarrow[\text{Newton}]{\text{retrieving}} \quad F_{\text{grav}}(r \gg R) \;=\; \frac{-\,m_1 m_2}{\Lambda_n^{n-1}\, R^{n-3}\, r^2} \;\equiv\; \frac{-\,m_1 m_2}{(\Lambda_{\text{P}}\, r)^2}\ ,$$

where the Planck scale can be expressed in terms of Newton's contstant as $\Lambda_{\text{P}} = 1/\sqrt{G}$.

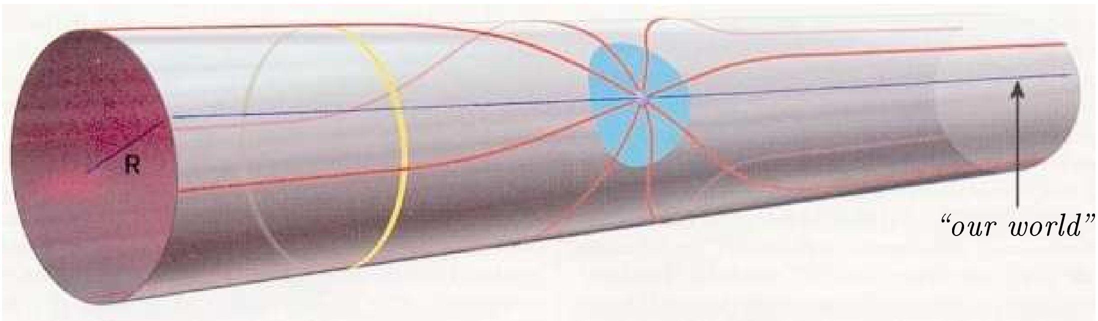


Figure 6: *As an illustrative example consider an infinite cylindrical shell (tube) with small radius R. At $r < R$ distances (blue region) the force lines (red) spread more rapidly as a result of the wrapped extra dimension of size R. At $r > R$ distances the spreading of the force lines in the extra dimension will start to saturate and for $r \gg R$ the 1-dimensional case (representing "our world") is approached asymptotically (yellow circle).*

The fundamental Planck scale in n spatial dimensions then becomes

$$\Lambda_n \;=\; \big(\Lambda_{\text{P}}^2/R^{n-3}\big)^{1/(n-1)} \;=\; \Lambda_{\text{P}}/\big(\Lambda_{\text{P}}\, R\big)^{(n-3)/(n-1)}\ .$$

By making $\Lambda_{\text{P}}\, R = R/10^{-35}\,\text{m}$ sufficiently large, which is usually referred to as models with "large extra dimensions", the effective Planck scale can be lowered from $\mathcal{O}(10^{19}\,\text{GeV})$ to $\mathcal{O}(\text{TeV})$. For $n - 3 = 2, \cdots, 6$ extra dimensions we can achieve this by setting the size of the compact extra dimensions to $R = 10^{-3}\,\text{m}, \cdots, 10^{-14}\,\text{m}$. This would imply that in those scenarios gravity would become strong at the $\mathcal{O}(10^{-19}\,\text{m})$ length scales probed at the LHC, giving rise to the production of microscopic black holes. Alternatively, the idea of extra dimensions can be tested by performing dedicated submillimeter gravity experiments.

2.8 Translation into probabilities (§ 4.5 in the book)

8 *At this point we know how to calculate amplitudes for decay processes and scattering reactions by means of Feynman diagrams and Feynman rules. In the next step we derive the probabilistic interpretation belonging to these amplitudes.*

2.8.1 Decay widths

Consider an initial state consisting of a single particle in the momentum state $|\vec{k}_A\rangle$, decaying into a final state consisting of n particles in the momentum state $|\vec{p}_1 \cdots \vec{p}_n\rangle$. The probability density for this decay to occur is given by

$$\frac{|\langle \vec{p}_1 \cdots \vec{p}_n|\hat{S}|\vec{k}_A\rangle|^2}{\langle \vec{k}_A|\vec{k}_A\rangle\langle \vec{p}_1 \cdots \vec{p}_n|\vec{p}_1 \cdots \vec{p}_n\rangle}\ ,$$

with

$$\langle \vec{k}_A|\vec{k}_A\rangle \ = \ 2E_{\vec{k}_A}(2\pi)^3\,\delta(\vec{0}\,) \ \overset{\text{p. 15}}{=\!=} \ 2E_{\vec{k}_A}V \qquad \text{and} \qquad \langle \vec{p}_1 \cdots \vec{p}_n|\vec{p}_1 \cdots \vec{p}_n\rangle \ = \ \prod_{j=1}^{n}(2E_{\vec{p}_j}V)\,.$$

This is also valid for identical particles in the final state. Finding a set of particles with the required momenta effectively identifies the particles. Since the initial and final states are different in a decay process, the S-matrix element is in fact equivalent with the corresponding T-matrix element. In the rest frame of the decaying particle $\vec{k}_A = \vec{0}$ and $E_{\vec{k}_A} = m_A$, hence

$$\frac{|\langle \vec{p}_1 \cdots \vec{p}_n|i\hat{T}|\vec{k}_A\rangle|^2}{\langle \vec{k}_A|\vec{k}_A\rangle\langle \vec{p}_1 \cdots \vec{p}_n|\vec{p}_1 \cdots \vec{p}_n\rangle} \ = \ \frac{|\mathcal{M}(k_A \to \{p_j\})|^2}{2m_A V}\Big[(2\pi)^4\,\delta^{(4)}\big(k_A - \sum_{j=1}^{n}p_j\big)\Big]^2\,\frac{1}{\prod_{j=1}^{n}(2E_{\vec{p}_j}V)}$$

$$\overset{(2\pi)^4\delta^{(4)}(0)=VT}{=\!=\!=\!=} \ \frac{|\mathcal{M}(k_A \to \{p_j\})|^2}{2m_A V}\,(2\pi)^4\,\delta^{(4)}\big(k_A - \sum_{j=1}^{n}p_j\big)\,\frac{VT}{\prod_{j=1}^{n}(2E_{\vec{p}_j}V)}\ .$$

The linear time factor $T = \int_{t_-}^{t_+}dt$ in this expression was to be expected from Fermi's Golden Rule! This factor can be divided out in order to obtain the corresponding constant decay rate.

Next we integrate over all possible momenta of the n final-state particles. This time it does matter whether there are identical particles in the final state. In order to avoid double counting we have to restrict the integration to inequivalent configurations or divide by $1/n_k!$ factors for any group of n_k identical final-state particles. Generically we will indicate this combinatorial final-state identical-particle factor by C_f. The final expression for the integrated constant decay rate then becomes

$$\Gamma_n = C_f \Bigg(\overbrace{\prod_{j=1}^{n} V \int \frac{d\vec{p}_j}{(2\pi)^3}}^{\int \text{density of states}} \Bigg) \frac{(2\pi)^4 \delta^{(4)}\big(k_A - \sum_{j=1}^{n} p_j\big) \, |\mathcal{M}(k_A \to \{p_j\})|^2}{2m_A \Big(\prod_{j=1}^{n} 2E_{\vec{p}_j} V \Big)}$$

$$= \frac{1}{2m_A} \, C_f \, \overbrace{\int d\Pi_n \, |\mathcal{M}(k_A \to \{p_j\})|^2}^{\text{Lorentz invariant}} \; ,$$

in terms of the relativistically invariant n-body phase-space element

$$d\Pi_n \equiv \Bigg(\prod_{j=1}^{n} \frac{d\vec{p}_j}{(2\pi)^3} \frac{1}{2E_{\vec{p}_j}} \Bigg) (2\pi)^4 \delta^{(4)}\big(k_A - \sum_{j=1}^{n} p_j\big) \; , \tag{3}$$

which is sometimes denoted by dPS_n in other textbooks. This decay rate is called the partial decay width for the decay mode into the considered n-particle final state.

After summation over all possible final states one obtains the so-called total decay width

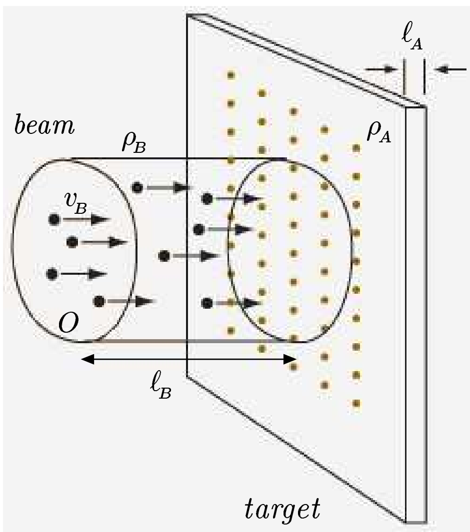
$$\boxed{\Gamma = \frac{1}{2m_A} \sum_{\text{final states}} C_f \int d\Pi_f \, |\mathcal{M}(k_A \to \{p_f\})|^2} \; ,$$

with $d\Pi_f$ corresponding to a given final state.

8a *This total decay width is related to the half-life of the decaying particle through the relation $\tau = 1/\Gamma$. If the decaying particle is not at rest, the decay width is reduced by a factor $m_A/E_{\vec{k}_A}$. This leads to an increased half-life $\tau E_{\vec{k}_A}/m_A = \tau/\sqrt{1-\vec{v}^{\,2}} \equiv \gamma\tau$, where $\vec{v}$ is the velocity of the decaying particle.*

2.8.2 Cross sections for scattering reactions

Consider a beam of B particles hitting a target at rest consisting of A particles. The case of two colliding particle beams can be obtained from this by an appropriate Lorentz boost. Let's start by assuming constant densities ρ_A and ρ_B in target and beam. The number of scattering events will be proportional to $(\rho_A \ell_A)(\rho_B \ell_B) O$, with O the cross-sectional overlap area common to both the beam and the target. The ratio



$$\frac{\#\text{ scattering events}}{(O\ell_A\rho_A)(O\ell_B\rho_B)/O} \equiv \frac{1}{N_A} \frac{\#\text{ scattering events}}{N_B/O}$$

$$\equiv \sigma$$

defines the cross section σ as the effective area of a chunk taken out of the beam by each particle in the target. The quantities N_A and N_B are the numbers of A and B particles that are relevant for scattering, i.e. the particles that at some point in time belong to the overlap between beam and target. All of this can be equally well formulated in terms of time-related quantities like the scattering rate and the incoming particle flux: simply replace the number of scattering events by the number of scattering events per second and $\ell_B \rho_B$ by the flux $v_B \rho_B$ of beam particles. Hence,

$$\sigma \;=\; \frac{1}{N_A} \frac{\text{scattering rate}}{\text{beam flux}}$$

Approximate plane-wave states: in reality ρ_A and ρ_B are not constant, since the colliding particles are described quantum mechanically by wave packets and both beam and target have a density profile. However,

> *the studied range of the interaction between the colliding particles is usually much smaller than the width of the individual wave packets perpendicular to the beam, which in turn is much maller than the actual diameter of the beam.*

Therefore, in good approximation ρ_A and ρ_B can be considered as locally constant on quantum mechanical (i.e. interaction) length scales[3], whereas the density profiles inside the beam and target can be incorporated properly by averaging over the overlap region:

$$\ell_A \ell_B \int \mathrm{d}^2 x_\perp \, \rho_A(x_\perp) \rho_B(x_\perp) \;\equiv\; N_A N_B / O \; .$$

Here N_A and N_B are the effective numbers of A and B particles that are relevant for scattering and $x_\perp$ is the spatial coordinate perpendicular to the beam. From this it follows that

$$\#\ \text{scattering events} \;=\; \sigma \, N_A N_B / O \; ,$$

where σ can be calculated for effectively constant values of ρ_A and ρ_B corresponding to approximately plane-wave initial states. By the way, we don't have to restrict ourselves to the total number of scattering events. In a similar way we can study the cross section for scattering into the region $\mathrm{d}\vec{p}_1 \cdots \mathrm{d}\vec{p}_n$ around the n-particle final-state momentum point $\vec{p}_1, \cdots, \vec{p}_n$. This is actually what detectors usually do: they detect particles with energy and momentum in certain finite bins, which are given by the detector resolution. These bins cannot resolve the momentum spread of any of the wave packets, just like the detector cells can in general not resolve the particle positions at the level of the de Broglie wavelengths. For all practical purposes detectors observe classical point-like particles with well-defined momenta (in direction and magnitude). So, in the final state it makes sense to use plane waves as well.

[3]These (slowly changing) densities can even be locally zero!

8b *Calculating cross sections therefore amounts to calculating transition probabilities in momentum space. These transition probabilities are universal in the sense that they are independent of details of the experiment, such as the properties of the beams, the targets or the preparation of the initial-state particles.*

The differential cross section: consider an initial state consisting of one target particle and one beam particle in the momentum state $|\vec{k}_A, \vec{k}_B\rangle$ scattering into a final state consisting of n particles in a momentum state with momenta inside the bin $\mathrm{d}\vec{p}_1 \cdots \mathrm{d}\vec{p}_n$ around $\vec{p}_1, \cdots, \vec{p}_n$. In analogy with the calculation in § 2.8.1, the corresponding differential transition probability per unit time and per unit flux is given by

$$\mathrm{d}\sigma \;=\; \frac{1}{F}\,\frac{1}{4E_{\vec{k}_A}E_{\vec{k}_B}V}\,|\mathcal{M}(k_A, k_B \to \{p_j\})|^2\,\mathrm{d}\Pi_n \;,$$

which is usually referred to as the differential cross section. As explained in § 2.8.1 this result for $\mathrm{d}\sigma$ is also valid for identical particles in the final state. In this expression F stands for the flux associated with the incoming beam particle:

$$F \;=\; \frac{1}{V}\,|\vec{v}_{\rm rel}| \;=\; \frac{|\vec{v}_A - \vec{v}_B|}{V} \;\stackrel{\vec{v}=\vec{p}/E}{=\!=}\; \frac{|\vec{k}_A/E_{\vec{k}_A} - \vec{k}_B/E_{\vec{k}_B}|}{V} \;,$$

where we have chosen $\vec{e}_z$ along the beam axis. Furthermore, we have used that the four-momentum of a massive particle reads $p_0^\mu = (m, \vec{0})$ in its rest frame, which becomes $p^\mu = \big(\gamma(E_0 + \vec{v}\cdot\vec{p}_0), \gamma(\vec{p}_0 + E_0\vec{v})\big) \xrightarrow{E_0=m,\,\vec{p}_0=\vec{0}} (m\gamma, m\gamma\vec{v})$ upon boosting with velocity v along the $\vec{e}_p$-direction. We therefore find

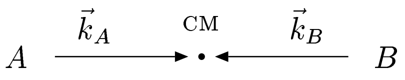
$$\boxed{\mathrm{d}\sigma \;=\; \frac{|\mathcal{M}(k_A, k_B \to \{p_j\})|^2\,\mathrm{d}\Pi_n}{4\,|E_{\vec{k}_B}\vec{k}_A - E_{\vec{k}_A}\vec{k}_B|}}$$

for the differential cross section.

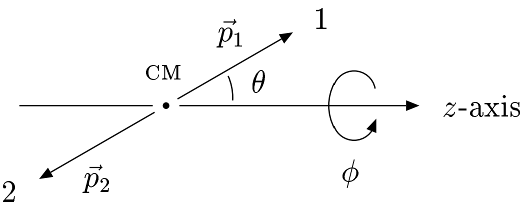
8b *The so-called flux factor* $\frac{1}{4}|E_{\vec{k}_B}\vec{k}_A - E_{\vec{k}_A}\vec{k}_B|^{-1}$ *is invariant under boosts along the beam direction and the same goes for the differential cross section* $\mathrm{d}\sigma$*, as expected for a cross-sectional area perpendicular to the beam.*

2.8.3 CM kinematics and Mandelstam variables for $2 \to 2$ reactions

Consider a $2 \to 2$ reaction with matrix element $\mathcal{M}\big(k_A, k_B \to p_1, p_2\big)$. In the CM frame with the z-direction taken along the beam axis and oriented parallel to the incoming A particles the corresponding kinematics reads:



before



after

$$k_A^\mu = (E_A, 0, 0, k) \ \ , \ \ k_B^\mu = (E_B, 0, 0, -k) \qquad p_1^\mu = (E_1, \vec{p}\,) \ \ , \ \ p_2^\mu = (E_2, -\vec{p}\,) \ .$$

Hence, the two final-state particles are produced back-to-back in the CM frame. Written in compact notation the CM momenta and energies are given by

$$k \ = \ \sqrt{E_{A,B}^2 - m_{A,B}^2} \quad , \quad p \ = \ |\vec{p}\,| \ = \ \sqrt{E_{1,2}^2 - m_{1,2}^2} \quad \text{and} \quad E_A + E_B \ = \ E_1 + E_2 \ \equiv \ E_{\text{CM}}$$

$$\Rightarrow \quad E_{A,B} \ = \ \frac{E_{\text{CM}}^2 + m_{A,B}^2 - m_{B,A}^2}{2E_{\text{CM}}} \quad , \quad k \ = \ \frac{\sqrt{(E_{\text{CM}}^2 - m_A^2 - m_B^2)^2 - 4m_A^2 m_B^2}}{2E_{\text{CM}}} \ ,$$

$$E_{1,2} \ = \ \frac{E_{\text{CM}}^2 + m_{1,2}^2 - m_{2,1}^2}{2E_{\text{CM}}} \quad , \quad p \ = \ \frac{\sqrt{(E_{\text{CM}}^2 - m_1^2 - m_2^2)^2 - 4m_1^2 m_2^2}}{2E_{\text{CM}}} \ .$$

The matrix element is Lorentz invariant, so it can be expressed in terms of invariant combinations of the particle momenta. Since only three out of four particle momenta are independent, this leaves six kinematical variables: the squared masses of the four particles, three so-called Mandelstam variables that combine two of the particle momenta and one condition. We start with the Mandelstam variable

$$s \ \equiv \ (k_A + k_B)^2 \ = \ (p_1 + p_2)^2 \ = \ E_{\text{CM}}^2 \ .$$

In order to guarantee that both $k, p \geq 0$ and $E_{A,B} \geq m_{A,B}$ this variable has to satisfy the inequalities $s \geq (m_A + m_B)^2$ and $s \geq (m_1 + m_2)^2$. The expressions for the CM energies and momenta then become

$$E_{A,B} \ = \ \frac{s + m_{A,B}^2 - m_{B,A}^2}{2\sqrt{s}} \quad , \quad E_{1,2} \ = \ \frac{s + m_{1,2}^2 - m_{2,1}^2}{2\sqrt{s}} \ ,$$

$$k \ = \ \frac{\sqrt{(s - m_A^2 - m_B^2)^2 - 4m_A^2 m_B^2}}{2\sqrt{s}} \qquad \text{and} \qquad p \ = \ \frac{\sqrt{(s - m_1^2 - m_2^2)^2 - 4m_1^2 m_2^2}}{2\sqrt{s}} \ .$$

The other two Mandelstam variables are

$$t \ \equiv \ (k_A - p_1)^2 \ = \ (k_B - p_2)^2 \qquad \text{and} \qquad u \ \equiv \ (k_A - p_2)^2 \ = \ (k_B - p_1)^2 \ ,$$

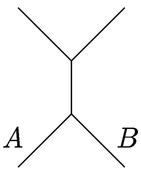
which contain the angular dependence of the reaction through

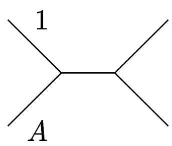
$$2\vec{k}_A \cdot \vec{p}_1 \ = \ 2k\,\vec{e}_z \cdot \vec{p} \ = \ 2kp\cos\theta \qquad \text{and} \qquad 2\vec{k}_A \cdot \vec{p}_2 \ = \ -\,2k\,\vec{e}_z \cdot \vec{p} \ = \ -\,2kp\cos\theta \ .$$

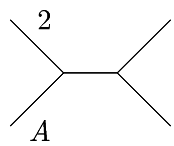
These three Mandelstam variables satisfy the energy-momentum conservation condition

$$s + t + u = m_A^2 + m_B^2 + m_1^2 + m_2^2 \ .$$

A few conventions: in general $2 \to 2$ reactions the most similar initial- and final-state particles are combined into the t-variable. For instance, in the reaction $e^+e^- \to \mu^+\mu^-$ one should combine the momenta of the electron (e^-) and muon (μ^-), or equivalently the momenta of the positron (e^+) and antimuon (μ^+). A reaction channel is referred to as s-channel (or t-channel, or u-channel) if the Mandelstam variable s (or t, or u) features in the propagator at lowest order.

s-channel: 

t-channel: 

u-channel: 

The 2-body phase-space element: in the CM frame, where $\vec{k}_A = -\vec{k}_B \equiv k\,\vec{e}_z$, the beam flux reads

$$F_{\text{CM}} = \frac{k(E_A + E_B)}{E_A E_B V} \equiv \frac{k E_{\text{CM}}}{E_A E_B V} \ .$$

The differential cross section for a $2 \to 2$ reaction in the CM frame therefore becomes

$$\mathrm{d}\sigma_{\text{CM}} = \frac{|\mathcal{M}(k_A, k_B \to p_1, p_2)|^2 \, \mathrm{d}\Pi_2}{4 k E_{\text{CM}}} \ .$$

8b *Note that the differential cross section falls off as $1/E_{CM}^2$ at high energies. This is a destructive interference effect caused by probing the relevant interaction length scale with particles that have a much smaller de Broglie wavelength.*

In analogy with equation (3) the Lorentz invariant phase-space element for two final-state particles becomes

$$\int \mathrm{d}\Pi_2 = \int \frac{\mathrm{d}\vec{p}_1}{(2\pi)^3} \frac{1}{2E_1} \int \frac{\mathrm{d}\vec{p}_2}{(2\pi)^3} \frac{1}{2E_2} \, (2\pi)^4 \, \overbrace{\delta^{(4)}(k_A + k_B - p_1 - p_2)}^{\text{CM: } \delta(E_{\text{CM}} - E_1 - E_2)\,\delta(\vec{p}_1 + \vec{p}_2)}$$

$$\stackrel{\text{CM}}{=\!=} \int \frac{\mathrm{d}p}{16\pi^2} \frac{p^2}{E_1 E_2} \int \mathrm{d}\Omega \; \delta(E_{\text{CM}} - E_1 - E_2) \ .$$

Replacing the integration variable p by $E_1 + E_2 = \sqrt{p^2 + m_1^2} + \sqrt{p^2 + m_2^2}$ this becomes

$$\int \mathrm{d}\Pi_2 = \int \frac{\mathrm{d}(E_1 + E_2)}{16\pi^2 E_1 E_2} \frac{p^2}{p/E_1 + p/E_2} \int \mathrm{d}\Omega \; \delta(E_{\text{CM}} - E_1 - E_2)$$

$$= \frac{p}{16\pi^2 E_{\text{CM}}} \int \mathrm{d}\Omega = \frac{p}{16\pi^2 E_{\text{CM}}} \int_0^{2\pi} \mathrm{d}\phi \int_{-1}^{1} \mathrm{d}\cos\theta \ ,$$

where θ is the polar scattering angle with respect to the beam axis and ϕ the azimuthal scattering angle around the beam axis (as displayed in the figure at the start of this paragraph). From this the following angular differential cross section can be obtained:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} = \frac{p}{64\pi^2 k E_{\text{CM}}^2} \left|\mathcal{M}\left(k_A, k_B \to p_1, p_2\right)\right|^2 . \tag{4}$$

In view of rotational symmetry about the z-axis there will be no ϕ-dependence and the ϕ-integral will straightforwardly yield a factor 2π. Once we also integrate over θ to obtain the total cross section σ, one has to restrict this integration to inequivalent configurations or multiply by the appropriate final-state identical-particle factor C_f.

To give a simple example, we again consider the process

$$\phi(k_A) + \phi(k_B) \;\to\; \phi(p_1) + \phi(p_2)$$

in the scalar ϕ^4-theory. As we have seen on page 48, the lowest-order matrix element for this process is given by $-\lambda$. Hence,

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} = \frac{|\mathcal{M}|^2}{64\pi^2 E_{\text{CM}}^2}\,\frac{p}{k} = \frac{\lambda^2}{64\pi^2 E_{\text{CM}}^2} = \frac{\lambda^2}{64\pi^2 s} \quad \text{and} \quad \sigma = \frac{1}{2}\int d\Omega \left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} = \frac{\lambda^2}{32\pi s} ,$$

where the factor $1/2$ occurring in the last expression is the identical-particle factor for two identical final-state particles. Further examples of cross sections for $2 \to 2$ reactions can for instance be found in Ex. 11.

2.9 Dealing with states in the interacting theory (§7.1 in the book)

> 9 *In order to close the gaps that were left behind during previous steps, we now have to address some of the non-perturbative properties of the interacting theory.*

We study these issues by considering n-point correlation functions (Green's functions) in the scalar ϕ^4-theory:

$$G^{(n)}(x_1, \cdots, x_n) \;\equiv\; \frac{\langle\Omega|T\left(\hat{\phi}(x_1)\cdots\hat{\phi}(x_n)\right)|\Omega\rangle}{\langle\Omega|\Omega\rangle} .$$

Here $\hat{\phi}(x_1), \cdots, \hat{\phi}(x_n)$ are Heisenberg fields in the interacting theory and $|\Omega\rangle$ is the ground state of the interacting theory, which satisfies

$$\hat{H}\,|\Omega\rangle = E_0\,|\Omega\rangle \qquad \text{and} \qquad \langle\Omega|\Omega\rangle = 1 .$$

These Green's functions play an important role in the derivation of scattering amplitudes and are interesting objects in their own right, for instance for studying density perturbations. Without loss of generality we can take $x_1^0 = t_1 \geq x_2^0 = t_2 \geq \cdots \geq x_n^0 = t_n$, so that

$$\langle\Omega|T\big(\hat{\phi}(x_1)\cdots\hat{\phi}(x_n)\big)|\Omega\rangle \;=\; \langle\Omega|\hat{\phi}(x_1)\cdots\hat{\phi}(x_n)|\Omega\rangle$$

$$\overset{\text{p. }34}{=\!=\!=} \;\langle\Omega|\hat{U}^{-1}(t_1,0)\,\hat{\phi}_I(x_1)\,\overbrace{\hat{U}(t_1,0)\hat{U}^{-1}(t_2,0)}^{\hat{U}(t_1,t_2)}\;\cdots\;\overbrace{\hat{U}(t_{n-1},0)\hat{U}^{-1}(t_n,0)}^{\hat{U}(t_{n-1},t_n)}\;\hat{\phi}_I(x_n)\hat{U}(t_n,0)|\Omega\rangle\;.$$

Projecting on the free-particle vacuum: for an arbitrary state $|\psi\rangle$ it will prove handy to consider

$$\langle\psi|\hat{U}(0,t_-)|0\rangle \;\overset{\hat{H}_0|0\rangle\,\equiv\,0}{=\!=\!=\!=}\; \langle\psi|e^{i\hat{H}t_-}|0\rangle \;\overset{\text{completeness relation for }\hat{H}}{=\!=\!=\!=\!=\!=\!=\!=\!=}\; \sum_n\langle\psi|e^{i\hat{H}t_-}|n\rangle\langle n|0\rangle$$

$$=\; e^{iE_0t_-}\langle\psi|\Omega\rangle\langle\Omega|0\rangle \;+\sum_{n\neq\Omega} e^{iE_nt_-}\langle\psi|n\rangle\langle n|0\rangle$$

and subsequently take the limit $t_-\to-\infty$. The "summation" over the excited states $\{|n\rangle\neq|\Omega\rangle\}$ is just a shorthand notation, in fact it will involve an integration over energy (see later). Provided that there is a finite energy gap between the ground state $|\Omega\rangle$ and the excited states $|n\!\neq\!\Omega\rangle$, as is for instance the case for massive excitations, we can employ the Riemann–Lebesgue lemma. This lemma states that

$$\lim_{\beta\to\pm\infty}\int_{v_0}^{v_1}\mathrm{d}v\, f(v)\, e^{i\beta v} \;=\; 0$$

for any integrable function f and any compact or non-compact interval $[v_0,v_1]$. Using this lemma one finds for an arbitrary state $|\psi\rangle$ the identity

$$\lim_{t_-\to-\infty}\frac{e^{-iE_0t_-}\langle\psi|\hat{U}(0,t_-)|0\rangle}{\langle\Omega|0\rangle} \;=\; \langle\psi|\Omega\rangle \;+\; \lim_{t_-\to-\infty}\sum_{n\neq\Omega} e^{i(E_n-E_0)t_-}\frac{\langle\psi|n\rangle\langle n|0\rangle}{\langle\Omega|0\rangle} \;=\; \langle\psi|\Omega\rangle\;.$$

Similarly we can derive the identity

$$\lim_{t_+\to+\infty}\frac{e^{iE_0t_+}\langle 0|\hat{U}(t_+,0)|\psi\rangle}{\langle 0|\Omega\rangle} \;=\; \langle\Omega|\psi\rangle\;.$$

This procedure closely resembles Fermi's Golden Rule for time-dependent perturbation theory. By supplying $|0\rangle$ with the right frequency factor and waiting long enough, only the $|\Omega\rangle$ component of $|0\rangle$ survives as a result of destructive phase interference.

Note: on pages 86 and 87 of the textbook by Peskin & Schroeder the same identities are obtained by tilting the time axis according to $t\to t(1-i\epsilon)$ *with* $\epsilon\in\mathbb{R}$ *infinitesimal. This procedure is closely related to the* $i\epsilon$ *prescription for obtaining the Feynman propagator in § 1.6.*

Inserting these identities in the numerator and denominator of the Green's functions yields

$$G^{(n)}(x_1,\cdots,x_n)$$

$$= \lim_{t_\pm\to\pm\infty} \frac{\langle 0|\,\overbrace{\hat{U}(t_+,0)\hat{U}^{-1}(t_1,0)}^{\hat{U}(t_+,t_1)}\,\hat{\phi}_I(x_1)\hat{U}(t_1,t_2)\cdots\hat{U}(t_{n-1},t_n)\hat{\phi}_I(x_n)\,\overbrace{\hat{U}(t_n,0)\hat{U}(0,t_-)}^{\hat{U}(t_n,t_-)}\,|0\rangle}{\langle 0|\hat{U}(t_+,0)\hat{U}(0,t_-)|0\rangle}$$

$$= \lim_{t_\pm\to\pm\infty} \frac{\langle 0|T\big(\hat{\phi}_I(x_1)\cdots\hat{\phi}_I(x_n)\hat{U}(t_+,t_-)\big)|0\rangle}{\langle 0|\hat{U}(t_+,t_-)|0\rangle} = \frac{\langle 0|T\big(\hat{\phi}_I(x_1)\cdots\hat{\phi}_I(x_n)\hat{S}\big)|0\rangle}{\langle 0|\hat{S}|0\rangle} ,$$

9a *linking Green's functions and the time-ordered vacuum expectation values with vacuum bubbles removed that we studied in § 2.5.*

Interpretation of the vacuum bubbles: we have seen that

$$\lim_{t_\pm\to\pm\infty} e^{iE_0(t_+-t_-)}\,\overbrace{\langle 0|\hat{U}(t_+,t_-)|0\rangle}^{e^{\sum_j V_j}} = \langle 0|\Omega\rangle\langle\Omega|\Omega\rangle\langle\Omega|0\rangle = |\langle\Omega|0\rangle|^2 .$$

From this it follows that $e^{\sum_j V_j} \propto e^{-iE_0(t_+-t_-)} = e^{-iE_0 T}$. The sum of all vacuum bubbles is therefore related to the difference in the ground-state zero-point energies of the interacting theory and the free theory, the latter of which was defined to be 0 in the discussion above. Bearing in mind that V_j contains an infinite spacetime factor $(2\pi)^4\delta^{(4)}(0) = VT$, the energy density of the ground state of the interacting theory reads

$$\frac{E_0}{V} = -\sum_j \frac{\mathrm{Im}(V_j)}{VT} = -\frac{\mathrm{Im}\big(\sum_j V_j\big)}{(2\pi)^4\delta^{(4)}(0)} .$$

The long-distance infinity from the infinite extent of spacetime has been removed in this way, leaving behind the UV infinity that reflects our ignorance about the physics governing the ultra-high-energy regime.

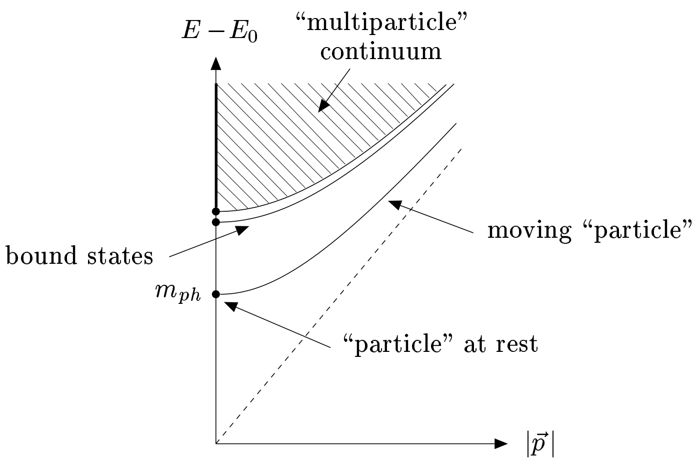
2.9.1 Källén–Lehmann spectral representation

9b *In the free theory $\langle 0|T\big(\hat{\phi}_I(x)\hat{\phi}_I^\dagger(y)\big)|0\rangle$ could be interpreted as the amplitude for a particle to propagate from y to x. The question now is: how should the corresponding 2-point Green's function $\langle\Omega|T\big(\hat{\phi}(x)\hat{\phi}^\dagger(y)\big)|\Omega\rangle$ be interpreted in the interacting theory? This question is related to the particle interpretation of the interacting theory.*

Complete set of interacting states: we start out by having a generic look at the excited states of the interacting theory, with the corresponding energies being defined relative to the ground-state energy E_0. This analysis will be based on the fact that $\big[\hat{H},\hat{\vec{P}}\big] = 0$, which implies that there is a simultaneous set of eigenfunctions of $\hat{H}-E_0\,\hat{1}$ and $\hat{\vec{P}}$. These states can consist of an arbitrary number of particles or they can even be bound states.

1) Zero-momentum states: let $\{|\lambda_{\vec{0}}\rangle\}$ be the set of excited eigenstates of $\hat{H}$ with vanishing total three-momentum, i.e. $\hat{\vec{P}}\,|\lambda_{\vec{0}}\rangle = \vec{0}$. These simultaneous eigenvalues of $\hat{H}-E_0\,\hat{1}$ and $\hat{\vec{P}}$ can be combined into the four-vector $p_0^\mu = (m_\lambda, \vec{0}\,)$, where $m_\lambda > 0$ is the "mass" associated with the particular zero-momentum state.

2) Finite-momentum states: the generator of spacetime translations $\hat{P}^\mu \equiv (\hat{H}-E_0\,\hat{1}, \hat{\vec{P}})$ transforms as a contravariant four-vector under boosts: $\hat{U}^{-1}(\Lambda)\hat{P}^\mu\hat{U}(\Lambda) = \Lambda^\mu{}_\nu\hat{P}^\nu$. This implies that all boosts of the states $|\lambda_{\vec{0}}\rangle$ have all possible total three-momenta $\vec{p}$ and are also eigenstates of $\hat{H}-E_0\,\hat{1}$ with energy $E_{\vec{p}}(\lambda) \equiv \sqrt{\vec{p}^{\,2}+m_\lambda^2}$. The other way round, any eigenstate with explicit three-momentum can be boosted to a zero-momentum eigenstate provided that $m_\lambda > 0$. The sets of eigenvalues $p^\mu = (E-E_0\,, \vec{p}\,)$ are thus organized into hyperboloids, as shown in the figure below. The lowest-lying isolated hyperboloid corresponds to the "1-particle" states of the interacting theory, whereas the other ones correspond to possible bound states. Above a certain threshold value of m_λ a continuum of "multiparticle" states starts (see later).



Proof of the boost statement: consider the Lorentz transformation Λ that transforms $p_0^\mu = (m_\lambda, \vec{0}\,)$ into $p^\mu = \Lambda^\mu{}_\nu p_0^\nu = (E_{\vec{p}}(\lambda), \vec{p}\,)$. Then $|\lambda_{\vec{p}}\rangle \equiv \hat{U}(\Lambda)|\lambda_{\vec{0}}\rangle$ indeed satisfies

$$\hat{P}^\mu|\lambda_{\vec{p}}\rangle \;=\; \hat{U}(\Lambda)\hat{U}^{-1}(\Lambda)\hat{P}^\mu\hat{U}(\Lambda)|\lambda_{\vec{0}}\rangle \;=\; \Lambda^\mu{}_\nu\hat{U}(\Lambda)\hat{P}^\nu|\lambda_{\vec{0}}\rangle \;=\; \Lambda^\mu{}_\nu\, p_0^\nu\,\hat{U}(\Lambda)|\lambda_{\vec{0}}\rangle \;=\; p^\mu\,|\lambda_{\vec{p}}\rangle\ .$$

By reversing the argument, the reversed statement can be proven as well, bearing in mind that $E-E_0 > 0$ for the excited states so that the combined four-momentum eigenvalues $p^\mu = (E-E_0\,, \vec{p}\,)$ have to satisfy $p^2 \geq 0$.

Completeness relation: in the interacting theory we can therefore use the following completeness relation associated with this complete set of states:

$$\hat{1} \;=\; |\Omega\rangle\langle\Omega| \;+\; \sum_\lambda\int\frac{\mathrm{d}\vec{p}}{(2\pi)^3}\,\frac{|\lambda_{\vec{p}}\rangle\langle\lambda_{\vec{p}}|}{2E_{\vec{p}}(\lambda)}\ ,$$

where the first term corresponds to the ground state and the second one to all excited states.

The 2-point Green's function: next we take $x^0 > y^0$ and insert the above-given completeness relation into the 2-point Green's function. This results in the following split-up:

$$\begin{aligned}\langle\Omega|T\big(\hat{\phi}(x)\hat{\phi}^\dagger(y)\big)|\Omega\rangle &= \langle\Omega|\hat{\phi}(x)|\Omega\rangle\langle\Omega|\hat{\phi}^\dagger(y)|\Omega\rangle \\ &\quad + \sum_\lambda\int\frac{\mathrm{d}\vec{p}}{(2\pi)^3}\,\frac{1}{2E_{\vec{p}}(\lambda)}\,\langle\Omega|\hat{\phi}(x)|\lambda_{\vec{p}}\rangle\langle\lambda_{\vec{p}}|\hat{\phi}^\dagger(y)|\Omega\rangle\ .\end{aligned}$$

In the absence of preferred directions in the universe, the ground state $|\Omega\rangle$ should be invariant under spacetime translations and Lorentz transformations, i.e. $e^{i\hat{P}\cdot x}|\Omega\rangle = |\Omega\rangle$ and $\hat{U}(\Lambda)|\Omega\rangle = |\Omega\rangle$. Therefore

$$\begin{aligned}\langle\Omega|\hat{\phi}(x)|\lambda_{\vec{p}}\rangle &\stackrel{\text{p. 19}}{=\!=} \langle\Omega|e^{i\hat{P}\cdot x}\hat{\phi}(0)e^{-i\hat{P}\cdot x}|\lambda_{\vec{p}}\rangle = e^{-ip\cdot x}\langle\Omega|\hat{\phi}(0)|\lambda_{\vec{p}}\rangle\Big|_{p^0=E_{\vec{p}}(\lambda)} \\ &= e^{-ip\cdot x}\langle\Omega|\hat{U}^{-1}(\Lambda)\hat{U}(\Lambda)\hat{\phi}(0)\hat{U}^{-1}(\Lambda)\hat{U}(\Lambda)|\lambda_{\vec{p}}\rangle\Big|_{p^0=E_{\vec{p}}(\lambda)} \\ &\stackrel{\text{p. 22}}{=\!=} e^{-ip\cdot x}\langle\Omega|\hat{\phi}(\Lambda 0)|\lambda_{\Lambda\vec{p}}\rangle\Big|_{p^0=E_{\vec{p}}(\lambda)} \stackrel{\text{take }\Lambda\text{ such that }\vec{\Lambda p}=\vec{0}}{=\!=\!=\!=\!=\!=} e^{-ip\cdot x}\langle\Omega|\hat{\phi}(0)|\lambda_{\vec{0}}\rangle\Big|_{p^0=E_{\vec{p}}(\lambda)}\end{aligned}$$

and similarly

$$\langle\Omega|\hat{\phi}(x)|\Omega\rangle = \langle\Omega|\hat{\phi}(0)|\Omega\rangle \equiv v\ .$$

The ground-state expectation value v, which in the literature is sloppily called the "vacuum expectation value" or short vev of the field $\hat{\phi}$, usually is taken to be 0. If this is not the case then one should reformulate the theory in terms of the field $\hat{\phi}'(x) = \hat{\phi}(x) - v$, which has a vanishing vev. The rest goes in the same way as described below. Leaving out the vev we now obtain

$$\begin{aligned}\langle\Omega|\hat{\phi}(x)\hat{\phi}^\dagger(y)|\Omega\rangle &= \sum_\lambda|\langle\Omega|\hat{\phi}(0)|\lambda_{\vec{0}}\rangle|^2\int\frac{\mathrm{d}\vec{p}}{(2\pi)^3}\,\frac{e^{-ip\cdot(x-y)}}{2E_{\vec{p}}(\lambda)}\Big|_{p^0=E_{\vec{p}}(\lambda)} \\ &\stackrel{x^0>y^0,\,p.\,25}{=\!=\!=\!=} \sum_\lambda|\langle\Omega|\hat{\phi}(0)|\lambda_{\vec{0}}\rangle|^2\int\frac{\mathrm{d}^4p}{(2\pi)^4}\,\frac{i\,e^{-ip\cdot(x-y)}}{p^2-m_\lambda^2+i\epsilon}\ .\end{aligned}$$

The integral on the last line we recognize as the Feynman propagator belonging to a "ϕ-particle" with mass m_λ, i.e. $D_F(x-y;m_\lambda^2)$.

> 9b *The particle interpretation has in fact changed in the interacting theory from free particles to dressed particles (quasi-particles), so the "particles" we are dealing with here are not the particles that we know from the free theory!*

Källén−Lehmann spectral representation: a similar procedure can be applied in the case that $x^0 < y^0$. Combining both cases one arrives at the so-called Källén−Lehmann spectral representation of the 2-point Green's function:

$$\langle\Omega|T\big(\hat{\phi}(x)\hat{\phi}^\dagger(y)\big)|\Omega\rangle = \int_0^\infty\frac{\mathrm{d}s}{2\pi}\,\rho(s)\,D_F(x-y;s)\ ,$$

where the function $\rho(s)$ in the squared invariant mass s is a positive spectral density function given by

$$\rho(s) \;=\; \sum_{\lambda} 2\pi\,\delta(s-m_\lambda^2)\,|\langle\Omega|\hat{\phi}(0)|\lambda_{\vec{0}}\rangle|^2 \;.$$

The states in the interacting theory that describe a single dressed particle correspond to an isolated δ-function in the spectral density function:

$$\rho_{\text{1-part.}}(s) \;=\; 2\pi\,\delta(s-m_{ph}^2)\,\big|\langle\Omega|\hat{\phi}(0)|\lambda_{\vec{0}}\rangle_{\text{1-part.}}\big|^2 \;\equiv\; 2\pi Z\,\delta(s-m_{ph}^2)\;.$$

9b *The field-strength/wave-function renormalization Z is the probability for $\hat{\phi}^\dagger(0)$ to create a state that describes a single dressed particle from the ground state, whereas m_{ph} is the observable physical mass of the dressed particle, being the energy eigenvalue in its rest frame. This physical (dressed) mass is in general not equal to the (bare) mass parameter m occurring in the Lagrangian, which is not observable directly!*

In momentum space: the Källén–Lehmann spectral representation trivially reads

$$\begin{aligned}\int \mathrm{d}^4x\; e^{ip\cdot x}\,\langle\Omega|T\big(\hat{\phi}(x)\hat{\phi}^\dagger(0)\big)|\Omega\rangle \;&=\; \int_0^\infty \frac{\mathrm{d}s}{2\pi}\,\rho(s)\,\frac{i}{p^2-s+i\epsilon}\\ &=\; \frac{iZ}{p^2-m_{ph}^2+i\epsilon} \;+\; \int_{\sim s_{th}}^\infty \frac{\mathrm{d}s}{2\pi}\,\rho(s)\,\frac{i}{p^2-s+i\epsilon}\end{aligned}$$

in momentum space, where s_{th} denotes the threshold for the creation of the continuum of "multiparticle" states. The fact that the last integral does not start exactly at s_{th} is caused by the possible existence of multiparticle bound states. Graphically the analytic (pole/cut) structure in the complex p^2-plane can be depicted as follows:

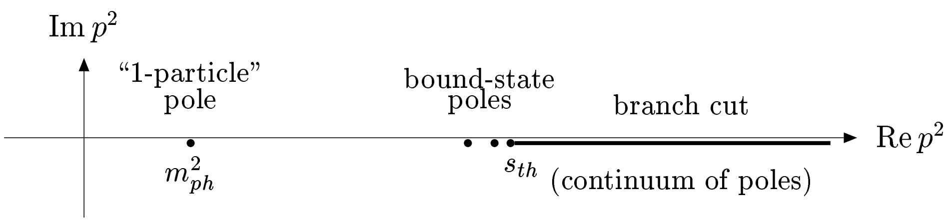


Figure 7: Poles and cuts of the 2-point Green's function.

Interacting theory vs free theory:

- In the interacting theory $|\langle\Omega|\hat{\phi}(0)|\lambda_{\vec{0}}\rangle|^2 = |\langle\Omega|\hat{\phi}(0)|\lambda_{\vec{p}}\rangle|^2$ represents the probability for the field $\hat{\phi}^\dagger(0)$ to create a given dressed state from the ground state, with the factor Z being the associated probability for creating a "1-dressed-particle" state.

The factor Z differs from unity since in the interacting theory $\hat{\phi}^\dagger(0)$ can also create "multiparticle" intermediate states with a continuous mass spectrum, unlike in the free theory.

- In the free theory $\rho(s) = 2\pi\,\delta(s-m^2)$ and $Z=1$, since

$$\langle\vec{p}\,|\hat{\phi}_{I}^\dagger(0)|0\rangle \;=\; \langle 0|\sqrt{2E_{\vec{p}}}\;\hat{a}_{\vec{p}}\int\frac{d\vec{q}}{(2\pi)^3}\,\frac{\hat{b}_{\vec{q}}+\hat{a}_{\vec{q}}^\dagger}{\sqrt{2E_{\vec{q}}}}\,|0\rangle \;=\; \langle 0|0\rangle \;=\; 1\,.$$

For $x^0>0$ the quantity

$$\int d^4x\; e^{ip\cdot x}\,\langle 0|T\big(\hat{\phi}(x)\hat{\phi}^\dagger(0)\big)|0\rangle \;=\; \frac{i}{p^2-m^2+i\epsilon}$$

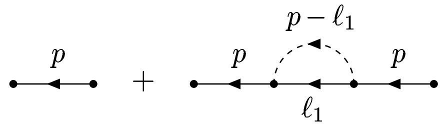
is interpreted as the amplitude for a particle to propagate from 0 to x.

2.9.2 2-point Green's functions in momentum space (§ 6.3 and 7.1 in the book)

9c *Question: does all this also follow from an explicit diagrammatic calculation within perturbation theory?*

In order to address this question we consider the 2-point Green's function for ψ-particles in the scalar Yukawa theory (with tadpole diagrams excluded, as will be explained later):

$$\int d^4x\; e^{ip\cdot x}\,\langle\Omega|T\big(\hat{\psi}(x)\hat{\psi}^\dagger(0)\big)|\Omega\rangle \;=\;$$



$$=\;\frac{i}{p^2-M^2+i\epsilon}+\frac{i}{p^2-M^2+i\epsilon}\Big(-i\Sigma_2(p^2)\Big)\frac{i}{p^2-M^2+i\epsilon}+\cdots\,,$$

where

$$-i\Sigma_2(p^2) \;=\; (-ig)^2\int\frac{d^4\ell_1}{(2\pi)^4}\,\frac{i}{\ell_1^2-M^2+i\epsilon}\,\frac{i}{(p-\ell_1)^2-m^2+i\epsilon}$$

is the so-called ψ-particle self-energy at $\mathcal{O}(g^2)$. Since the corresponding diagram involves one loop and therefore one energy-momentum integration, we usually refer to this self-energy as the 1-loop self-energy.

9c *There are two main approaches to calculate such an integral:*

1. *perform the ℓ_1^0-integration in the complex plane, involving four complex poles, and work out the resulting $\vec{\ell}_1$-integration;*
2. *apply the following two calculational tricks.*

Trick 1: use Feynman parameters. Writing the denominators in the integral as

$$D_1 \;\equiv\; \ell_1^2 - M^2 + i\epsilon \qquad \text{and} \qquad D_2 \;\equiv\; (p-\ell_1)^2 - m^2 + i\epsilon \ ,$$

we can combine the two denominators into

$$\frac{1}{D_1 D_2} \;=\; \frac{1}{D_1 - D_2}\Big(\frac{1}{D_2} - \frac{1}{D_1}\Big) \;=\; \Big[\frac{1}{D_1-D_2}\,\frac{1}{\alpha_2 D_2 + (1-\alpha_2)D_1}\Big]_{\alpha_2=0}^{\alpha_2=1}$$

$$= \int_0^1 \mathrm{d}\alpha_2\,\frac{1}{[\alpha_2 D_2 + (1-\alpha_2)D_1]^2} \;=\; \int_0^1 \mathrm{d}\alpha_1 \int_0^1 \mathrm{d}\alpha_2\;\delta(\alpha_1+\alpha_2-1)\,\frac{1}{(\alpha_1 D_1 + \alpha_2 D_2)^2}\ .$$

The parameters $\alpha_{1,2}$ are called Feynman parameters. Inserting the specific expressions for the denominators we then obtain

$$-\,i\Sigma_2(p^2) \;=\; g^2\int_0^1 \mathrm{d}\alpha_2 \int\frac{\mathrm{d}^4\ell_1}{(2\pi)^4}\,\big[\ell_1^2 - 2\alpha_2\,p\cdot\ell_1 + \alpha_2 p^2 - \alpha_2 m^2 - (1-\alpha_2)M^2 + i\epsilon\big]^{-2}$$

$$\equiv\; g^2\int_0^1 \mathrm{d}\alpha_2 \int\frac{\mathrm{d}^4\ell}{(2\pi)^4}\,\frac{1}{(\ell^2-\Delta+i\epsilon)^2}\ ,$$

with

$$\ell \;\equiv\; \ell_1 - \alpha_2 p \qquad \text{and} \qquad \Delta \;\equiv\; \alpha_2 m^2 + (1-\alpha_2)M^2 - \alpha_2(1-\alpha_2)p^2\ .$$

We have gained the following in this first step:

- The original integrand had four poles in the complex ℓ_1^0-plane, whereas now we have only two poles in the complex ℓ^0-plane.
- The integrand has become spherically invariant, implying that integrals with an odd numerator in ℓ should vanish, i.e.

  $$\int \mathrm{d}^4\ell\ f(\ell^2)\ell_\mu \;=\; \int \mathrm{d}^4\ell\ f(\ell^2)\ell_\mu\ell_\nu\ell_\rho \;=\; \cdots \;=\; 0\ .$$

  In contrast, integrals with an even numerator in ℓ can be simplified. For instance

  $$\int \mathrm{d}^4\ell\ f(\ell^2)\ell_\mu\ell_\nu \ \overset{0 \text{ if } \mu\neq\nu}{=\!=\!=\!=}\ \frac{g_{\mu\nu}}{4}\int \mathrm{d}^4\ell\ f(\ell^2)\,\ell^2\ ,$$

  using that

  $$\int \mathrm{d}^4\ell\ f(\ell^2)(\ell_1)^2 \;=\; \int \mathrm{d}^4\ell\ f(\ell^2)(\ell_2)^2 \;=\; \int \mathrm{d}^4\ell\ f(\ell^2)(\ell_3)^2 \;=\; -\int \mathrm{d}^4\ell\ f(\ell^2)(\ell_0)^2\ .$$

  These properties will in particular prove important for non-scalar particles.
- The trick works equally well for an arbitrary number of propagators occurring in the loop:

  $$\frac{1}{D_1\cdots D_n} \;=\; \int_0^1 \mathrm{d}\alpha_1 \cdots \int_0^1 \mathrm{d}\alpha_n\,\frac{(n-1)!\,\delta(\alpha_1+\cdots+\alpha_n-1)}{(\alpha_1 D_1 + \cdots + \alpha_n D_n)^n}\ .$$

Trick 2: perform Wick rotation. In order to perform the ℓ^0-part of the integral $\int \mathrm{d}^4\ell\,(\ell^2 - \Delta + i\epsilon)^{-j}/(2\pi)^4$ the integration contour C indicated in figure 8 is used. Since the poles are situated outside the integration contour in the complex ℓ^0-plane, the integral along the real ℓ^0-axis is transformed into an integral along the imaginary axis.

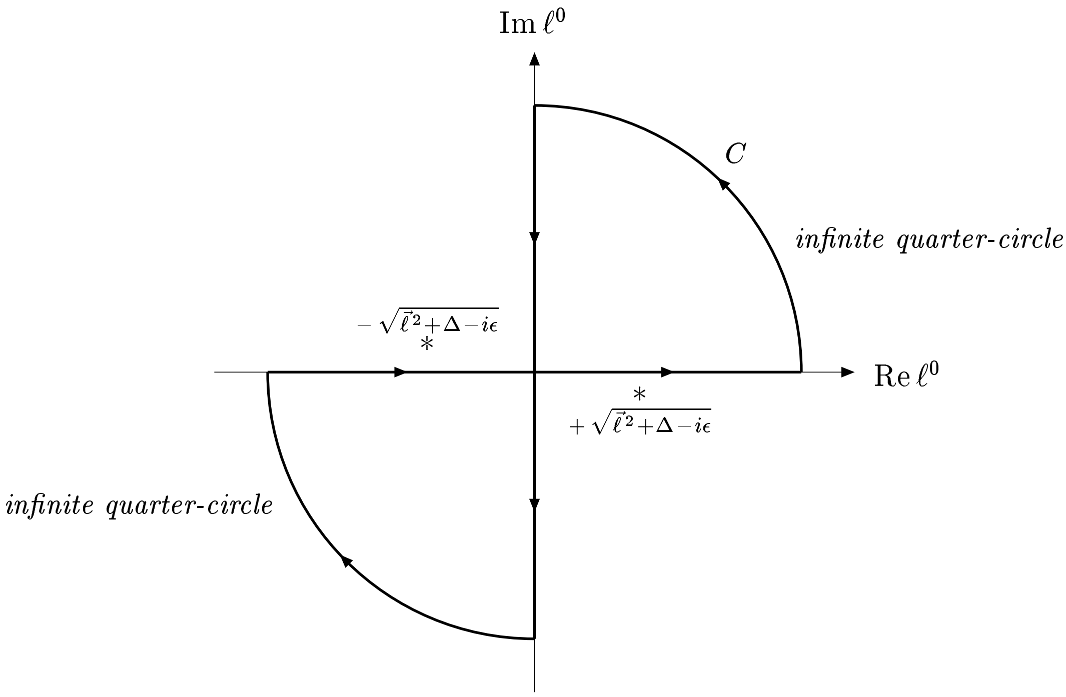


Figure 8: Closed integration contour used for performing Wick rotation.

In this way a Minkowskian integral can be transformed into a Euclidean one:

$$\int\limits_{-\infty}^{\infty}\mathrm{d}\ell^0 \;\to\; -\int\limits_{i\infty}^{-i\infty}\mathrm{d}\ell^0 \;\stackrel{\ell^0\equiv i\,\ell^0_E}{=\!=\!=}\; -i\int\limits_{\infty}^{-\infty}\mathrm{d}\ell^0_E \;=\; i\int\limits_{-\infty}^{\infty}\mathrm{d}\ell^0_E \qquad \text{and} \qquad \int\mathrm{d}\vec{\ell} \;\stackrel{\vec{\ell}\equiv\vec{\ell}_E}{=\!=\!=}\; \int\mathrm{d}\vec{\ell}_E\;.$$

This results in

$$\begin{aligned}
\int\frac{\mathrm{d}^4\ell}{(2\pi)^4}\,\frac{1}{(\ell^2-\Delta+i\epsilon)^j} \;&=\; \frac{i}{(2\pi)^4}\int\limits_{-\infty}^{\infty}\mathrm{d}\ell^0_E\int\mathrm{d}\vec{\ell}_E\;\frac{1}{\big[-(\ell^0_E)^2-\vec{\ell}^{\,2}_E-\Delta+i\epsilon\big]^j}\\
&=\; \frac{i}{16\pi^4}\,(-1)^j\int\frac{\mathrm{d}^4\ell_E}{(\ell^2_E+\Delta-i\epsilon)^j}\\
&=\; \frac{i}{16\pi^4}\,(-1)^j\int_0^{2\pi}\mathrm{d}\theta_1\int_0^{\pi}\mathrm{d}\theta_2\,\sin(\theta_2)\int_0^{\pi}\mathrm{d}\theta_3\,\sin^2(\theta_3)\;\frac{1}{2}\int_0^{\infty}\mathrm{d}\ell^2_E\;\frac{\ell^2_E}{(\ell^2_E+\Delta-i\epsilon)^j}\\
&=\; \frac{i}{16\pi^2}\,(-1)^j\int_0^{\infty}\mathrm{d}\ell^2_E\;\frac{\ell^2_E}{(\ell^2_E+\Delta-i\epsilon)^j}\;,
\end{aligned}$$

where the norm $\ell_E^2 = (\ell_E^0)^2 + (\ell_E^1)^2 + (\ell_E^2)^2 + (\ell_E^3)^2$ is positive definite in Euclidean space. In the penultimate step it was used that in an n-dimensional Euclidean space the transition to spherical coordinates is given by

$$\int \mathrm{d}\vec{r}\, f(r) \;=\; \int_0^\infty \mathrm{d}r\, r^{n-1} f(r) \int_0^{2\pi} \mathrm{d}\theta_1 \int_0^\pi \mathrm{d}\theta_2\, \sin(\theta_2) \cdots \int_0^\pi \mathrm{d}\theta_{n-1}\, \sin^{n-2}(\theta_{n-1})$$
$$= \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_0^\infty \mathrm{d}r\, r^{n-1} f(r)\ ,$$

where the gamma function $\Gamma(z)$ satisfies

$$\Gamma(1/2) = \sqrt{\pi}\quad ,\quad \Gamma(1) = 1 \qquad \text{and} \qquad \Gamma(z+1) = z\Gamma(z)\ .$$

The result after applying both tricks:

$$-\,i\Sigma_2(p^2) \;=\; \frac{ig^2}{16\pi^2} \int_0^1 \mathrm{d}\alpha_2 \int_0^\infty \mathrm{d}\ell_E^2\, \frac{\ell_E^2}{(\ell_E^2 + \Delta - i\epsilon)^2}$$
$$= \frac{ig^2}{16\pi^2} \int_0^1 \mathrm{d}\alpha_2 \Big(-1 - \log\big[\Delta(\alpha_2) - i\epsilon\big] + \text{UV infinity}\Big)\ ,$$

where the infinity originates from the large-momentum regime $\ell_E^2 \to \infty$. The logarithm

$$\log(z) \;\equiv\; \log\big(|z|e^{i\phi}\big) \;=\; \log\big(|z|\big) + i\phi$$

gives rise to a branch cut for $z \in \mathbb{R}^-$, since $\log(-x \pm i\epsilon) = \log(x e^{\pm i\pi}) = \log(x) \pm i\pi$ for $x > 0$. This corresponds to situations where $\Delta(\alpha_2) = \alpha_2^2 p^2 + \alpha_2(m^2 - M^2 - p^2) + M^2 < 0$ on the interval $\alpha_2 \in [0,1]$. Since $\Delta(\alpha_2 = 1) = m^2$ and $\Delta(\alpha_2 = 0) = M^2$, this happens when $\Delta(\alpha_2) = 0$ has both roots

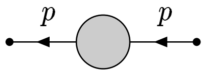
$$\alpha_2 \;=\; \frac{p^2 + M^2 - m^2 \pm \sqrt{(p^2 + M^2 - m^2)^2 - 4p^2M^2}}{2p^2}$$
$$= \frac{p^2 + M^2 - m^2 \pm \sqrt{\big[p^2 - (M+m)^2\big]\big[p^2 - (M-m)^2\big]}}{2p^2}$$

on the interval $\alpha_2 \in [0,1]$, which results in the requirement that $p^2 > (M+m)^2$.

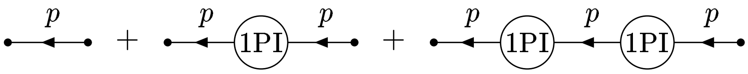
9c *There is a minimal value $p^2_{min} = (M+m)^2$ of p^2 for which the branch cut of the 2-point Green's function in the scalar Yukawa theory starts, being the threshold for the creation of a two-particle state with masses M and m. This is precisely what we would expect based on the Källén–Lehmann spectral representation.*

Dyson series: to all orders in perturbation theory the 2-point Green's function (a.k.a. the full propagator or dressed propagator) is given by the Dyson series

$$\int \mathrm{d}^4x\; e^{ip\cdot x}\,\langle\Omega|T\big(\hat{\psi}(x)\hat{\psi}^\dagger(0)\big)|\Omega\rangle \quad \equiv$$



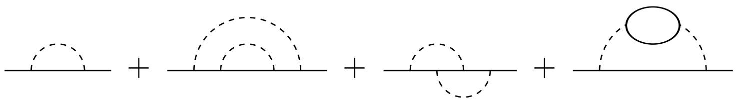
$=$



$+ \;\cdots\;,$

where

$\equiv\; -\,i\Sigma(p^2) \;=$

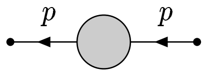


$+ \;\cdots$

is the collection of all 1-particle irreducible (1PI) self-energy diagrams. Diagrams are called 1-particle irreducible if they cannot be split in two by removing a single line.

The single-particle pole and physical mass: the Dyson series is in fact a geometric series, which can be summed according to

$$\int \mathrm{d}^4x\; e^{ip\cdot x}\,\langle\Omega|T\big(\hat{\psi}(x)\hat{\psi}^\dagger(0)\big)|\Omega\rangle \quad =$$



$$= \frac{i}{p^2-M^2+i\epsilon} + \frac{i}{p^2-M^2+i\epsilon}\Big(-\,i\Sigma(p^2)\Big)\frac{i}{p^2-M^2+i\epsilon} + \cdots$$

$$= \frac{i}{p^2-M^2-\Sigma(p^2)+i\epsilon}\;.$$

The full propagator has a simple pole located at the physical mass M_{ph}, which is shifted away from M by the self-energy:

$$\Big[p^2-M^2-\Sigma(p^2)\Big]\bigg|_{p^2=M_{ph}^2} = 0 \quad\Rightarrow\quad M_{ph}^2-M^2-\Sigma(M_{ph}^2) = 0\;.$$

Close to this pole the denominator of the full propagator can be expanded according to

$$p^2-M^2-\Sigma(p^2) \;\approx\; (p^2-M_{ph}^2)\big[1-\Sigma'(M_{ph}^2)\big] + \mathcal{O}\big([p^2-M_{ph}^2]^2\big) \quad \text{for} \quad p^2\approx M_{ph}^2\;,$$

where $\Sigma'(p^2)$ stands for the derivative of the self-energy with respect to p^2.

> 9c *Just like in the Källén–Lehmann spectral representation, the full propagator has a single-particle pole of the form $iZ/(p^2-M_{ph}^2+i\epsilon)$ with residue $Z=1/\big[1-\Sigma'(M_{ph}^2)\big]$. This observed close connection to the non-perturbative analytic structure of the 2-point Green's function serves as justification for our procedure, which involved summing the geometric series outside its formal radius of convergence.*

2.9.3 Deriving n-particle matrix elements from n-point Green's functions

For real scalar fields $\hat{\phi}(x)$ we have seen that

$$\int \mathrm{d}^4x\, e^{ip\cdot x}\, \langle\Omega|T\big(\hat{\phi}(x)\hat{\phi}(0)\big)|\Omega\rangle \quad \overset{p^2\to m_{ph}^2}{\sim} \quad \frac{iZ}{p^2 - m_{ph}^2 + i\epsilon}\ ,$$

by which is meant that the quantities on either side have the same single-particle poles and residues at the physical mass squared m_{ph}^2. The wave-function renormalization factor Z can be obtained straightforwardly from the 2-point Green's function in momentum space by multiplying by $(p^2 - m_{ph}^2)/i$ and taking the limit $p^2 \to m_{ph}^2$.

9d *We now wish to use this single-particle pole structure to obtain the asymptotic "in" and "out" states of the theory and in particular their matrix elements.*

Consider to this end

$$\int \mathrm{d}^4x\, e^{ip\cdot x}\, \langle\Omega|T\big(\hat{\phi}(x)\hat{\phi}(z_1)\hat{\phi}(z_2)\cdots\big)|\Omega\rangle \qquad \text{with} \qquad \int \mathrm{d}x^0 = \int_{T_+}^{\infty} \mathrm{d}x^0 + \int_{T_-}^{T_+} \mathrm{d}x^0 + \int_{-\infty}^{T_-} \mathrm{d}x^0\ ,$$

where $T_- < \min z_j^0$ and $T_+ > \max z_j^0$.

What can we say about the pole structure of this integrated Green's function?

- The integration region $x^0 \in [T_-, T_+]$: since the temporal integration interval is bounded and the integrand has no p^0-poles, the result of the integral is an analytic function in p^0 without any poles.
- The other two integration regions: the integrand still has no poles, but the integration intervals are unbounded. Therefore singularities in p^0 may develop upon integration!

The integration interval $x^0 \in [T_+, \infty)$: we again insert the completeness relation for $\hat{H}$ and assume that the field $\hat{\phi}(x)$ has a vanishing vev. If $\langle\Omega|\hat{\phi}(x)|\Omega\rangle = v \neq 0$, then $\hat{\phi}(x)$ should be rewritten as $\hat{\phi}(x) \equiv v + \hat{\phi}'(x)$ and the particle interpretation should be obtained from $\hat{\phi}'(x)$ rather than $\hat{\phi}(x)$. The integral then takes the form

$$\int_{T_+}^{\infty} \mathrm{d}x^0 \int \mathrm{d}\vec{x}\, e^{ip^0x^0} e^{-i\vec{p}\cdot\vec{x}} \sum_{\lambda} \int \frac{\mathrm{d}\vec{q}}{(2\pi)^3}\, \frac{1}{2E_{\vec{q}}(\lambda)}\, \langle\Omega|\hat{\phi}(x)|\lambda_{\vec{q}}\rangle\langle\lambda_{\vec{q}}|T\big(\hat{\phi}(z_1)\hat{\phi}(z_2)\cdots\big)|\Omega\rangle$$

$$\overset{\text{p. 66}}{=\!=} \int_{T_+}^{\infty} \mathrm{d}x^0 \int \mathrm{d}\vec{x}\, e^{-i\vec{p}\cdot\vec{x}} \sum_{\lambda} \int \frac{\mathrm{d}\vec{q}}{(2\pi)^3}\, \frac{\sqrt{Z(\lambda)}}{2E_{\vec{q}}(\lambda)}\, \langle\lambda_{\vec{q}}|T\big(\hat{\phi}(z_1)\hat{\phi}(z_2)\cdots\big)|\Omega\rangle\, e^{i\vec{q}\cdot\vec{x}}\, e^{ix^0[p^0 - E_{\vec{q}}(\lambda)]}\ ,$$

where $\langle\Omega|\hat{\phi}(x)|\lambda_{\vec{q}}\rangle \stackrel{\text{p. 66}}{=\!=} e^{-iq\cdot x}\langle\Omega|\hat{\phi}(0)|\lambda_{\vec{0}}\rangle\Big|_{q^0=E_{\vec{q}}(\lambda)} \equiv \sqrt{Z(\lambda)}\;e^{-iq\cdot x}\Big|_{q^0=E_{\vec{q}}(\lambda)}$. The phase of $\langle\Omega|\hat{\phi}(0)|\lambda_{\vec{0}}\rangle$ does not matter in this context, since it can be absorbed in the definition of $|\lambda_{\vec{0}}\rangle$. Now the Riemann–Lebesgue lemma can be invoked, which states that the larger x^0 becomes the sharper this integral is peaked around $p^0 = E_{\vec{q}}(\lambda)$. This fact can be quantified explicitly by adding a damping factor $e^{-\epsilon x^0}$ (with infinitesimal $\epsilon > 0$) to the integral, in order to ensure that it is well-defined. This procedure is equivalent with the $i\epsilon$ prescription for obtaining the Feynman propagator in § 1.6 and the tilted time axis prescription in the textbook by Peskin & Schroeder. After performing the trivial $\vec{x}$ integration we get

$$\sum_{\lambda} \frac{\sqrt{Z(\lambda)}}{2E_{\vec{p}}(\lambda)}\,\langle\lambda_{\vec{p}}|T\big(\hat{\phi}(z_1)\hat{\phi}(z_2)\cdots\big)|\Omega\rangle \int\limits_{T_+}^{\infty} \mathrm{d}x^0\; e^{ix^0[p^0-E_{\vec{p}}(\lambda)+i\epsilon]}$$

$$= \sum_{\lambda} \frac{\sqrt{Z(\lambda)}}{2E_{\vec{p}}(\lambda)}\,\langle\lambda_{\vec{p}}|T\big(\hat{\phi}(z_1)\hat{\phi}(z_2)\cdots\big)|\Omega\rangle\, \frac{i\,e^{iT_+[p^0-E_{\vec{p}}(\lambda)+i\epsilon]}}{p^0-E_{\vec{p}}(\lambda)+i\epsilon}\ ,$$

which corresponds to isolated 1-particle poles, isolated bound-state poles or multiparticle branch-cut poles. Subsequently we note that

$$\frac{i}{p^2-m_\lambda^2+i\epsilon} = \frac{i}{p_0^2-E_{\vec{p}}^2(\lambda)+i\epsilon} \qquad \text{and} \qquad \frac{1}{2E_{\vec{p}}(\lambda)}\,\frac{ie^{iT_+[p^0-E_{\vec{p}}(\lambda)+i\epsilon]}}{p^0-E_{\vec{p}}(\lambda)+i\epsilon}$$

have the same residues at the pole $p^0 = E_{\vec{p}}(\lambda) - i\epsilon$.

> *The 1-particle state in the far future corresponds to an isolated pole at the on-shell energy* $p^0 = E_{\vec{p}} = \sqrt{\vec{p}^{\,2}+m_{ph}^2}$ *:*

$$\int \mathrm{d}^4x\; e^{ip\cdot x}\langle\Omega|T\big(\hat{\phi}(x)\hat{\phi}(z_1)\hat{\phi}(z_2)\cdots\big)|\Omega\rangle \stackrel{p^0\to E_{\vec{p}}}{\sim} \frac{i\sqrt{Z}}{p^2-m_{ph}^2+i\epsilon}\;{}_{out}\langle\vec{p}|T\big(\hat{\phi}(z_1)\hat{\phi}(z_2)\cdots\big)|\Omega\rangle\ ,$$

using the notation $|\vec{p}\rangle_{out} \equiv |\lambda_{\vec{p}}\rangle_{\text{1-part.}}$ for a 1-particle eigenstate with momentum $\vec{p}$ that is created at asymptotically large times.

The integration interval $x^0 \in (-\infty, T_-]$: in this case the steps are similar to the ones for the previous integration interval. The following changes should be made though: the damping factor $e^{-\epsilon x^0}$ should be replaced by $e^{+\epsilon x^0}$, $\hat{\phi}(x)$ is now situated at the end of the operator chain, $e^{-iq\cdot x}$ should be replaced by $e^{+iq\cdot x}$ and the pole energy $p^0 = E_{\vec{p}}(\lambda) - i\epsilon$ now changes to $p^0 = -\,E_{\vec{p}}(\lambda) + i\epsilon$.

> *The 1-particle state in the far past corresponds to an isolated pole at the on-shell energy* $p^0 = -\,E_{\vec{p}} = -\sqrt{\vec{p}^{\,2}+m_{ph}^2}$ *:*

$$\int \mathrm{d}^4x\; e^{ip\cdot x}\langle\Omega|T\big(\hat{\phi}(x)\hat{\phi}(z_1)\hat{\phi}(z_2)\cdots\big)|\Omega\rangle \stackrel{p^0\to -E_{\vec{p}}}{\sim} \frac{i\sqrt{Z}}{p^2-m_{ph}^2+i\epsilon}\;\langle\Omega|T\big(\hat{\phi}(z_1)\hat{\phi}(z_2)\cdots\big)|-\vec{p}\,\rangle_{in}\ .$$

LSZ reduction formula: the procedure described above can actually be worked out for situations with as many 1-particle poles as there are field operators in the Green's function. This leads to the so-called LSZ (H. Lehmann, K. Symanzik, W. Zimmermann) reduction formula:

$$\Big(\prod_{j=1}^{n}\int \mathrm{d}^4x_j\; e^{ip_j\cdot x_j}\Big)\Big(\prod_{j'=1}^{n'}\int \mathrm{d}^4y_{j'}\; e^{-ik_{j'}\cdot y_{j'}}\Big)\langle\Omega|T\big(\hat{\phi}(x_1)\cdots\hat{\phi}(x_n)\hat{\phi}(y_1)\cdots\hat{\phi}(y_{n'})\big)|\Omega\rangle$$

$$\underset{k^0_{j'}\to E_{\vec{k}_{j'}}}{\overset{p^0_j\to E_{\vec{p}_j}}{\sim}}\quad \Big(\prod_{j=1}^{n}\frac{i\sqrt{Z}}{p_j^2-m_{ph}^2+i\epsilon}\Big)\Big(\prod_{j'=1}^{n'}\frac{i\sqrt{Z}}{k_{j'}^2-m_{ph}^2+i\epsilon}\Big)\overbrace{{}_{out}\langle\vec{p}_1\cdots\vec{p}_n|\vec{k}_1\cdots\vec{k}_{n'}\rangle_{in}}^{\langle\vec{p}_1\cdots\vec{p}_n|\hat{S}|\vec{k}_1\cdots\vec{k}_{n'}\rangle}\ , \qquad (5)$$

where the use of $e^{-ik_{j'}\cdot y_{j'}}$ ensures that the particles in the "in" state have positive energy.

The S-matrix element involving n' particles in the "in" state and n particles in the "out" state can be obtained from the corresponding $(n+n')$-point Green's function by extracting the leading singularities in the energies $k^0_{j'}$ and p^0_j, which coincide with the situations where the external particles become on-shell.

9d *The pole structure of the Green's functions emerging at asymptotic times contains all relevant information about the scattering amplitudes of the theory! To select the required information one should project on the right singularities by using appropriate plane waves.*

Wave packets instead of plane waves:

- In the asymptotic treatment of multiparticle states it is better to use normalized wave packets. In that case x is constrained to lie within a small band about the trajectory of a particle with momentum $\vec{p}$, with the spatial extent of the band being determined by the wave packet. In this way the particles do not interfere and can effectively be considered free at asymptotic times, unlike plane-wave states. Therefore we formally should have made the replacement

$$\int \mathrm{d}^4x\; e^{ip\cdot x} \quad\to\quad \int\frac{\mathrm{d}\vec{k}}{(2\pi)^3}\int \mathrm{d}^4x\; e^{ip^0x^0}\varphi(\vec{k}\,)e^{-i\vec{k}\cdot\vec{x}}\ ,$$

  with $\varphi(\vec{k}\,)$ a function that is peaked around $\vec{p}$, and we should have taken the limit of a sharply peaked wave packet $\varphi(\vec{k}\,)\to(2\pi)^3\delta(\vec{k}-\vec{p})$ at the end of the calculation.

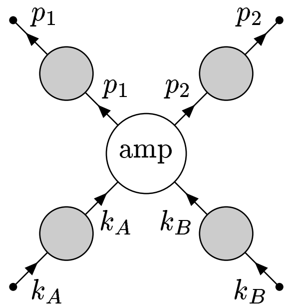
- A 1-particle wave packet spreads out differently than a multiparticle wave packet, so the overlap between them goes to zero as the elapsed time goes to infinity. Although $\hat{\phi}(x)$ creates some multiparticle states, we can "select" the 1-particle state that we want by using an appropriate wave packet. By waiting long enough we can make the multiparticle contribution to the matrix element as small as we like (cf. Fermi's Golden Rule for time-dependent perturbation theory).

- An n-particle asymptotic state is created/annihilated by n field operators that are constrained to lie in distant wave packets and therefore are effectively localized. Under these conditions an n-particle excitation in the continuum can be represented by n distinct (independent) 1-particle excitations of the ground state.

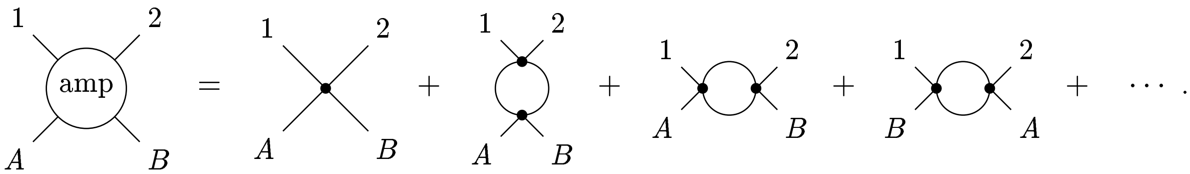
Translated in terms of Feynman diagrams: in order to investigate the implications of the LSZ reduction formula we consider the 4-point Green's function

$$\int d^4x_1\, e^{ip_1\cdot x_1} \int d^4x_2\, e^{ip_2\cdot x_2} \int d^4y_1\, e^{-ik_A\cdot y_1} \int d^4y_2\, e^{-ik_B\cdot y_2} \langle\Omega|T\big(\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(y_1)\hat{\phi}(y_2)\big)|\Omega\rangle$$

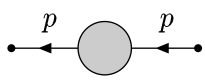
in the scalar ϕ^4-theory. From this we want to derive the T-matrix element for the scattering process $\phi(k_A)\phi(k_B) \to \phi(p_1)\phi(p_2)$. To this end we need to consider the contributions from fully connected diagrams, as was explained in § 2.6. These diagrams can be represented generically by



The blob in the centre of the diagram represents the sum of all amputated 4-point diagrams:



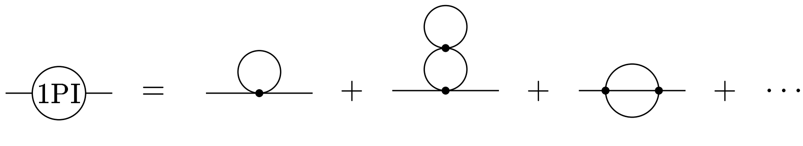
The shaded circles indicate that the corresponding full propagators



$$= \frac{i}{p^2 - m^2 - \Sigma(p^2)}$$

should be used, where

$$-\,i\,\Sigma(p^2) =$$



represents the 1-particle irreducible scalar self-energy diagrams in ϕ^4-theory. Near the physical particle pole $p^2 = m^2_{ph}$ the full propagator can be expanded according to

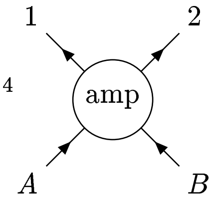
$$p^2 - m^2 - \Sigma(p^2) \;\approx\; (p^2 - m^2_{ph})\big[1 - \Sigma'(m^2_{ph})\big] + \mathcal{O}\big([p^2 - m^2_{ph}]^2\big) \;\equiv\; \frac{p^2 - m^2_{ph}}{Z} + \mathcal{O}\big([p^2 - m^2_{ph}]^2\big)\,.$$

As a result, the sum of all fully connected diagrams contains a product of four poles:

$$\frac{iZ}{p_1^2 - m_{ph}^2}\,\frac{iZ}{p_2^2 - m_{ph}^2}\,\frac{iZ}{k_A^2 - m_{ph}^2}\,\frac{iZ}{k_B^2 - m_{ph}^2}\ ,$$

multiplying the amputated 4-point diagrams. According to the LSZ reduction formula (5) the T-matrix element for the scattering process $\phi(k_A)\phi(k_B) \to \phi(p_1)\phi(p_2)$ thus reads

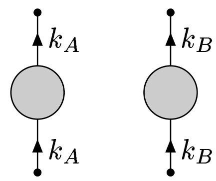
$$\langle \vec{p}_1\vec{p}_2|i\hat{T}|\vec{k}_A\vec{k}_B\rangle \;=\; (\sqrt{Z})^4$$



,

with all external momenta being on-shell.

Any 4-point diagram that is not fully connected, like the one displayed in the figure on the right, does not contain a product of four poles. Such diagrams are therefore projected out in the transition from the Green's function to the T-matrix.



9d *This completes the derivation of the connection between scattering matrix elements and fully connected amputated Feynman diagrams that was given on page 50 of these lecture notes. In fact we have also obtained the missing ingredient in the Feynman rules for the scalar ϕ^4-theory on page 50.*

Multiply the sum of all possible fully connected amputated Feynman diagrams in position/momentum space by a factor $(\sqrt{Z})^{n+n'}$ for $n+n'$ external particles.

2.9.4 The optical theorem (§ 7.3 in the book)

From the unitarity of the S-operator it follows that

$$\hat{S}^\dagger\hat{S} \;=\; \hat{1} \;\xrightarrow{\hat{S}=\hat{1}+i\hat{T}}\; (\hat{1}-i\hat{T}^\dagger)(\hat{1}+i\hat{T}) \;=\; \hat{1}+i(\hat{T}-\hat{T}^\dagger)+\hat{T}^\dagger\hat{T} \;=\; \hat{1} \;\Rightarrow\; -i(\hat{T}-\hat{T}^\dagger) \;=\; \hat{T}^\dagger\hat{T}\,.$$

In order to investigate the implications of this equation we consider the scattering process $\phi(k_A)\phi(k_B) \to \phi(p_1)\phi(p_2)$ in the scalar ϕ^4-theory:

$$-\,i\,\langle \vec{p}_1\vec{p}_2|\hat{T}|\vec{k}_A\vec{k}_B\rangle \,+\, i\,\langle \vec{p}_1\vec{p}_2|\hat{T}^\dagger|\vec{k}_A\vec{k}_B\rangle \;=\; \langle \vec{p}_1\vec{p}_2|\hat{T}^\dagger\hat{T}|\vec{k}_A\vec{k}_B\rangle$$

$$=\; \sum_n \frac{1}{n!}\left(\prod_{j=1}^{n}\int\frac{\mathrm{d}\vec{q}_j}{(2\pi)^3}\,\frac{1}{2E_j}\right)\langle \vec{p}_1\vec{p}_2|\hat{T}^\dagger|\{\vec{q}_j\}\rangle\langle\{\vec{q}_j\}|\hat{T}|\vec{k}_A\vec{k}_B\rangle\ ,$$

where in the last step a complete set of intermediate plane-wave states has been inserted. In terms of matrix elements this becomes:

$$-\,i\mathcal{M}(k_A\,,k_B \to p_1\,,p_2) + i\mathcal{M}^*(p_1\,,p_2 \to k_A\,,k_B)$$
$$= \sum_n \frac{1}{n!}\int \mathrm{d}\Pi_n\, \mathcal{M}^*\big(p_1\,,p_2 \to \{q_j\}\big)\mathcal{M}\big(k_A\,,k_B \to \{q_j\}\big)\ ,$$

containing the n-body phase-space element that was defined in equation (3). Using the abbreviations $a \equiv k_A\,,k_B\,,\ b \equiv p_1\,,p_2$ and $f \equiv \{q_j\}$ this results in the generalized optical theorem

$$-\,i\mathcal{M}(a \to b) + i\mathcal{M}^*(b \to a) \;=\; \sum_f C_f \int \mathrm{d}\Pi_f\, \mathcal{M}^*(b \to f)\mathcal{M}(a \to f)\ ,$$

where C_f stands for the combinatorial identical-particle factor belonging to the state f (i.e. the factors $1/n!$ in this ϕ^4 example). This generalized optical theorem is equally valid for initial/final states consisting of one particle or more than two particles. In more complicated theories the summation on the right-hand-side of the optical theorem runs over all possible sets of “final-state” particles that can be created by the initial state a.

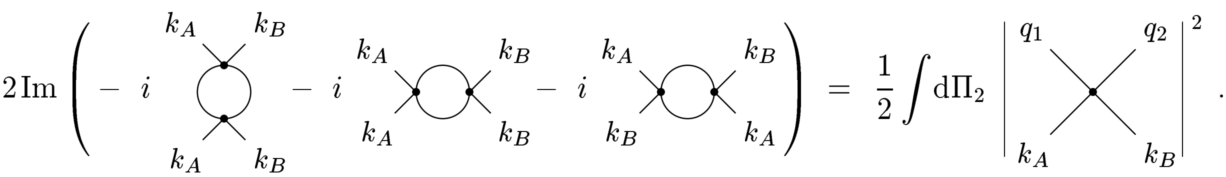
Specialized to forward scattering, i.e. $p_1 = k_A$ and $p_2 = k_B$ ($\Rightarrow\ a = b$), this yields the optical theorem in its standard form:

$$2\,\mathrm{Im}\,\mathcal{M}(a \to a) \;=\; \sum_f C_f \int \mathrm{d}\Pi_f\, |\mathcal{M}(a \to f)|^2 \;=\; \text{inverse flux factor} * \sigma_{\text{tot}}(a \to \text{anything})\ ,$$

where the inverse flux factor reads $4E_{\text{CM}}|\vec{k}\,|$ in the CM frame of the reaction.

> 9e *The optical theorem expresses the total cross section for scattering in terms of the attenuation (reduction) of the forward-going wave as the beams pass through each other. This is caused by the destructive interference between the scattered wave and the beam.*

Diagrammatic example for ϕ^4-theory at first non-trivial order: in Ex. 12 it is worked out that



The factors $-\,i$ on the left-hand-side are in fact cancelled by the factor i from Wick-rotating the loop integral. Note the absence of the lowest-order matrix element on the left-hand-side, because it has no imaginary part. This is nicely consistent with the right-hand-side, which contributes at $\mathcal{O}(\lambda^2)$ rather than at $\mathcal{O}(\lambda)$.

Sources of imaginary parts: the imaginary parts that feature in the optical theorem originate from the $i\epsilon$ parts of the propagators. For instance

$$\frac{1}{p^2-m^2\pm i\epsilon} = \frac{p^2-m^2}{(p^2-m^2)^2+\epsilon^2} \mp \frac{i\epsilon}{(p^2-m^2)^2+\epsilon^2} = \mathcal{P}\Big(\frac{1}{p^2-m^2}\Big) \mp i\pi\,\delta(p^2-m^2)\ ,$$

where $\mathcal{P}$ stands for the principal value. When going from $p^2-i\epsilon$ to $p^2+i\epsilon$ there is a $-2\pi i\,\delta(p^2-m^2)$ jump (discontinuity) in the propagator.

> 9e *Non-vanishing imaginary parts correspond to those situations where intermediate particles inside the loop(s) become on-shell. The associated lines of the diagram are in that case referred to as being "cut". The imaginary parts are the result of branch-cut discontinuities, marking invariant-mass values for which certain multiparticle intermediate states become physically possible.*

The Cutkosky cutting rules (without proof): the discontinuities of an arbitrary Feynman diagram can be obtained by means of a general method that is based on the discontinuities of the individual propagators. It involves the following three-step procedure (usually referred to as the Cutkosky cutting rules):

- cut the diagram in all possible ways, with all cut propagators becoming on-shell simultaneously;
- replace $1/(p^2-m^2+i\epsilon)$ by $-2\pi i\,\delta(p^2-m^2)$ in each cut propagator and perform the loop integrals;
- sum the contributions of all (kinematically) possible cuts.

2.10 The concept of renormalization (chapter 10 in the book)

> *Before we close this chapter on interacting scalar field theories, there is one final issue to be addressed.*

As we have already observed in the previous discussion, there still is the issue of UV divergences from the loop integrals $\int_0^\infty \mathrm{d}\ell_E^2\, \ell_E^2/(\ell_E^2+\Delta-i\epsilon)^j$ for $j\leq 2$.

> 10 *Question: how should we deal with UV divergences that occur at loop level in the perturbative expansion of interacting quantum field theories, bearing in mind that physical observables should be finite?*

The occurrence of singularities should not come as a surprise, though. Inside the loops particles of all energies are taken into account as being described by the same theory, i.e. we treat them as point-particles at all length scales, which is rather unrealistic.

Regularization: before we can continue the discussion we first have to quantify the UV divergence. This is called regularization.

10a *An obvious way to quantify UV divergences is by using a cutoff method:*

$$\int_0^{\infty} d\ell_E^2 \quad \xrightarrow{\text{to be replaced by}} \quad \int_0^{\Lambda^2} d\ell_E^2 \ ,$$

which removes all Fourier modes with momentum larger than Λ.

This means that the corresponding fields are not allowed to fluctuate too energetically. In this way we look at the physics through blurry glasses: we are interested in length scales $L \gtrsim 1/\Lambda$, but we do not care about length scales $L < 1/\Lambda$. This approach reflects that quantum field theory is in some sense an effective field theory with Λ marking the threshold of our ignorance beyond which quantum field theory ceases to be valid. As such, the cutoff Λ plays the role that $1/a$ played in the 1-dimensional quantum chain in Ex. 1, although Λ does not correspond to a specific energy/mass scale in the theory and should in fact be taken much larger than any such scale.

10e *We speak of a renormalizable quantum field theory if it keeps its predictive power in spite of its shortcomings at small length scales.*

Technically this means that we should be able to absorb all UV divergences of the theory into a finite number of parameters of the theory (like couplings and masses).

Example: consider the ϕ^4-process $\phi(k_A)\phi(k_B) \to \phi(p_1)\phi(p_2)$ at 1-loop order in the CM frame of the reaction. To make life easy we will neglect the mass of the particles in this study, which will not affect the outcome. Indicating the relevant invariant-mass scale of the process by s, the matrix element reads

$$\begin{aligned}\mathcal{M}_{\phi\phi\to\phi\phi}(s,\theta) \ &= \ -\lambda + \frac{\lambda^2}{32\pi^2}\left[\log\Big(\frac{\Lambda^2}{-s-i\epsilon}\Big) + \log\Big(\frac{\Lambda^2}{-t}\Big) + \log\Big(\frac{\Lambda^2}{-u}\Big) + 3\right] + \mathcal{O}(\lambda^3) \\ &\overset{\text{CM}}{=\!=} \ -\lambda + \frac{\lambda^2}{32\pi^2}\left[3\log\Big(\frac{\Lambda^2}{s}\Big) + \log\Big(\frac{4}{\sin^2\theta}\Big) + i\pi + 3\right] + \mathcal{O}(\lambda^3) \ .\end{aligned}$$

Details of the calculation are worked out in Ex. 12. As we will see later $Z = 1 + \mathcal{O}(\lambda^2)$, so there will be no 1-loop contribution from the wave-function renormalization factor $(\sqrt{Z}\,)^4$ in ϕ^4-theory.

From this result a few interesting observations follow.

1. The Lagrangian parameter (bare coupling) λ is not an observable quantity! The quantum corrections are an integral part of the effective coupling, which can be measured through $|\mathcal{M}_{\phi\phi\to\phi\phi}(s,\theta)|^2$.

10b *This effective coupling is energy-dependent due to the creation and annihilation of virtual particles (quantum fluctuations) at 1-loop order. So, the effective strength of the ϕ^4-interaction changes with energy!*

2. $\mathcal{M}_{\phi\phi\to\phi\phi}(s,\theta)$ depends logarithmically on the cutoff at $\mathcal{O}(\lambda^2)$. A short but sloppy way of saying this is that "$\mathcal{M}_{\phi\phi\to\phi\phi}(s,\theta)$ is logarithmically divergent".

3. $|\mathcal{M}_{\phi\phi\to\phi\phi}(s,\theta)|^2$ is observable and should therefore be independent of Λ. After all, Λ can be chosen arbitrarily and as such an observable cannot depend on it. To achieve this, the unobservable bare coupling λ should depend on the cutoff Λ:

$$0 \;=\; \frac{\mathrm{d}\mathcal{M}_{\phi\phi\to\phi\phi}(s,\theta)}{\mathrm{d}\Lambda^2} \;=\; \frac{\mathrm{d}\lambda}{\mathrm{d}\Lambda^2}\left(-1 + \frac{\lambda}{16\pi^2}\left[3\log\Big(\frac{\Lambda^2}{s}\Big) + \log\Big(\frac{4}{\sin^2\theta}\Big) + i\pi + 3\right]\right) + \frac{3\lambda^2}{32\pi^2}\,\frac{1}{\Lambda^2} + \mathcal{O}(\lambda^3)$$

$$\Rightarrow\; \frac{\mathrm{d}(1/\lambda)}{\mathrm{d}\log(\Lambda^2)} \;=\; -\,\frac{\Lambda^2}{\lambda^2}\,\frac{\mathrm{d}\lambda}{\mathrm{d}\Lambda^2} \;\approx\; -\,\frac{3}{32\pi^2} \;\Rightarrow\; \frac{1}{\lambda(\Lambda^2)} \;\approx\; \frac{1}{\lambda(\mu^2)} - \frac{3}{32\pi^2}\log\Big(\frac{\Lambda^2}{\mu^2}\Big)$$

$$\Rightarrow\; \lambda(\Lambda^2) \;\approx\; \frac{\lambda(\mu^2)}{1 - \frac{3\lambda(\mu^2)}{32\pi^2}\log(\Lambda^2/\mu^2)}\,.$$

10b *This is an example of a so-called Renormalization Group Equation (or short: RGE), which tells us that $\lambda(\Lambda^2)$ grows with Λ^2 if $\lambda(\mu^2) > 0$.*

The miracle of vanishing divergences: renormalization

Suppose we measure the above-given effective 4-point coupling at $s = \mu^2$ and $\theta = \pi/2$, and let's call this physical observable λ_{ph}:

$$\mathcal{M}_{\phi\phi\to\phi\phi}(s=\mu^2,\pi/2) \;\equiv\; -\,\lambda_{ph} \;=\; -\,\lambda + \frac{\lambda^2}{32\pi^2}\left[3\log\Big(\frac{\Lambda^2}{\mu^2}\Big) + \log(4) + i\pi + 3\right] + \mathcal{O}(\lambda^3)\,.$$

The bare coupling λ can then be expressed in terms of the physical coupling λ_{ph} and the divergence $\log(\Lambda^2/\mu^2)$ according to

$$-\,\lambda \;=\; -\,\lambda_{ph} - \frac{\lambda_{ph}^2}{32\pi^2}\left[3\log\Big(\frac{\Lambda^2}{\mu^2}\Big) + \log(4) + i\pi + 3\right] + \mathcal{O}(\lambda_{ph}^3)\,.$$

If we now want to know the effective 4-point coupling at an arbitrary scale s and scattering angle θ, then we can simply write

$$\mathcal{M}_{\phi\phi\to\phi\phi}(s,\theta) \;=\; -\,\lambda_{ph} + \frac{\lambda_{ph}^2}{32\pi^2}\left[3\log\Big(\frac{\mu^2}{s}\Big) - \log(\sin^2\theta)\right] + \mathcal{O}(\lambda_{ph}^3)\,,$$

where the $\log(\mu^2/s)$ term is completely governed by the above-given RGE for λ. This reflects that the observable effective 4-point coupling should not depend on the choice of reference scale μ.

> *The reference scale μ labels an entire equivalence class of parametrizations of the ϕ^4-theory and it should not matter which element of the class we choose for setting up the theory.*

When expressed in terms of the physical coupling λ_{ph}, the effective coupling $|\mathcal{M}_{\phi\phi\to\phi\phi}(s,\theta)|^2$ is independent of the cutoff Λ, as expected for a correct observable! The cutoff dependence has been absorbed into a redefinition of the unobservable Lagrangian parameter (bare coupling) λ in terms of the observable physical parameter (effective coupling) λ_{ph}. In the literature this physical observable is usually referred to as the renormalized coupling λ_R, although this terminology is a bit strange bearing in mind that the original coupling was not normalized to begin with. This is an example of the concept of renormalization.

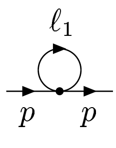
> (*10c*) *Renormalization: express physically measurable quantities in terms of physically measurable quantities and not in terms of bare Lagrangian parameters.*

- For setting up a perturbative expansion, the bare Lagrangian parameters are in fact not the right parameters. Instead the physically measurable parameters should be used (cf. the discussion about m and m_{ph} in § 2.9.2).
- The occurrence of infinities in the loop integrals is linked to this. Our initial perturbative expansion consisted of taking $\Lambda \to \infty$ while keeping λ and m finite. From the renormalization group viewpoint, however, the set $(\mu = \Lambda = \infty, \lambda < \infty, m < \infty)$ does not belong to the equivalence class of the ϕ^4-theory!
- The convergence of the perturbative series can be further improved by using physical quantities at the "right scale", thereby avoiding large logarithmic factors like $\log(\mu^2/s)$ in the example above. This choice of scale has no consequence for all-order calculations, but it does if the series is truncated at a certain perturbative order.

To complete the story for the scalar ϕ^4-theory we consider the UV divergences that are present in the scalar self-energy. This time the mass parameter is essential and therefore should not be neglected.

Scalar self-energy at $\mathcal{O}(\lambda)$:

$$-i\Sigma(p^2) \overset{\mathcal{O}(\lambda)}{=\!=}$$



$$= \frac{-i\lambda}{2}\int \frac{\mathrm{d}^4\ell_1}{(2\pi)^4}\,\frac{i}{\ell_1^2 - m^2 + i\epsilon}$$

$$\xrightarrow[\text{Wick rotation}]{\text{cutoff }\Lambda \gg m} \frac{-i\lambda}{32\pi^2}\int_0^{\Lambda^2} \mathrm{d}\ell_E^2\,\frac{\ell_E^2}{\ell_E^2 + m^2 - i\epsilon} = \frac{-i\lambda}{32\pi^2}\left[\Lambda^2 - m^2\log\left(\frac{\Lambda^2}{m^2}\right)\right].$$

After Dyson summation the full propagator becomes

$$\frac{i}{p^2 - m^2 - \Sigma(p^2) + i\epsilon} \;\equiv\; \frac{iZ}{p^2 - m_{ph}^2} + \text{regular terms} \; .$$

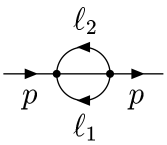
Since the 1-loop scalar self-energy does not depend on p^2, it is absorbed completely into the physical mass:

$$m_{ph}^2 \;=\; m^2 + \Sigma(m_{ph}^2) \;\overset{\mathcal{O}(\lambda)}{=\!=\!=}\; m^2 + \frac{\lambda}{32\pi^2}\left[\Lambda^2 - m^2 \log\Big(\frac{\Lambda^2}{m^2}\Big)\right] ,$$

whereas the residue of the pole remains 1.

> 10d *Note the strong Λ^2 dependence of the scalar mass, which implies that this mass is very sensitive to high-scale quantum corrections. This is in fact a general feature of scalar particles, like the Higgs boson: intrinsically the quantum corrections to the mass of a scalar particle are dominated by the highest mass scale the scalar particle couples to!*

Scalar self-energy at $\mathcal{O}(\lambda^2)$: the residue of the pole is affected at 2-loop level by the contribution



$$= \frac{(-i\lambda)^2}{6}\int\frac{\mathrm{d}^4\ell_1}{(2\pi)^4}\int\frac{\mathrm{d}^4\ell_2}{(2\pi)^4}\,\frac{i}{\ell_1^2 - m^2 + i\epsilon}\,\frac{i}{\ell_2^2 - m^2 + i\epsilon}\,\frac{i}{(\ell_1+\ell_2+p)^2 - m^2 + i\epsilon}$$

$$= a + bp^2 + cp^4 + \cdots \; .$$

To assess the UV behaviour of this diagram we perform naive power counting, which involves treating all loop momenta as being of the same order of magnitude. For $\ell_{1,2} \to \infty$ we obtain an integral of the order $\int \mathrm{d}^8\ell_E/\ell_E^6 \xrightarrow{\ell_E \,\leq\, \Lambda} \Lambda^{8-6} = \Lambda^2$.

- $a = \mathcal{O}(\Lambda^2)$ is obtained by setting $p = 0$;
- $b = \mathcal{O}(\log\Lambda)$ is obtained by taking $\frac{1}{2}\partial^2/\partial p_0^2$ and then setting $p = 0$. In naive power counting this logarithmically divergent term corresponds to integrals of order Λ^0.
- $c = \mathcal{O}(1)$ is obtained by taking $\frac{1}{4!}\partial^4/\partial p_0^4$ and then setting $p = 0$.

Adding all self-energy contributions and focussing on the diverging terms

$$\frac{i}{p^2 - m^2 - \Sigma(p^2) + i\epsilon} \;\to\; \frac{i}{p^2 - m^2 - A - Bp^2} \;\equiv\; \frac{iZ}{p^2 - m_{ph}^2} + \text{regular terms} \; ,$$

$$Z = \frac{1}{1-B} = \mathcal{O}(\log\Lambda) \quad , \quad m_{ph}^2 = \frac{m^2+A}{1-B} \equiv Zm^2 + \delta m^2 \quad , \quad \delta m^2 = \frac{A}{1-B} = \mathcal{O}(\Lambda^2) \; .$$

This leads to an $\mathcal{O}(\Lambda^2)$ shift in the mass and an $\mathcal{O}(\log\Lambda)$ contribution to the wave-function renormalization, which can be absorbed in the field ϕ itself.

So, UV divergent loop corrections in ϕ^4-theory are present in $\Sigma(p^2)$ and $\mathcal{M}_{\phi\phi\to\phi\phi}(s,\theta)$, with

$$\Sigma(m_{ph}^2) \;=\; m_{ph}^2 - m^2 \;=\; (Z-1)m^2 + \delta m^2 \;\equiv\; m^2\delta_Z + \delta m^2 \quad , \quad \Sigma'(m_{ph}^2) \;=\; 1 - 1/Z$$

$$\text{and} \qquad \mathcal{M}_{\phi\phi\to\phi\phi}(s=\mu^2,\pi/2) \;=\; -\lambda_{ph} \;\equiv\; -Z^2\lambda - \delta_\lambda \; .$$

The occurrence of the factor Z^2 in the last expression originates from the multiplicative factor $(\sqrt{Z})^4$ that should be added according to the Feynman rules.

2.10.1 Physical perturbation theory (a.k.a. renormalized perturbation theory)

> 10c *The lowest-order ϕ^4-theory should have been written in terms of the experimentally measurable physical parameters m_{ph} and λ_{ph}, and perturbation theory should have been defined with respect to this lowest-order theory.*

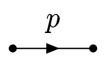
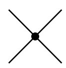
This is done as follows: take the original Lagrangian and write

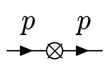
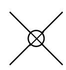
$$\phi \;=\; \phi_R\sqrt{Z} \quad , \quad m^2 Z \;=\; m_{ph}^2 - \delta m^2 \quad , \quad \lambda Z^2 \;=\; \lambda_{ph} - \delta_\lambda \quad \text{and} \quad Z \;\equiv\; 1 + \delta_Z$$

so that

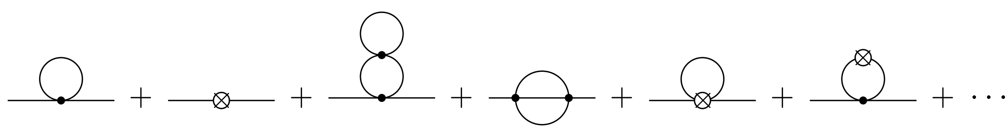
$$\mathcal{L} \;=\; \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \;=\; \frac{1}{2}(\partial_\mu\phi_R)(\partial^\mu\phi_R) - \frac{1}{2}m_{ph}^2\phi_R^2 - \frac{\lambda_{ph}}{4!}\phi_R^4$$
$$+\;\frac{1}{2}\delta_Z(\partial_\mu\phi_R)(\partial^\mu\phi_R) + \frac{1}{2}\delta m^2\phi_R^2 + \frac{\delta_\lambda}{4!}\phi_R^4 \; .$$

We get back the original Lagrangian in terms of renormalized objects (first line) and we obtain extra interactions that are called counterterms (second line), since their purpose is to cancel the divergences in the theory. The Feynman rules for the propagators and vertices including counterterms are now given by

 $= \dfrac{i}{p^2 - m_{ph}^2 + i\epsilon}$,  $= -i\lambda_{ph}$,

 $= i(p^2\delta_Z + \delta m^2)$,  $= i\delta_\lambda$.

Renormalization conditions: as an explicit example, the full propagator now reads $i/\big[p^2 - m_{ph}^2 - \Sigma_R(p^2)\big]$, with the renormalized self-energy given by

$-i\Sigma_R(p^2) \;=$ 

The parameters δ_Z and δm^2 can be fixed by imposing the renormalization conditions

$$\Sigma_R(m_{ph}^2) = 0 \quad \text{and} \quad \Sigma_R'(m_{ph}^2) = 0 \quad \Rightarrow \quad \text{full propagator} = \frac{i}{p^2 - m_{ph}^2} + \text{regular terms}\,.$$

The pole structure of the full propagator then resembles that of a free particle, so in that sense the physical 1-particle states have been re-normalized by this procedure. Adding one more renormalization condition based on $\mathcal{M}_{\phi\phi\to\phi\phi}$ in order to fix δ_λ, we have three conditions fixing three counterterm parameters. This will in fact be sufficient to make all observables of the ϕ^4-theory finite.

> *10e* *The scalar ϕ^4-theory is called renormalizable: "the infinities of the theory can be absorbed into a finite number of parameters".*

2.10.2 What has happened?

The above procedure seems odd: we calculated something that turned out to be infinite, then subtracted infinity from our original mass and coupling in an arbitrary way and ended up with something finite. Moreover, we have added divergent terms to our Lagrangian and we have suddenly ended up with a scale-dependent coupling. Why would a procedure consisting of such ill-defined mathematical tricks be legitimate? To see what has really happened, let us closely examine the starting point of our calculation.

In general, we start with a Lagrangian containing all possible terms that are compatible with basic assumptions such as relativity, causality, locality, etc. It still contains a few parameters such as m and λ in the case of ϕ^4-theory. It is tempting to call them "mass" and "coupling", as they turn out to be just that in the classical (i.e. lowest-order) theory. However, up to this point they are just free parameters. In order to make the theory predictive, the parameters need to be fixed by a set of measurements: we should calculate a set of cross sections at a given order in perturbation theory, measure their values and then fit the parameters so that they reproduce the experimental data. After this procedure, the theory is completely determined and becomes predictive.

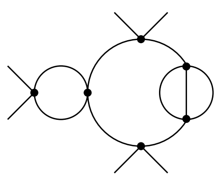
The bare parameters m and λ are only useful in intermediate calculations and will be replaced by physical (i.e. measured) quantities in the end anyway. So, we might as well parametrize the theory in terms of the latter. The renormalizability hypothesis is that this reparametrization of the theory is enough to turn the perturbation expansion into a well-defined expansion. The divergence problem then has nothing to do with the perturbation expansion itself: we have just chosen unsuitable parameters to perform it. Also, the fact that our physical coupling is scale-dependent should not surprise us. The physical reason for this "running" is the existence of quantum fluctuations, which were not there in the classical theory. These fluctuations correspond to intermediate particle states: at sufficiently high (i.e. relativistic) energies, new particles can be created and annihilated. As the available energy increases, more and more energetic particles can be created. This effectively changes the couplings.

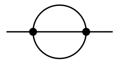
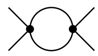
Having traded the bare parameters m and λ for renormalized parameters m_{ph} and λ_{ph}, let us take a closer look at the internal consistency of the renormalization procedure. We have introduced the physical coupling at a reference scale μ, but we could equally well have chosen an energy scale μ' with corresponding effective coupling λ'_{ph}. Physical processes should not depend on our choice of reference scale, hence the couplings should be related in such a way that for any observable O we have $O = O(m_{ph}, \mu, \lambda_{ph}) = O(m_{ph}, \mu', \lambda'_{ph})$. In other words, there should exist an equivalence class of parametrizations of the theory and it should not matter which element of the class we choose. This observation clarifies where the divergences came from: our initial perturbation expansion consisted of taking $\Lambda \to \infty$ while keeping m and λ finite. From the viewpoint of the renormalization group, however, the set $(\mu = \Lambda = \infty\,, m < \infty\,, \lambda < \infty)$ does not belong to any equivalence class of the ϕ^4-theory.

2.10.3 Superficial degree of divergence and renormalizability

10e *The statement at the end of §2.10.1 was a bit premature. In fact we still have to prove that amplitudes with more than four external particles do not introduce a new type of infinity that cannot be absorbed into the 2- and 4-point terms in the Lagrangian.*

A 6-point diagram like



will contain singular building blocks like  and  that should become finite once we perform the afore-mentioned renormalization procedure. The question that remains is whether the overall 6-point diagram can give rise to a new type of infinity. To assess this we perform naive power counting, i.e. we treat all loop momenta as being of the same order of magnitude. The outcome of this power counting is called the superficial degree of divergence D of the diagram, with $D = 0$ denoting logarithmic divergence.

Consider a 1PI amputated diagram with N external lines, P propagators and V vertices.

- In ϕ^4-theory four lines enter each vertex, each propagator counts twice towards the total number of lines entering vertices and each external line counts once. This results in the condition

$$4V = N + 2P \quad \Rightarrow \quad P = 2V - N/2 \qquad \text{and} \qquad N = \text{ even number}\,.$$

- The number of loop momenta is given by the number of propagators $-$ the number of four-momentum δ-functions $+\,1$, since one of the δ-functions corresponds to the external momenta and will not fix an internal loop momentum. This results in

$$L \;=\; P - V + 1 \;=\; V - N/2 + 1$$

independent loop momenta. So, loop diagrams require $V \geq N/2$.

Power counting: assume for argument's sake that the loop momenta are d-dimensional. That means that in the context of naive power counting each loop momentum contributes Λ^d and each propagator Λ^{-2}. The superficial degree of divergence of the diagram then reads

$$D \;=\; dL - 2P \;=\; d(V - N/2 + 1) - 2(2V - N/2) \;=\; d + V(d-4) + N(1 - d/2)\ ,$$

whereas the coupling λ has mass dimension $[\lambda] = 4 - d$ in d dimensions.

Superficially the diagram diverges like Λ^D *if* $D > 0$ *and like* $\log(\Lambda)$ *if* $D = 0$, *provided it contains a loop. The diagram does not diverge superficially if* $D < 0$.

Let's now consider a few values for the dimensionality d of spacetime.

$\boldsymbol{d = 4}$: $D = 4 - N$ is independent of V and $[\lambda] = 0 \;\Rightarrow\;$ the theory is renormalizable. Divergences occur at all orders, but only a finite number of amplitudes diverges superficially (i.e. amplitudes with $N = 2$ or 4)! The theory keeps its predictive power in spite of the infinities that occur if we assume it to be valid at all energies.

$\boldsymbol{d = 3}$: $D = 3 - N/2 - V$ and $[\lambda] = 1 \;\Rightarrow\;$ the theory is superrenormalizable. At most a finite number of diagrams diverges superficially (i.e. the diagrams with $N = 2$ and $V = 1$ or $V = 2$), as the diagrams get less divergent if the loop order is increased!

$\boldsymbol{d = 5}$: $D = 5 - 3N/2 + V$ and $[\lambda] = -1 \;\Rightarrow\;$ the theory is nonrenormalizable. Now all amplitudes will diverge superficially at a sufficiently high loop order! An infinite amount of counterterms would be required to remove all divergences, which means that all predictive power is lost if we assume the theory to be valid at all energies!

10e If we express the superficial degree of divergence in terms of V *and* N, *the coefficient in front of* V *determines whether the theory is superrenormalizable (negative coefficient), renormalizable (zero coefficient) or nonrenormalizable (positive coefficient)!*

In conclusion: for $d > 4$ the scalar ϕ^4-theory is nonrenormalizable and $[\lambda] < 0$, for $d = 4$ it is renormalizable and $[\lambda] = 0$, and for $d < 4$ it is superrenormalizable and $[\lambda] > 0$. These conclusions agree nicely with the general discussion on page 28 of these lecture notes.

3 The Dirac field

During the next three and a half lectures Chapter 3 of Peskin & Schroeder will be covered. We have seen various aspects of scalar theories, describing spin-0 particles. However, most particles in nature have spin $\neq 0$.

> 11a *Question: how should we find Lorentz-invariant equations of motion for fields that do not transform as scalars?*

Consider to this end an n-component multiplet field $\Phi_a(x)$ with $a = 1, \cdots, n$, which has the following linear transformation characteristic under Lorentz transformations:

$$\Phi_a(x) \xrightarrow{\text{Lorentz transf.}} M_{ab}(\Lambda)\,\Phi_b(\Lambda^{-1}x)$$

with summation over the repeated index implied. A compact way of writing this is

$$\Phi(x) \xrightarrow{\text{Lorentz transf.}} M(\Lambda)\Phi(\Lambda^{-1}x) \ .$$

In the case of scalar fields the transformation matrix $M(\Lambda)$ was simply the identity matrix. In order to find different solutions, we make use of the fact that the Lorentz transformations form a group: $\Lambda^{\mu}{}_{\nu} = g^{\mu}{}_{\nu}$ is the unit element, $\Lambda^{-1} = \Lambda^T$ is the inverse, and for Λ_1 and Λ_2 being Lorentz transformations also $\Lambda_3 = \Lambda_2\Lambda_1$ is a Lorentz transformation. The transformation matrices $M(\Lambda)$ should reflect this group structure:

$$M(g) \ = \ I_n \quad , \quad M(\Lambda^{-1}) \ = \ M^{-1}(\Lambda) \qquad \text{and} \qquad M(\Lambda_2\Lambda_1) \ = \ M(\Lambda_2)M(\Lambda_1) \ ,$$

where I_n is the $n \times n$ identity matrix. To phrase it differently, <u>the transformation matrices $M(\Lambda)$ should form an n-dimensional representation of the Lorentz group!</u>

The continuous Lorentz group (rotations and boosts): transformations that lie infinitesimally close to the identity transformation define a vector space, called the <u>Lie algebra</u> of the group. The <u>basis vectors</u> for this vector space are called the <u>generators</u> of the Lie algebra. The Lorentz group has six generators $J^{\mu\nu} = -J^{\nu\mu}$, three for boosts and three for rotations. These generators are antisymmetric, as a result of $\Lambda^{-1} = \Lambda^T$, and they satisfy the following set of <u>fundamental commutation relations</u>:

$$\left[J^{\mu\nu}, J^{\rho\sigma}\right] \ = \ i\left(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}\right) \ .$$

The three generators belonging to the boosts and the three generators belonging to the rotations are given by

$$K^j \ \equiv \ J^{0j} \quad \text{respectively} \quad J^j \ \equiv \ \frac{1}{2}\epsilon^{jkl}J^{kl} \ \Rightarrow \ J^{jk} \ = \ \epsilon^{jkl}J^l \qquad (j,k,l = 1,\cdots,3) \ ,$$

with summation over the repeated spatial indices implied. The latter generators, which span the Lie algebra of the rotation group, satisfy the fundamental commutation relations

$$\left[J^j, J^k\right] \ = \ i\epsilon^{jkl}J^l \ .$$

(11a) *In fact it is proven in Ex. 15 that all finite-dimensional representations of the Lorentz group correspond to pairs of integers or half integers (j_+, j_-), where both j_+ and j_- correspond to a representation of the rotation group. The sum $j_+ + j_-$ should be interpreted as the spin of the representation, since it corresponds to the actual rotations contained in the Lorentz group.*

A finite Lorentz transformation is then in general given by $\exp(-i\omega_{\mu\nu}J^{\mu\nu}/2)$, where the antisymmetric tensor $\omega_{\mu\nu} \in \mathbb{R}$ represents the Lorentz transformation. For instance:

$$\omega_{12} \,=\, -\,\omega_{21} \,=\, \delta\theta \quad , \quad \text{rest} = 0 \quad \Rightarrow \quad \omega^{\mu}{}_{\nu} \,=\, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\,\delta\theta & 0 \\ 0 & \delta\theta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

for an infinitesimal rotation about the z-axis (see Ex. 14), and

$$\omega_{01} \,=\, -\,\omega_{10} \,=\, \delta v \quad , \quad \text{rest} = 0 \quad \Rightarrow \quad \omega^{\mu}{}_{\nu} \,=\, \begin{pmatrix} 0 & \delta v & 0 & 0 \\ \delta v & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

for an infinitesimal boost along the x-direction (see Ex. 14).

The task at hand is now to find the matrix representations of the generators of the Lorentz group.

Examples: in Ex. 14 it is proven that

- $(J^{\mu\nu})^{\alpha}{}_{\beta} = i(g^{\mu\alpha}g^{\nu}{}_{\beta} - g^{\mu}{}_{\beta}\,g^{\nu\alpha})$ are the six generators that describe Lorentz transformations of contravariant four-vectors:

  $$x^{\alpha} \xrightarrow{\text{Lorentz transf.}} \Lambda^{\alpha}{}_{\beta}\, x^{\beta} \,=\, \big[\exp(-\,i\omega_{\mu\nu}J^{\mu\nu}/2)\big]^{\alpha}{}_{\beta}\, x^{\beta} \,\approx\, \big[g^{\alpha}{}_{\beta} \,-\, \frac{i}{2}\,\omega_{\mu\nu}(J^{\mu\nu})^{\alpha}{}_{\beta}\big]\, x^{\beta} \ .$$

  This implies that $g^{\alpha}{}_{\beta} - \frac{i}{2}\,\omega_{\mu\nu}(J^{\mu\nu})^{\alpha}{}_{\beta} = g^{\alpha}{}_{\beta} + \omega^{\alpha}{}_{\beta}$ represents the infinitesimal form of the Lorentz transformation matrix $\Lambda^{\alpha}{}_{\beta}$, as is indeed the case.

- $J^{\mu\nu} = i(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu})$ are the six generators in coordinate space, which describe the infinitesimal Lorentz transformations of scalar fields

  $$\phi(x) \xrightarrow{\text{Lorentz transf.}} \phi(\Lambda^{-1}x) \,\approx\, \phi(x) - \frac{1}{2}\,\omega_{\rho\sigma}\,[x^{\sigma}\partial^{\rho} - x^{\rho}\partial^{\sigma}]\,\phi(x) \ ,$$

  as derived on page 11.

(11b) **Dirac's trick:** *introduce four $n \times n$ matrices γ^μ that are referred to as the γ-matrices of Dirac, which satisfy the Dirac algebra (Clifford algebra)*

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} I_n \ ,$$

with I_n the $n \times n$ identity matrix. In Ex. 14 it is proven that this implies that the $n \times n$ matrices $S^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu, \gamma^\nu]$ form a representation of the generators $J^{\mu\nu}$ of the Lorentz group.[4]

Four-dimensional solution to the Dirac algebra: since there are no solutions for $n = 2$ or 3, the first solution can be found for $n = 4$. Written in 2×2 block form in terms of the 2×2 identity matrix I_2 and the Pauli spin matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ,$$

the solution reads

$$\gamma^0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \quad \text{and} \quad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} \qquad (j = 1, 2, 3)$$

in the Weyl representation, which is also known as the chiral representation. In fact there is an infinite number of such four-dimensional representations, since for any invertable 4×4 matrix V also $V\gamma^\mu V^{-1}$ is a solution. In the Weyl representation the generators of the Lorentz group have a block-diagonal form. The generators for boosts are given by

$$S^{0j} = \frac{i}{4}[\gamma^0, \gamma^j] = \frac{i}{2}\gamma^0\gamma^j = -\frac{i}{2}\begin{pmatrix} \sigma^j & 0 \\ 0 & -\sigma^j \end{pmatrix} \qquad (j = 1, 2, 3) ,$$

whereas the generators S^1, S^2 and S^3 for rotations follow from

$$S^{jk} \overset{j \neq k}{=\!=} \frac{i}{4}[\gamma^j, \gamma^k] = -\frac{i}{4}\begin{pmatrix} [\sigma^j, \sigma^k] & 0 \\ 0 & [\sigma^j, \sigma^k] \end{pmatrix} = \epsilon^{jkl}\begin{pmatrix} \frac{1}{2}\sigma^l & 0 \\ 0 & \frac{1}{2}\sigma^l \end{pmatrix} \equiv \epsilon^{jkl} S^l \quad (j, k = 1, 2, 3)$$

$$\Rightarrow \quad S^l = \begin{pmatrix} \frac{1}{2}\sigma^l & 0 \\ 0 & \frac{1}{2}\sigma^l \end{pmatrix} \equiv \frac{1}{2}\Sigma^l \qquad (l = 1, 2, 3) .$$

The generators for rotations look like twice replicated two-dimensional representations of the rotation group. We will come back to this point later on. As a result of the properties

$$(\gamma^0)^\dagger = \gamma^0 \quad , \quad (\gamma^j)^\dagger = -\gamma^j \qquad (j = 1, 2, 3) \quad \Rightarrow \quad (\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0 ,$$

the generators of the Lorentz group satisfy

$$(S^{\mu\nu})^\dagger = -\frac{i}{4}\big[(\gamma^\nu)^\dagger, (\gamma^\mu)^\dagger\big] = \frac{i}{4}\big[(\gamma^\mu)^\dagger, (\gamma^\nu)^\dagger\big] = \gamma^0 S^{\mu\nu} \gamma^0 .$$

[4] In fact this is true for any spacetime dimensionality

This means that the generators of rotations are hermitian, since $(S^{jk})^\dagger = S^{jk}$, indicating that rotations preserve normalization. On the other hand, the generators of boosts are non-hermitian, since $(S^{0j})^\dagger = -S^{0j}$, indicating that boosts do not preserve normalization owing to the Lorentz contraction of spatial volumes.

Dirac spinors and adjoint Dirac spinors: a four-component field $\psi(x)$ that Lorentz transforms according to this four-dimensional representation of the Lorentz group is called a Dirac spinor:

$$\psi(x) \xrightarrow{\text{Lorentz transf.}} \Lambda_{1/2}\,\psi(\Lambda^{-1}x) \qquad \text{with} \qquad \Lambda_{1/2} = \exp(-i\omega_{\mu\nu}S^{\mu\nu}/2)\ .$$

The adjoint Dirac spinor $\bar{\psi}(x)$ is defined as

$$\bar{\psi}(x) \equiv \psi^\dagger(x)\gamma^0$$

and therefore transforms as

$$\bar{\psi}(x) \xrightarrow{\text{Lorentz transf.}} \bar{\psi}(\Lambda^{-1}x)\Lambda_{1/2}^{-1}\ ,$$

since

$$\gamma^0\Lambda_{1/2}^\dagger\gamma^0 = \gamma^0\exp\bigl(i\omega_{\mu\nu}[S^{\mu\nu}]^\dagger/2\bigr)\gamma^0 \xlongequal{(S^{\mu\nu})^\dagger = \gamma^0 S^{\mu\nu}\gamma^0} \exp(i\omega_{\mu\nu}S^{\mu\nu}/2) = \Lambda_{1/2}^{-1}\ .$$

Using the important γ-matrix property

$$\begin{aligned}\bigl[\gamma^\mu, S^{\rho\sigma}\bigr] &= \frac{i}{2}\bigl[\gamma^\mu, \gamma^\rho\gamma^\sigma\bigr] = \frac{i}{2}\,(\gamma^\mu\gamma^\rho\gamma^\sigma - \gamma^\rho\gamma^\sigma\gamma^\mu) = i(g^{\mu\rho}\gamma^\sigma - g^{\mu\sigma}\gamma^\rho) \\ &= i(g^{\rho\mu}g^\sigma{}_\nu - g^\rho{}_\nu g^{\sigma\mu})\gamma^\nu = (J^{\rho\sigma})^\mu{}_\nu\gamma^\nu\ ,\end{aligned}$$

the following infinitesimal Lorentz-transformation identity holds up to $\mathcal{O}(\omega)$:

$$\bigl(I_4 + \frac{i}{2}\,\omega_{\rho\sigma}S^{\rho\sigma}\bigr)\gamma^\mu\bigl(I_4 - \frac{i}{2}\,\omega_{\alpha\beta}S^{\alpha\beta}\bigr) \approx \bigl[g^\mu{}_\nu - \frac{i}{2}\,\omega_{\rho\sigma}(J^{\rho\sigma})^\mu{}_\nu\bigr]\gamma^\nu\ .$$

This reflects that for finite transformations

$$\Lambda_{1/2}^{-1}\gamma^\mu\Lambda_{1/2} = \Lambda^\mu{}_\nu\gamma^\nu\ ,$$

which indicates that γ^μ transforms like a contravariant four-vector provided it is properly contracted with Dirac spinors and adjoint Dirac spinors.

11d *Consequently,* $\psi(x)$, $\gamma^\mu\partial_\mu\psi(x)$, $\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu\psi(x), \cdots$ *are good building blocks for constructing a Lorentz-invariant wave equation for Dirac spinors, whereas* $\bar{\psi}(x)\psi(x)$, $\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x), \cdots$ *are scalar building blocks for obtaining the corresponding Lagrangian.*

3.1 Towards the Dirac equation (§ 3.2 and 3.4 in the book)

11d **Dirac-field bilinears (currents)**: *the interesting objects in spinor space are of the form* $\bar{\psi}\Gamma\psi$, *with* Γ *a* 4×4 *matrix that consists of a sequence of* γ*-matrices. These objects are called bilinears or currents. They will be needed to construct Lagrangians that include interactions with other fields, like* $\bar{\psi}(x)\gamma^{\mu}\psi(x)A_{\mu}(x)$ *for interactions with a vector field and* $\bar{\psi}(x)\gamma^{\mu}\gamma^{\nu}\psi(x)h_{\mu\nu}(x)$ *for interactions with a tensor field. A basis for* Γ *that satisfies* $\Gamma^{\dagger} = \gamma^0\Gamma\gamma^0$ *is given by the following combinations of* γ*-matrices:*

$$I_4\,,\,\gamma^{\mu},\,\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^{\mu},\gamma^{\nu}\right],\,\gamma^{\mu}\gamma^5,\,i\gamma^5\,,$$

where

$$\gamma^5 \;\equiv\; i\gamma^0\gamma^1\gamma^2\gamma^3 \;=\; -\frac{i}{4!}\,\epsilon^{\mu\nu\rho\sigma}\gamma_{\mu}\gamma_{\nu}\gamma_{\rho}\gamma_{\sigma}$$

in terms of the totally antisymmetric tensor

$$\epsilon^{\mu\nu\rho\sigma} \;=\; \begin{cases} +1 & \text{if } (\mu\nu\rho\sigma) = \text{even permutation of } (0123) \\ -1 & \text{if } (\mu\nu\rho\sigma) = \text{odd permutation of } (0123) \\ 0 & \text{else} \end{cases} .$$

Properties of γ^5: the properties of the matrix γ^5 will prove important for the description of weak interactions. They read:

$$(\gamma^5)^{\dagger} \;=\; \gamma^5 \quad , \quad (\gamma^5)^2 \;=\; I_4 \qquad \text{and} \qquad \left\{\gamma^5,\gamma^{\mu}\right\} \;=\; 0 \qquad (\mu = 0,\cdots,3)$$

$$\Rightarrow \quad \left[\gamma^5, S^{\mu\nu}\right] \;=\; 0 \quad \Rightarrow \quad \left[\gamma^5, \Lambda_{1/2}\right] \;=\; 0\,.$$

This means that γ^5 is a "Lorentz scalar" if it is properly contracted with Dirac spinors and adjoint Dirac spinors. Since γ^5 commutes with the generators of Lorentz transformations in spinor space, eigenvectors of γ^5 corresponding to different eigenvalues transform independently (i.e. without mixing).

11c *According to Schur's lemma this implies that the Dirac representation of the Lorentz group is reducible, i.e. we should be able to write it in terms of two independent lower-dimensional chiral representations.*

In the Weyl representation of the γ-matrices, the matrix γ^5 has the following form in terms of 2×2 blocks:

$$\gamma^5 \;=\; \begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix}.$$

As a result,

$$P_R \equiv \frac{1}{2}(I_4+\gamma^5) = \begin{pmatrix} 0 & 0 \\ 0 & I_2 \end{pmatrix} \qquad \text{and} \qquad P_L \equiv \frac{1}{2}(I_4-\gamma^5) = \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix}$$

are (chiral) projection operators on 2-dimensional vectors ψ_R and ψ_L:

$$\psi \equiv \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \quad \rightarrow \quad P_R\,\psi = \begin{pmatrix} 0 \\ \psi_R \end{pmatrix} \qquad \text{and} \qquad P_L\,\psi = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix},$$

which are eigenvectors of γ^5 corresponding to the <u>chirality</u> eigenvalues $+1$ and -1. In terms of these <u>right-handed Weyl spinors ψ_R</u> and <u>left-handed Weyl spinors ψ_L</u> the infinitesimal Lorentz transformations of ψ can be rewritten as (cf. Ex. 15 and the generators that are given on page 90)

$$\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \xrightarrow{\text{Lorentz transf.}} \begin{pmatrix} [I_2 - i\vec{\theta}\cdot\vec{\sigma}/2 - \vec{\beta}\cdot\vec{\sigma}/2]\,\psi_L \\ [I_2 - i\vec{\theta}\cdot\vec{\sigma}/2 + \vec{\beta}\cdot\vec{\sigma}/2]\,\psi_R \end{pmatrix}.$$

The real infinitesimal parameters $\vec{\theta}$ and $\vec{\beta}$ coincide with the parameters $\delta\vec{\alpha}$ and $\delta\vec{v}$ that were used in Ex. 14. We see that the Weyl spinors transform independently, which indeed implies that the four-dimensional Dirac representation of the Lorentz group is reducible and can be split into two two-dimensional representations. For later use we mention the following identity for the Pauli spin matrices:

$$\sigma^2\vec{\sigma}^* = -\vec{\sigma}\sigma^2 \quad \Rightarrow \quad \sigma^2\psi_L^* \xrightarrow{\text{Lorentz transf.}} \sigma^2[\,I_2 + i\vec{\theta}\cdot\vec{\sigma}^*/2 - \vec{\beta}\cdot\vec{\sigma}^*/2\,]\,\psi_L^*$$
$$= [\,I_2 - i\vec{\theta}\cdot\vec{\sigma}/2 + \vec{\beta}\cdot\vec{\sigma}/2\,]\,\sigma^2\psi_L^*\ ,$$

which indicates that $\sigma^2\psi_L^*$ transforms like a right-handed Weyl spinor.

Chirality and currents: from the 4×4 matrix basis on the previous page all possible hermitian currents can be obtained as $\bar{\psi}\Gamma\psi$, since $(\bar{\psi}\Gamma\psi)^\dagger = \psi^\dagger\Gamma^\dagger\gamma^0\psi \overset{\Gamma^\dagger=\gamma^0\Gamma\gamma^0}{=\!=\!=} \bar{\psi}\Gamma\psi$. These currents and their associated continuous Lorentz transformations read:

$$\underline{\textit{scalar current}}:\ j_S(x) \equiv \bar{\psi}(x)\psi(x) \xrightarrow{\text{Lorentz transf.}} j_S(\Lambda^{-1}x)\ ,$$

$$\underline{\textit{vector current}}:\ j_V^\mu(x) \equiv \bar{\psi}(x)\gamma^\mu\psi(x) \xrightarrow{\text{Lorentz transf.}} \Lambda^\mu{}_\alpha\, j_V^\alpha(\Lambda^{-1}x)\ ,$$

$$\underline{\textit{tensor current}}:\ j_T^{\mu\nu}(x) \equiv \bar{\psi}(x)\sigma^{\mu\nu}\psi(x) \xrightarrow{\text{Lorentz transf.}} \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta\, j_T^{\alpha\beta}(\Lambda^{-1}x)\ ,$$

$$\underline{\textit{axial vector current}}:\ j_A^\mu(x) \equiv \bar{\psi}(x)\gamma^\mu\gamma^5\psi(x) \xrightarrow{\text{Lorentz transf.}} \Lambda^\mu{}_\alpha\, j_A^\alpha(\Lambda^{-1}x)\ ,$$

$$\underline{\textit{pseudo scalar current}}:\ j_P(x) \equiv i\bar{\psi}(x)\gamma^5\psi(x) \xrightarrow{\text{Lorentz transf.}} j_P(\Lambda^{-1}x)\ ,$$

making use of the fact that $\Lambda_{1/2}^{-1}\gamma^\mu\Lambda_{1/2} = \Lambda^\mu{}_\nu\gamma^\nu$ and $\Lambda_{1/2}^{-1}\gamma^5\Lambda_{1/2} = \gamma^5$.

Using the chiral projection operators $P_{L/R}$, the Dirac spinors can be decomposed into chiral components according to

$$P_{L/R}\,\psi(x) \;\equiv\; \psi_{L/R} \quad \Rightarrow \quad \bar{\psi}_{L/R} \equiv (\psi_{L/R})^\dagger\gamma^0 \;=\; \psi^\dagger P_{L/R}\gamma^0 \;=\; \psi^\dagger\gamma^0 P_{R/L} \;=\; \bar{\psi}P_{R/L}\ .$$

This results in the following chiral decompositions of the currents.

- The scalar current mixes left- and right-handed Weyl spinors, since

  $$\bar{\psi}\psi \;=\; \bar{\psi}(P_R+P_L)\psi \;=\; \bar{\psi}(P_R^2+P_L^2)\psi \;=\; \bar{\psi}_L\psi_R+\bar{\psi}_R\psi_L\ .$$

  This will prove important for the description of massive spin-1/2 particles.

- The vector current treats left- and right-handed Weyl spinors on equal footing, since

  $$\bar{\psi}\gamma^\mu\psi \;=\; \bar{\psi}\gamma^\mu(P_R^2+P_L^2)\psi \;=\; \bar{\psi}(P_L\gamma^\mu P_R+P_R\gamma^\mu P_L)\psi \;=\; \bar{\psi}_R\gamma^\mu\psi_R+\bar{\psi}_L\gamma^\mu\psi_L\ .$$

  This will prove important for vector-like theories, describing for instance the electromagnetic and strong interactions.

- Similarly the tensor current mixes left- and right-handed Weyl spinors:

  $$\bar{\psi}\sigma^{\mu\nu}\psi \;=\; \bar{\psi}_L\sigma^{\mu\nu}\psi_R+\bar{\psi}_R\sigma^{\mu\nu}\psi_L\ .$$

  This is needed for describing Lorentz transformations, as we have seen already.

- The axial vector current treats left- and right-handed Weyl spinors in opposite ways:

  $$\begin{aligned}\bar{\psi}\gamma^\mu\gamma^5\psi \;&=\; \bar{\psi}\gamma^\mu\gamma^5(P_R^2+P_L^2)\psi \;=\; \bar{\psi}(P_L\gamma^\mu\gamma^5P_R+P_R\gamma^\mu\gamma^5P_L)\psi \\ &=\; \bar{\psi}_R\gamma^\mu\gamma^5\psi_R+\bar{\psi}_L\gamma^\mu\gamma^5\psi_L \;\overset{\gamma^5\psi_{R/L}\,=\,\pm\,\psi_{R/L}}{=\!=\!=\!=\!=}\; \bar{\psi}_R\gamma^\mu\psi_R-\bar{\psi}_L\gamma^\mu\psi_L\ .\end{aligned}$$

  This will prove important for chiral theories, like the one that describes weak interactions.

- Similarly the pseudo scalar current decomposes according to

  $$i\bar{\psi}\gamma^5\psi \;=\; i\bar{\psi}_L\gamma^5\psi_R+i\bar{\psi}_R\gamma^5\psi_L \;=\; i\bar{\psi}_L\psi_R-i\bar{\psi}_R\psi_L\ .$$

  This will prove important in describing interactions between spin-0 and spin-1/2 particles.

Handy combinations of such currents are given by the left/right-handed vector currents

$$j^\mu_{L/R}(x) \;\equiv\; \bar{\psi}(x)\gamma^\mu P_{L/R}\psi(x) \;=\; \bar{\psi}_{L/R}(x)\gamma^\mu\psi_{L/R}(x)\ ,$$

which will feature in the Standard Model of electroweak interactions.

11e **Dirac equation**: *let's now try to construct a Lorentz-invariant wave equation that has the Klein-Gordon equation built in. The simplest candidate is the Dirac equation*

$$(i\gamma^\mu \partial_\mu - m)\psi(x) \;=\; 0\ .$$

This is a first order differential equation, whereas the Klein-Gordon equation was a second order equation. This is possible because γ^μ behaves like a vector without actually introducing a preferred direction, which is not possible in scalar theories!

Proof: first of all

$$\begin{aligned} 0 \;&=\; (i\gamma^\nu \partial_\nu + m)(i\gamma^\mu \partial_\mu - m)\psi(x) \;=\; -\,(\gamma^\nu\gamma^\mu \partial_\nu \partial_\mu + m^2)\,\psi(x) \\ &=\; -\,\big(\tfrac{1}{2}\{\gamma^\mu,\gamma^\nu\}\partial_\mu\partial_\nu + m^2\big)\,\psi(x) \;\xlongequal{\{\gamma^\mu,\gamma^\nu\}=2g^{\mu\nu}I_4}\; -\,(\Box + m^2)\,\psi(x)\ , \end{aligned}$$

so the Klein-Gordon equation is indeed built in! Secondly, under continuous Lorentz transformations a Dirac spinor transforms according to $\psi(x) \to \psi'(x) = \Lambda_{1/2}\psi(\Lambda^{-1}x)$. If $\psi(x)$ satisfies the Dirac equation then it follows that

$$\begin{aligned} \forall_x\ (i\gamma^\mu \partial_\mu - m)\psi(x) \;=\; 0 \;\Rightarrow\; (i\gamma^\mu \partial_\mu - m)\Lambda_{1/2}\psi(\Lambda^{-1}x) \;&=\; \Lambda_{1/2}(i\Lambda^\mu{}_\sigma\gamma^\sigma \partial_\mu - m)\psi(\Lambda^{-1}x) \\ &=\; \Lambda_{1/2}\big[i\Lambda^\mu{}_\sigma\gamma^\sigma(\Lambda^{-1})^\nu{}_\mu(\partial_\nu\psi)(\Lambda^{-1}x) - m\psi(\Lambda^{-1}x)\big] \\ &=\; \Lambda_{1/2}\big[i\gamma^\nu\partial_\nu\psi - m\psi\big](\Lambda^{-1}x) \;=\; 0 \end{aligned}$$

$$\Rightarrow \quad (i\gamma^\mu \partial_\mu - m)\psi'(x) \;=\; 0\ .$$

If the field $\psi(x)$ satisfies the Dirac equation then so does the Lorentz transformed field $\psi'(x)$, as required for having a Lorentz invariant wave equation.

In the Weyl representation the Dirac equation reads

$$0 = (i\gamma^\mu\partial_\mu - m)\psi \;=\; \begin{pmatrix} -\,mI_2 & i(I_2\partial_0 + \vec{\sigma}\cdot\vec{\nabla}) \\ i(I_2\partial_0 - \vec{\sigma}\cdot\vec{\nabla}) & -\,mI_2 \end{pmatrix}\begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix} \equiv \begin{pmatrix} -\,mI_2 & i\sigma^\mu\partial_\mu \\ i\bar{\sigma}^\mu\partial_\mu & -\,mI_2 \end{pmatrix}\begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix}$$

using the compact notation

$$\sigma^\mu \;\equiv\; (I_2, \vec{\sigma}\,) \qquad \text{and} \qquad \bar{\sigma}^\mu \;\equiv\; (I_2, -\,\vec{\sigma}\,) \qquad \Rightarrow \qquad \gamma^\mu \;=\; \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}.$$

From this we conclude that

11e *the two representations associated with ψ_L and ψ_R are mixed by the mass term in the Dirac equation! In the massless case the Dirac equation splits into two independent wave equations for ψ_L and ψ_R, the so-called Weyl equations*

$$i\bar{\sigma}^\mu\partial_\mu\psi_L(x) \;=\; 0 \qquad \text{and} \qquad i\sigma^\mu\partial_\mu\psi_R(x) \;=\; 0\ .$$

The Dirac Lagrangian: the Lagrangian that corresponds to the Dirac equation reads

$$\mathcal{L}_{\rm Dirac}(x) \;=\; \bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x) \ .$$

Proof: the Euler-Lagrange equations for the $\bar{\psi}$ and ψ fields are given by

$$\partial_\mu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}\Big) \;-\; \frac{\partial\mathcal{L}}{\partial\bar{\psi}} \;=\; -\,(i\gamma^\mu\partial_\mu - m)\,\psi \;=\; 0 \ ,$$

$$\partial_\mu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\Big) \;-\; \frac{\partial\mathcal{L}}{\partial\psi} \;=\; \partial_\mu\big(i\bar{\psi}\gamma^\mu\big) + m\bar{\psi} \;=\; \bar{\psi}(i\overleftarrow{\partial}_\mu\gamma^\mu + m) \;=\; 0 \ ,$$

which are indeed the Dirac equation and the corresponding adjoint equation

$$0 \;=\; \big[(i\gamma^\mu\partial_\mu - m)\,\psi(x)\big]^\dagger\gamma^0 \;=\; -\,i\,\big(\partial_\mu\psi^\dagger(x)\big)\,\gamma^{\mu\,\dagger}\gamma^0 - m\psi^\dagger(x)\gamma^0 \;=\; -\,\bar{\psi}(x)(i\overleftarrow{\partial}_\mu\gamma^\mu + m) \ .$$

11e **Conserved currents**: *in preparation for the quantization of the free Dirac theory and the derivation of its particle interpretation, we have a closer look at the conserved currents for the solutions $\psi(x)$ of the Dirac equation.*

1. **The vector current $\boldsymbol{j_V^\mu(x)}$ is conserved.**

   Proof 1: $\partial_\mu j_V^\mu = (\partial_\mu\bar{\psi})\gamma^\mu\psi + \bar{\psi}\gamma^\mu\partial_\mu\psi \;\overset{\text{Dirac eqns.}}{=\!=\!=}\; im\bar{\psi}\psi - im\bar{\psi}\psi \;=\; 0$.

   Proof 2: in Ex. 17 an alternative proof is given based on global U(1) invariance.

2. **The axial vector current $\boldsymbol{j_A^\mu(x)}$ is conserved if $\mathbf{m = 0}$.**

   Proof 1: $\partial_\mu j_A^\mu = (\partial_\mu\bar{\psi})\gamma^\mu\gamma^5\psi - \bar{\psi}\gamma^5\gamma^\mu\partial_\mu\psi \;\overset{\text{Dirac eqns.}}{=\!=\!=}\; 2im\bar{\psi}\gamma^5\psi \;=\; 0$ if $m = 0$.

   Proof 2: in Ex. 17 an alternative proof is given based on global chiral invariance.

3. **The energy-momentum tensor $\boldsymbol{T^{\mu\nu}}$ is conserved.**

   Only the spacetime coordinates of $\bar{\psi}(x)$ and $\psi(x)$ transform under translations, i.e. the spinors themselves do not transform. Consequently, the energy-momentum tensor $T^{\mu\nu}$ derived on page 8 will be conserved. This gives rise to four conserved charges, the field energy

$$H \;=\; \int\!{\rm d}\vec{x}\;\mathcal{H} \;=\; \int\!{\rm d}\vec{x}\;\Big[\pi_\psi\dot{\psi} + \dot{\bar{\psi}}\pi_{\bar{\psi}} - \mathcal{L}_{\rm Dirac}\Big] \;=\; \int\!{\rm d}\vec{x}\;\pi_\psi\dot{\psi}$$

   and field momentum

$$\vec{P} \;=\; -\int\!{\rm d}\vec{x}\;\Big[\pi_\psi\vec{\nabla}\psi + (\vec{\nabla}\bar{\psi})\pi_{\bar{\psi}}\Big] \;=\; -\int\!{\rm d}\vec{x}\;\pi_\psi\vec{\nabla}\psi.$$

   Here we used that in these Noether charges $\psi(x)$ should satisfy the Dirac equation, and that $\pi_\psi \;=\; \dfrac{\partial\mathcal{L}_{\rm Dirac}}{\partial(\partial_0\psi)} \;=\; i\bar{\psi}\gamma^0 \;=\; i\psi^\dagger$ as well as $\pi_{\bar{\psi}} \;=\; \dfrac{\partial\mathcal{L}_{\rm Dirac}}{\partial(\partial_0\bar{\psi})} \;=\; 0$.

   From these conjugate momenta we can read off that out of the eight real degrees of freedom of the Dirac spinor $\psi(x)$ in fact four belong to the conjugate momentum.

4. Under continuous Lorentz transformations a Dirac spinor transforms as

$$\psi(x) \xrightarrow{\text{Lorentz transf.}} \Lambda_{1/2}\,\psi(\Lambda^{-1}x) \overset{\text{inf.}}{\approx} \big[I_4 - \frac{i}{2}\,\omega_{\rho\sigma}S^{\rho\sigma}\big]\psi(x) - \frac{1}{2}\omega_{\rho\sigma}\big[x^\sigma\partial^\rho - x^\rho\partial^\sigma\big]\psi(x)\,,$$

where the first term is typical for Dirac spinors and the second term is the same as for scalar fields. Bearing in mind that the Dirac Lagrangian is a Lorentz scalar, we can generalize the derivation on page 11 to arrive at the following six conserved Noether currents:

$$\begin{aligned} J^{\mu\rho\sigma}(x) &= \frac{\partial\mathcal{L}_{\text{Dirac}}}{\partial(\partial_\mu\psi)}\big[x^\rho\partial^\sigma - x^\sigma\partial^\rho - i\,S^{\rho\sigma}\big]\psi(x) \;+\; \big[g^{\mu\rho}x^\sigma - g^{\mu\sigma}x^\rho\big]\mathcal{L}_{\text{Dirac}}(x) \\ &= T^{\mu\sigma}x^\rho - T^{\mu\rho}x^\sigma + \bar{\psi}(x)\gamma^\mu S^{\rho\sigma}\psi(x)\,. \end{aligned}$$

11e The last term in these conserved Noether currents is specific for Dirac theories. After quantization of the Dirac theory this term will help us to determine the spin of the particles described by the (free) Dirac field theory.

3.2 Solutions of the free Dirac equation (§ 3.3 in the book)

11f Since solutions of the (free) Dirac equation automatically satisfy the Klein-Gordon equation, we can use the standard plane-wave (Fourier) decomposition in order to decouple the degrees of freedom as much as possible.

The positive-energy case: according to this decomposition we introduce

$$\psi_p(x) \;\equiv\; u(p)e^{-ip\cdot x} \quad \text{with} \quad p^2 = m^2 \;\text{ and }\; p^0 > 0 \quad \Rightarrow \quad p^\mu = \big(\sqrt{\vec{p}^{\,2}+m^2}\,,\vec{p}\,\big) \equiv (E_{\vec{p}}\,,\vec{p}\,)\,.$$

The spinor $u(p)$ then has to satisfy the Dirac equation in momentum space:

$$(\gamma^\mu p_\mu - m)\,u(p) \;\equiv\; (\not{p} - m)\,u(p) \;=\; 0\,,$$

using Feynman slash notation. The claim is now that $u(p)$ can be written as

$$u(p) \;=\; \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar{\sigma}}\,\xi \end{pmatrix},$$

with ξ an arbitrary normalized 2-dimensional vector.

Proof: using that

$$\sqrt{(p\cdot\sigma)(p\cdot\bar{\sigma})} \;=\; \sqrt{(p^0I_2 - \vec{p}\cdot\vec{\sigma})(p^0I_2 + \vec{p}\cdot\vec{\sigma})} \;\overset{\{\sigma^j,\sigma^k\}=2\delta_{jk}I_2}{=\!=\!=} \; I_2\sqrt{p_0^2 - \vec{p}^{\,2}} \;=\; m\,I_2\,,$$

it easily follows that

$$(\not{p} - m)\,u(p) \;=\; \begin{pmatrix} -\,m\,I_2 & p\cdot\sigma \\ p\cdot\bar{\sigma} & -\,m\,I_2 \end{pmatrix}\begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar{\sigma}}\,\xi \end{pmatrix} \;=\; 0\,.$$

The negative-energy case: similarly we introduce

$$\psi_p(x) \;\equiv\; v(p) e^{+ip\cdot x} \quad \text{with again} \quad p^\mu = (E_{\vec{p}}\,, \vec{p}\,)$$

to get two more independent solutions of the Dirac equation. The spinor $v(p)$ has to satisfy

$$-\,(\gamma^\mu p_\mu + m)\,v(p) \;\equiv\; -\,(\not{p} + m)\,v(p) \;=\; 0$$

and is given by

$$v(p) \;=\; \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta \\ -\sqrt{p\cdot\bar{\sigma}}\,\eta \end{pmatrix},$$

with η another arbitrary normalized 2-dimensional vector.

Helicity: for the normalized base vectors ξ^1, ξ^2 and η^1, η^2 we could for instance choose the eigenvectors of $\vec{\sigma}\cdot\vec{p}/|\vec{p}\,| \equiv \vec{\sigma}\cdot\vec{e}_p$ with eigenvalues $+1\,, -1$. This results in

$$u^1(p) \;=\; \begin{pmatrix} \sqrt{E_{\vec{p}} - |\vec{p}\,|}\;\xi^1 \\ \sqrt{E_{\vec{p}} + |\vec{p}\,|}\;\xi^1 \end{pmatrix} \xrightarrow{|\vec{p}\,|\gg m} \sqrt{2E_{\vec{p}}}\begin{pmatrix} 0 \\ \xi^1 \end{pmatrix},$$

$$u^2(p) \;=\; \begin{pmatrix} \sqrt{E_{\vec{p}} + |\vec{p}\,|}\;\xi^2 \\ \sqrt{E_{\vec{p}} - |\vec{p}\,|}\;\xi^2 \end{pmatrix} \xrightarrow{|\vec{p}\,|\gg m} \sqrt{2E_{\vec{p}}}\begin{pmatrix} \xi^2 \\ 0 \end{pmatrix}$$

and

$$v^1(p) \;=\; \begin{pmatrix} \sqrt{E_{\vec{p}} - |\vec{p}\,|}\;\eta^1 \\ -\sqrt{E_{\vec{p}} + |\vec{p}\,|}\;\eta^1 \end{pmatrix} \xrightarrow{|\vec{p}\,|\gg m} -\sqrt{2E_{\vec{p}}}\begin{pmatrix} 0 \\ \eta^1 \end{pmatrix},$$

$$v^2(p) \;=\; \begin{pmatrix} \sqrt{E_{\vec{p}} + |\vec{p}\,|}\;\eta^2 \\ -\sqrt{E_{\vec{p}} - |\vec{p}\,|}\;\eta^2 \end{pmatrix} \xrightarrow{|\vec{p}\,|\gg m} \sqrt{2E_{\vec{p}}}\begin{pmatrix} \eta^2 \\ 0 \end{pmatrix}.$$

11f *In the ultrarelativistic limit the chiral states coincide with the eigenstates of the helicity operator*

$$\hat{h} \;=\; \vec{e}_p\cdot\hat{\vec{S}} \;=\; \frac{1}{2}\begin{pmatrix} \vec{\sigma}\cdot\vec{e}_p & 0 \\ 0 & \vec{\sigma}\cdot\vec{e}_p \end{pmatrix}.$$

In that case positive helicity ($h = +\,1/2$) corresponds to right-handed chirality (ψ_R) and negative helicity ($h = -\,1/2$) to left-handed chirality (ψ_L).

Helicity is frame dependent if $m \neq 0$, since $\vec{e}_p$ can be flipped by a boost along that direction. Helicity is frame independent if $m = 0$. The Lorentz invariance of helicity for $m = 0$ is manifest in the notation of Weyl spinors, since $\psi_{L/R}$ live in different representations of the Lorentz group.

Normalization and orthogonality of the u and v spinors: from the orthogonality properties $\xi^{r\,\dagger}\xi^s = \delta_{rs}$ and $\eta^{r\,\dagger}\eta^s = \delta_{rs}$ of the normalized 2-dimensional base vectors ξ^1, ξ^2 and η^1, η^2, it follows that

$$u^{r\,\dagger}(p)u^s(p) \;=\; \big(\xi^{r\,\dagger}\sqrt{p\cdot\sigma}\,,\,\xi^{r\,\dagger}\sqrt{p\cdot\bar{\sigma}}\big)\begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi^s\\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s\end{pmatrix} \;=\; \xi^{r\,\dagger}(p\cdot\sigma + p\cdot\bar{\sigma})\xi^s = 2E_{\vec{p}}\,\delta_{rs}\ ,$$

$$v^{r\,\dagger}(p)v^s(p) \;=\; \big(\eta^{r\,\dagger}\sqrt{p\cdot\sigma}\,,\,-\,\eta^{r\,\dagger}\sqrt{p\cdot\bar{\sigma}}\big)\begin{pmatrix}\sqrt{p\cdot\sigma}\,\eta^s\\ -\sqrt{p\cdot\bar{\sigma}}\,\eta^s\end{pmatrix} \;=\; \eta^{r\,\dagger}(p\cdot\sigma + p\cdot\bar{\sigma})\eta^s \;=\; 2E_{\vec{p}}\,\delta_{rs}\ ,$$

$$u^{r\,\dagger}(p)v^s(\tilde{p}) \;=\; \big(\xi^{r\,\dagger}\sqrt{p\cdot\sigma}\,,\,\xi^{r\,\dagger}\sqrt{p\cdot\bar{\sigma}}\big)\begin{pmatrix}\sqrt{p\cdot\bar{\sigma}}\,\eta^s\\ -\sqrt{p\cdot\sigma}\,\eta^s\end{pmatrix} \;=\; 0 \;=\; v^{r\,\dagger}(\tilde{p})u^s(p)\ ,$$

with $\tilde{p}^\mu \equiv (p^0, -\vec{p}) \;\Rightarrow\; \tilde{p}\cdot\bar{\sigma} = p\cdot\sigma$ and $\tilde{p}\cdot\sigma = p\cdot\bar{\sigma}$. This is obviously not boost-invariant. Lorentz invariant contractions are obtained through

$$\bar{u}^r(p)u^s(p) \;=\; u^{r\,\dagger}(p)\gamma^0 u^s(p) \;=\; \big(\xi^{r\,\dagger}\sqrt{p\cdot\sigma}\,,\,\xi^{r\,\dagger}\sqrt{p\cdot\bar{\sigma}}\big)\begin{pmatrix}\sqrt{p\cdot\bar{\sigma}}\,\xi^s\\ \sqrt{p\cdot\sigma}\,\xi^s\end{pmatrix} \;=\; 2m\,\delta_{rs}\ ,$$

$$\bar{v}^r(p)v^s(p) \;=\; v^{r\,\dagger}(p)\gamma^0 v^s(p) \;=\; \big(\eta^{r\,\dagger}\sqrt{p\cdot\sigma}\,,\,-\,\eta^{r\,\dagger}\sqrt{p\cdot\bar{\sigma}}\big)\begin{pmatrix}-\sqrt{p\cdot\bar{\sigma}}\,\eta^s\\ \sqrt{p\cdot\sigma}\,\eta^s\end{pmatrix} \;=\; -\,2m\,\delta_{rs}\ ,$$

$$\bar{u}^r(p)v^s(p) \;=\; \big(\xi^{r\,\dagger}\sqrt{p\cdot\sigma}\,,\,\xi^{r\,\dagger}\sqrt{p\cdot\bar{\sigma}}\big)\begin{pmatrix}-\sqrt{p\cdot\bar{\sigma}}\,\eta^s\\ \sqrt{p\cdot\sigma}\,\eta^s\end{pmatrix} \;=\; 0 \;=\; \bar{v}^r(p)u^s(p)\ .$$

Polarization sums: for dealing with Feynman diagrams that involve Dirac fermions, polarization sums (helicity sums) are an essential ingredient. These polarization sums read

$$\begin{aligned}
\sum_{s=1}^{2} u^s(p)\bar{u}^s(p) \;&=\; \sum_{s=1}^{2}\begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi^s\\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s\end{pmatrix}\big(\xi^{s\,\dagger}\sqrt{p\cdot\bar{\sigma}}\,,\,\xi^{s\,\dagger}\sqrt{p\cdot\sigma}\big)\\
&=\; \begin{pmatrix}\sqrt{p\cdot\sigma}\,\sum\limits_{s=1}^{2}\xi^s\xi^{s\,\dagger}\sqrt{p\cdot\bar{\sigma}} & \sqrt{p\cdot\sigma}\,\sum\limits_{s=1}^{2}\xi^s\xi^{s\,\dagger}\sqrt{p\cdot\sigma}\\ \sqrt{p\cdot\bar{\sigma}}\,\sum\limits_{s=1}^{2}\xi^s\xi^{s\,\dagger}\sqrt{p\cdot\bar{\sigma}} & \sqrt{p\cdot\bar{\sigma}}\,\sum\limits_{s=1}^{2}\xi^s\xi^{s\,\dagger}\sqrt{p\cdot\sigma}\end{pmatrix}\\
&\overset{\text{compl.}}{=\!=}\; \begin{pmatrix} mI_2 & p\cdot\sigma\\ p\cdot\bar{\sigma} & mI_2\end{pmatrix} \;=\; \gamma^\mu p_\mu + mI_4 \;=\; \not{p} + mI_4\ ,
\end{aligned}$$

$$\sum_{s=1}^{2} v^s(p)\bar{v}^s(p) \;=\; \not{p} - mI_4\ ,$$

where in the third step the completeness relation for the 2-dimensional basis ξ^1, ξ^2 is used.

3.3 Quantization of the free Dirac theory (§ 3.5 in the book)

11g *The same philosophy will be applied as in the Klein-Gordon case. We diagonalize the Hamiltonian $\hat{H}$ of the Dirac theory in its quantized form by expanding the solutions of the Dirac equation in spatial plane-wave modes, which are written in terms of creation and annihilation operators. The particle interpretation is obtained by letting these creation operators act on the vacuum state $|0\rangle$, which is defined to contain no particles (i.e. positive-energy quanta) and to have the lowest energy. This leads to the requirement that the spectrum of $\hat{H}$ should be bounded from below. On top of that, we again demand that causality should be preserved for having a viable theory.*

Derivation of the operator algebra: step 1. According to the discussion on page 96

$$\mathcal{H}_{\text{Dirac}}(x) \;=\; \pi_\psi(x)\dot{\psi}(x) \;=\; i\psi^\dagger(x)\dot{\psi}(x)\;.$$

In analogy with the scalar case we expand a solution of the Dirac equation in terms of plane-wave modes, bearing in mind that $\hat{\psi}(x)$ is non-hermitian and has spinorial degrees of freedom:

$$\hat{\psi}(x) \;=\; \int\frac{\mathrm{d}\vec{p}}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\vec{p}}}}\sum_{s=1}^{2}\Big(\hat{a}^s_{\vec{p}}\,u^s(p)e^{-ip\cdot x}+\hat{b}^{s\,\dagger}_{\vec{p}}\,v^s(p)e^{ip\cdot x}\Big)\Big|_{p_0=E_{\vec{p}}}\;.$$

The difference with the scalar case is the occurrence of the u and v spinors that span spinor space. The Hamilton operator of the free Dirac theory now reads

$$\hat{H} \;=\; \int\mathrm{d}\vec{x}\;i\hat{\psi}^\dagger(x)\dot{\hat{\psi}}(x) \;=\; \int\mathrm{d}\vec{x}\int\frac{\mathrm{d}\vec{p}\,\mathrm{d}\vec{p}\,'}{(2\pi)^6}\,\frac{\sqrt{E_{\vec{p}\,'}}}{2\sqrt{E_{\vec{p}}}}\sum_{s,s'=1}^{2}\Big(\hat{a}^{s\,\dagger}_{\vec{p}}\,u^{s\,\dagger}(p)e^{ip\cdot x}+\hat{b}^{s}_{\vec{p}}\,v^{s\,\dagger}(p)e^{-ip\cdot x}\Big)\times$$

$$\times\,\Big(\hat{a}^{s'}_{\vec{p}\,'}\,u^{s'}(p')e^{-ip'\cdot x}-\hat{b}^{s'\,\dagger}_{\vec{p}\,'}\,v^{s'}(p')e^{ip'\cdot x}\Big)\Big|_{p_0=E_{\vec{p}}\,,\,p_0'=E_{\vec{p}\,'}}$$

$$\overset{\vec{x}\text{ integral}}{=\!=\!=}\;\int\frac{d\vec{p}}{(2\pi)^3}\,\frac{1}{2}\sum_{s,s'=1}^{2}\Big(\hat{a}^{s\,\dagger}_{\vec{p}}\hat{a}^{s'}_{\vec{p}}\,u^{s\,\dagger}(p)u^{s'}(p)-\hat{b}^{s}_{\vec{p}}\hat{b}^{s'\,\dagger}_{\vec{p}}\,v^{s\,\dagger}(p)v^{s'}(p)$$

$$+\,\hat{b}^{s}_{\vec{p}}\hat{a}^{s'}_{-\vec{p}}\,v^{s\,\dagger}(p)u^{s'}(\tilde{p})e^{-2itE_{\vec{p}}}-\hat{a}^{s\,\dagger}_{\vec{p}}\hat{b}^{s'\,\dagger}_{-\vec{p}}\,u^{s\,\dagger}(p)v^{s'}(\tilde{p})e^{2itE_{\vec{p}}}\Big)\Big|_{p_0=E_{\vec{p}}}$$

$$\overset{\text{norm.}}{=\!=\!=}\;\int\frac{d\vec{p}}{(2\pi)^3}\sum_{s=1}^{2}E_{\vec{p}}\left(\hat{a}^{s\,\dagger}_{\vec{p}}\hat{a}^{s}_{\vec{p}}-\hat{b}^{s}_{\vec{p}}\hat{b}^{s\,\dagger}_{\vec{p}}\right)\;.$$

From this expression for the Hamilton operator of the free Dirac theory we can read off that

- the energy spectrum is not bounded from below if we use commutation relations like in the case of scalar theories;
- it does certainly not help if $\hat{b}_{\vec{p}}^{s\,\dagger}$ is replaced by $\hat{c}_{\vec{p}}^{s}$, since in that case the problem cannot be solved at all;
- 11g *we are forced to impose fermionic anticommutation relations on the creation and annihilation operators* $\hat{b}_{\vec{p}}^{s\,\dagger}$ *and* $\hat{b}_{\vec{p}}^{s}$*, being the alternative starting point for setting up a many-particle quantum theory:*

$$\big\{\hat{b}_{\vec{p}}^{s}\,,\hat{b}_{\vec{p}\,'}^{s'\,\dagger}\big\} \;=\; (2\pi)^3\delta(\vec{p}-\vec{p}\,')\,\delta_{ss'}\,\hat{1} \quad \text{and} \quad \big\{\hat{b}_{\vec{p}}^{s}\,,\hat{b}_{\vec{p}\,'}^{s'}\big\} \;=\; \big\{\hat{b}_{\vec{p}}^{s\,\dagger}\,,\hat{b}_{\vec{p}\,'}^{s'\,\dagger}\big\} \;=\; 0\ .$$

Upon implementing these anticommutation relations, the Hamilton operator indeed becomes bounded from below by a zero-point energy:

$$\hat{H} \;=\; \int \frac{d\vec{p}}{(2\pi)^3} \sum_{s=1}^{2} E_{\vec{p}}\left(\hat{a}_{\vec{p}}^{s\,\dagger}\hat{a}_{\vec{p}}^{s} + \hat{b}_{\vec{p}}^{s\,\dagger}\hat{b}_{\vec{p}}^{s} - (2\pi)^3\delta(\vec{0}\,)\hat{1}\right)\ .$$

- Again only positive-energy quanta feature in the Hamilton operator.
- This time we find an infinite zero-point energy with opposite sign, which can again be removed by normal ordering:

  $$\hat{b}^\dagger\hat{b} \;\to\; N(\hat{b}^\dagger\hat{b}) \;=\; \hat{b}^\dagger\hat{b} \quad , \quad \hat{b}\hat{b}^\dagger \;\to\; N(\hat{b}\hat{b}^\dagger) \;=\; -\,\hat{b}^\dagger\hat{b}\ .$$

  Note the extra minus sign that is required for normal ordering of fermionic operators. This will also have repercussions on the derivation of Wick's theorem and the ensuing Feynman rules.

  The opposite-sign fermionic zero-point energy could actually cancel the infinities originating from bosonic zero-point energies. So, there might be some profound physical concepts hidden in the zero-point sector ...!?

Let's ignore the latter issue from now on and proceed with the operator algebra.

Question: do the operators $\hat{a}\,,\hat{a}^\dagger$ *also obey fermionic anticommutation relations or bosonic commutation relations?*

Derivation of the operator algebra: step 2. To address the previous question we need to study the causal structure of the theory. In the scalar case this was intimately linked to the particle and antiparticle propagation amplitudes. This will involve both the Dirac operator field and the adjoint Dirac operator field, which is given by

$$\hat{\bar{\psi}}(x) \;=\; \int \frac{\mathrm{d}\vec{p}}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\vec{p}}}}\,\sum_{s=1}^{2}\Big(\hat{a}^{s\,\dagger}_{\vec{p}}\,\bar{u}^s(p)e^{ip\cdot x} + \hat{b}^s_{\vec{p}}\,\bar{v}^s(p)e^{-ip\cdot x}\Big)\Big|_{p_0=E_{\vec{p}}} \;=\; \hat{\psi}^\dagger(x)\gamma^0\ .$$

We start by having a look at the propagation of positive-energy particles from y to x. This is defined according to

$$\langle 0|\hat{\psi}_a(x)\hat{\bar{\psi}}_b(y)|0\rangle \;=\; \int \frac{\mathrm{d}\vec{p}\,\mathrm{d}\vec{p}\,'}{(2\pi)^6}\,\frac{e^{-ip\cdot x+ip'\cdot y}}{2\sqrt{E_{\vec{p}}\,E_{\vec{p}\,'}}}\,\sum_{s,s'=1}^{2} u^s_a(p)\bar{u}^{s'}_b(p')\langle 0|\hat{a}^s_{\vec{p}}\,\hat{a}^{s'\,\dagger}_{\vec{p}\,'}|0\rangle\Big|_{p_0=E_{\vec{p}},\,p_0'=E_{\vec{p}\,'}}$$

$$= \int \frac{\mathrm{d}\vec{p}}{(2\pi)^3}\,\frac{e^{-ip\cdot(x-y)}}{2E_{\vec{p}}}\,\sum_{s=1}^{2} u^s_a(p)\bar{u}^s_b(p)\Big|_{p_0=E_{\vec{p}}} \;=\; \int \frac{\mathrm{d}\vec{p}}{(2\pi)^3}\,\frac{e^{-ip\cdot(x-y)}}{2E_{\vec{p}}}\,(\not{p} + m\,I_4)_{ab}\Big|_{p_0=E_{\vec{p}}}$$

$$= (i\not{\partial}_x + m\,I_4)_{ab}\,D(x-y)\ ,$$

where a,b are spinor indices and $D(x-y)$ is given on page 19. This expression is valid irrespective of the statistics for the $\hat{a}$-operators:

$$\langle 0|\hat{a}^s_{\vec{p}}\,\hat{a}^{s'\,\dagger}_{\vec{p}\,'}|0\rangle \;=\; \langle 0|\big[(2\pi)^3\delta(\vec{p}-\vec{p}\,')\,\delta_{ss'}\,\hat{1} \,\pm\, \hat{a}^{s'\,\dagger}_{\vec{p}\,'}\hat{a}^s_{\vec{p}}\big]|0\rangle \;=\; (2\pi)^3\delta(\vec{p}-\vec{p}\,')\,\delta_{ss'}\ ,$$

where the $+/-$ sign occurring after the first step refers to bosonic/fermionic statistics. Similarly the propagation of positive-energy antiparticles from x to y is given by

$$\langle 0|\hat{\bar{\psi}}_b(y)\hat{\psi}_a(x)|0\rangle \;=\; \int \frac{\mathrm{d}\vec{p}\,\mathrm{d}\vec{p}\,'}{(2\pi)^6}\,\frac{e^{ip\cdot x-ip'\cdot y}}{2\sqrt{E_{\vec{p}}\,E_{\vec{p}\,'}}}\,\sum_{s,s'=1}^{2} \bar{v}^{s'}_b(p')v^s_a(p)\langle 0|\hat{b}^{s'}_{\vec{p}\,'}\,\hat{b}^{s\,\dagger}_{\vec{p}}|0\rangle\Big|_{p_0=E_{\vec{p}},\,p_0'=E_{\vec{p}\,'}}$$

$$= \int \frac{\mathrm{d}\vec{p}}{(2\pi)^3}\,\frac{e^{ip\cdot(x-y)}}{2E_{\vec{p}}}\,\sum_{s=1}^{2} v^s_a(p)\bar{v}^s_b(p)\Big|_{p_0=E_{\vec{p}}} \;=\; \int \frac{\mathrm{d}\vec{p}}{(2\pi)^3}\,\frac{e^{ip\cdot(x-y)}}{2E_{\vec{p}}}\,(\not{p} - m\,I_4)_{ab}\Big|_{p_0=E_{\vec{p}}}$$

$$= -\,(i\not{\partial}_x + m\,I_4)_{ab}\,D(y-x)\ .$$

Important observation for the causality discussion: for $(x-y)^2 < 0$ we know from the scalar case that $D(x-y) = D(y-x)$, hence we have to conclude that $\langle 0|\big[\hat{\psi}_a(x), \hat{\bar{\psi}}_b(y)\big]|0\rangle \neq 0$ and $\langle 0|\big\{\hat{\psi}_a(x), \hat{\bar{\psi}}_b(y)\big\}|0\rangle = 0$ in that case.

Causality: in the coordinate representation any observable that involves Dirac particles contains an even number of spinor fields, i.e. as many ψ as $\bar{\psi}$ fields, since such an observable should have no open spinor indices. So, if either $\big[\hat{\psi}_a(x)\,,\hat{\bar{\psi}}_b(y)\big] = \big[\hat{\psi}_a(x)\,,\hat{\psi}_b(y)\big] = 0$ or $\big\{\hat{\psi}_a(x)\,,\hat{\bar{\psi}}_b(y)\big\} = \big\{\hat{\psi}_a(x)\,,\hat{\psi}_b(y)\big\} = 0$ for $(x-y)^2 < 0$, then measurements do not influence each other for spacelike separations and causality is preserved! As we have seen above, the first option cannot be achieved but the second option is possible. Based on the previous discussion,

$$\begin{aligned}\big\{\hat{\psi}_a(x)\,,\hat{\bar{\psi}}_b(y)\big\} \;=\; &\int \frac{\mathrm{d}\vec{p}\,\mathrm{d}\vec{p}\,'}{(2\pi)^6}\,\frac{1}{2\sqrt{E_{\vec{p}}\,E_{\vec{p}\,'}}}\sum_{s,s'=1}^{2}\Big(u_a^s(p)\bar{u}_b^{s'}(p')\,e^{-ip\cdot x+ip'\cdot y}\big\{\hat{a}_{\vec{p}}^{s}\,,\hat{a}_{\vec{p}\,'}^{s'\,\dagger}\big\}\\ &+ v_a^s(p)\bar{v}_b^{s'}(p')\,e^{ip\cdot x-ip'\cdot y}\big\{\hat{b}_{\vec{p}}^{s\,\dagger},\hat{b}_{\vec{p}\,'}^{s'}\big\} + u_a^s(p)\bar{v}_b^{s'}(p')\,e^{-ip\cdot x-ip'\cdot y}\big\{\hat{a}_{\vec{p}}^{s}\,,\hat{b}_{\vec{p}\,'}^{s'}\big\}\\ &+ v_a^s(p)\bar{u}_b^{s'}(p')\,e^{ip\cdot x+ip'\cdot y}\big\{\hat{b}_{\vec{p}}^{s\,\dagger},\hat{a}_{\vec{p}\,'}^{s'\,\dagger}\big\}\Big)\bigg|_{p_0=E_{\vec{p}}\,,\,p_0'=E_{\vec{p}\,'}} \;=\; 0\end{aligned}$$

and

$$\begin{aligned}\big\{\hat{\psi}_a(x)\,,\hat{\psi}_b(y)\big\} \;=\; &\int \frac{\mathrm{d}\vec{p}\,\mathrm{d}\vec{p}\,'}{(2\pi)^6}\,\frac{1}{2\sqrt{E_{\vec{p}}\,E_{\vec{p}\,'}}}\sum_{s,s'=1}^{2}\Big(u_a^s(p)u_b^{s'}(p')\,e^{-ip\cdot x-ip'\cdot y}\big\{\hat{a}_{\vec{p}}^{s}\,,\hat{a}_{\vec{p}\,'}^{s'}\big\}\\ &+ v_a^s(p)v_b^{s'}(p')\,e^{ip\cdot x+ip'\cdot y}\big\{\hat{b}_{\vec{p}}^{s\,\dagger},\hat{b}_{\vec{p}\,'}^{s'\,\dagger}\big\} + u_a^s(p)v_b^{s'}(p')\,e^{-ip\cdot x+ip'\cdot y}\big\{\hat{a}_{\vec{p}}^{s}\,,\hat{b}_{\vec{p}\,'}^{s'\,\dagger}\big\}\\ &+ v_a^s(p)u_b^{s'}(p')\,e^{ip\cdot x-ip'\cdot y}\big\{\hat{b}_{\vec{p}}^{s\,\dagger},\hat{a}_{\vec{p}\,'}^{s'}\big\}\Big)\bigg|_{p_0=E_{\vec{p}}\,,\,p_0'=E_{\vec{p}\,'}} \;=\; 0\end{aligned}$$

is guaranteed for spacelike separations $(x-y)^2 < 0$ if

$$\big\{\hat{a}_{\vec{p}}^{s}\,,\hat{a}_{\vec{p}\,'}^{s'\,\dagger}\big\} = (2\pi)^3\delta(\vec{p}-\vec{p}\,')\,\delta_{ss'}\,\hat{1} = \big\{\hat{b}_{\vec{p}}^{s}\,,\hat{b}_{\vec{p}\,'}^{s'\,\dagger}\big\}\,, \qquad \text{with all other anticommutators being } 0\,.$$

Note: the anticommutation relation for $\hat{a}$ and $\hat{a}^\dagger$ follows from the first two terms in the first expression, bearing in mind the anticommutation relation for $\hat{b}$ and $\hat{b}^\dagger$ as well as the equality $D(x-y) = D(y-x)$ for $(x-y)^2 < 0$.

11g *In the free Dirac theory both particles and antiparticles have to be fermions. On top of that, the creation and annihilation operators for particles anticommute with those for antiparticles. This implies that particles and antiparticles are versions of the same object, differing merely by the quantum number charge (as we will see below).*

Canonical equal-time anticommutation relations: from the fundamental fermionic anticommutation relations for creation and annihilation operators it follows that

$$\{\hat{\psi}_a(\vec{x},t),\hat{\bar{\psi}}_b(\vec{y},t)\} \stackrel{p.102}{=\!=} \int\frac{d\vec{p}}{(2\pi)^3}\,\frac{\hat{1}}{2E_{\vec{p}}}\Big[e^{i\vec{p}\cdot(\vec{x}-\vec{y})}(\not{p}+mI_4)_{ab}+e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}(\not{p}-mI_4)_{ab}\Big]\Big|_{p_0=E_{\vec{p}}}$$

$$\stackrel{\vec{p}\to-\vec{p}\text{ in 2nd term}}{=\!=\!=\!=} \int\frac{d\vec{p}}{(2\pi)^3}\,e^{i\vec{p}\cdot(\vec{x}-\vec{y})}(\gamma^0)_{ab}\hat{1} \;=\; (\gamma^0)_{ab}\,\delta(\vec{x}-\vec{y})\,\hat{1}$$

$$\Rightarrow\quad \{\hat{\psi}_a(\vec{x},t),\hat{\pi}_{\psi_c}(\vec{y},t)\} \;=\; \sum_b\{\hat{\psi}_a(\vec{x},t),i\hat{\bar{\psi}}_b(\vec{y},t)\}(\gamma^0)_{bc} \stackrel{(\gamma^0)^2=I}{=\!=\!=} i\delta_{ac}\,\delta(\vec{x}-\vec{y})\,\hat{1}$$

and

$$\{\hat{\psi}_a(\vec{x},t),\hat{\psi}_c(\vec{y},t)\} \;=\; \{\hat{\pi}_{\psi_a}(\vec{x},t),\hat{\pi}_{\psi_c}(\vec{y},t)\} \;=\; 0\,.$$

11g *The quantization of the free Dirac theory could equally well have been performed by imposing canonical equal-time anticommutation relations for the fields and their corresponding conjugate momenta.*

Completing the particle interpretation: what else do we know about the particles and antiparticles in the Dirac theory?

- After quantization the momentum carried by the Dirac field becomes (cf. page 96)

$$\hat{\vec{P}} \;=\; -\int d\vec{x}\;\hat{\pi}_\psi(x)\vec{\nabla}\hat{\psi}(x) \;=\; \int d\vec{x}\;\hat{\psi}^\dagger(x)(-i\vec{\nabla})\hat{\psi}(x)$$

$$=\int\frac{d\vec{p}}{(2\pi)^3}\,\vec{p}\sum_{s=1}^{2}(\hat{a}^{s\,\dagger}_{\vec{p}}\hat{a}^s_{\vec{p}}-\hat{b}^s_{\vec{p}}\hat{b}^{s\,\dagger}_{\vec{p}}) \;=\; \int\frac{d\vec{p}}{(2\pi)^3}\,\vec{p}\sum_{s=1}^{2}(\hat{a}^{s\,\dagger}_{\vec{p}}\hat{a}^s_{\vec{p}}+\hat{b}^{s\,\dagger}_{\vec{p}}\hat{b}^s_{\vec{p}})\,,$$

  just like in the scalar case.

- After quantization the conserved charge (called particle number) originating from the global $U(1)$ gauge symmetry becomes

$$\hat{Q} \;=\; \int d\vec{x}\;\hat{j}^{\,0}_V(x) \;=\; \int d\vec{x}\;\hat{\psi}^\dagger(x)\hat{\psi}(x) \;=\; \int\frac{d\vec{p}}{(2\pi)^3}\sum_{s=1}^{2}(\hat{a}^{s\,\dagger}_{\vec{p}}\hat{a}^s_{\vec{p}}+\hat{b}^s_{\vec{p}}\hat{b}^{s\,\dagger}_{\vec{p}})$$

$$\Rightarrow\quad N(\hat{Q}) \;=\; \int\frac{d\vec{p}}{(2\pi)^3}\sum_{s=1}^{2}(\hat{a}^{s\,\dagger}_{\vec{p}}\hat{a}^s_{\vec{p}}-\hat{b}^{s\,\dagger}_{\vec{p}}\hat{b}^s_{\vec{p}})\,,$$

  which implies that particles/antiparticles have particle number $+/-\,1$. Multiplied by the electromagnetic charge q of the particles this yields the total charge operator for interactions with electromagnetic fields (see later). So, in that case we can read off that particles and antiparticles have opposite charge.

- Based on the discussion on page 97, the total spin operator is given by

$$\hat{\vec{S}} = \int d\vec{x}\, \hat{\psi}^\dagger(x) \begin{pmatrix} \frac{1}{2}\vec{\sigma} & 0 \\ 0 & \frac{1}{2}\vec{\sigma} \end{pmatrix} \hat{\psi}(x) .$$

  Just like in the previous case, the order of the $\hat{b}^s_{\vec{p}}$ and $\hat{b}^{s\,\dagger}_{\vec{p}}$ operators results in opposite spin quantum numbers for antiparticles if we would set $\eta^s = \xi^s$ in the v and u spinors.

This allows us to read off the particle content of the free Dirac theory. We already know that for anticommuting creation and annihilation operators there exists a groundstate (vacuum state) $|0\rangle$ such that $\langle 0|0\rangle = 1$ and $\hat{a}^s_{\vec{p}}|0\rangle = \hat{b}^s_{\vec{p}}|0\rangle = 0$ for all $\vec{p}$ and s. Then $N(\hat{H})|0\rangle = 0$, $\hat{\vec{P}}|0\rangle = \vec{0}$ and $N(\hat{Q})|0\rangle = 0$, i.e. the vacuum "has" energy $E = 0$, momentum $\vec{P} = \vec{0}$ and charge $Q = 0$. From this groundstate the 1-particle excitations can be obtained as $\hat{a}^{s\,\dagger}_{\vec{p}}|0\rangle$ and $\hat{b}^{s\,\dagger}_{\vec{p}}|0\rangle$, corresponding to an energy $E_{\vec{p}}$, a momentum $\vec{p}$, spin 1/2 and opposite charge/particle number. In view of the fermionic anticommutation relations, $\hat{a}^\dagger$ creates fermionic particles and $\hat{b}^\dagger$ creates fermionic antiparticles.

> 11h *We can summarize the particle interpretation of the free Dirac theory as follows: $\hat{a}^{s\,\dagger}_{\vec{p}}$ creates particles with energy $E_{\vec{p}}$, momentum $\vec{p}$, spin 1/2, charge q and polarization appropriate to ξ^s, whereas $\hat{b}^{s\,\dagger}_{\vec{p}}$ creates antiparticles with energy $E_{\vec{p}}$, momentum $\vec{p}$, spin 1/2, charge $-q$ and polarization opposite to η^s. Hence, if $m = 0$ then $\hat{\psi}_{L/R}(x)$ annihilates particles with negative/positive helicity and creates antiparticles with positive/negative helicity.*

Inversion of the Dirac equation: the retarded Green's function is obtained by

$$\big[S_R(x-y)\big]_{ab} \equiv \theta(x^0-y^0)\langle 0|\big\{\hat{\psi}_a(x), \hat{\bar{\psi}}_b(y)\big\}|0\rangle$$

$$\overset{\text{p. 102}}{=\!=} \theta(x^0-y^0)(i\partial\!\!\!/_x + m I_4)_{ab}\big(D(x-y) - D(y-x)\big) = (i\partial\!\!\!/_x + m I_4)_{ab} D_R(x-y) .$$

We have used in the last step that $\partial_0\theta(x^0-y^0) = \delta(x^0-y^0)$ causes $D(x-y) - D(y-x)$ to vanish according to property 2 on page 20, which implies that it is safe to interchange the order of $\theta(x^0-y^0)$ and $(i\partial\!\!\!/_x + m I_4)_{ab}$.

Proof that this Green's function indeed inverts the Dirac equation:

$$(i\partial\!\!\!/_x - m)S_R(x-y) = (i\partial\!\!\!/_x - m)(i\partial\!\!\!/_x + m)D_R(x-y) = -(\Box + m^2)D_R(x-y)I_4 = i\delta^{(4)}(x-y)I_4 .$$

In Fourier language this inversion reads:

$$(i\partial\!\!\!/_x - m)S_R(x-y) = \int \frac{d^4p}{(2\pi)^4}\,(p\!\!\!/ - m)\tilde{S}_R(p)e^{-ip\cdot(x-y)} = i\int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x-y)} I_4$$

$$\Rightarrow \quad \tilde{S}_R(p) = \frac{i(p\!\!\!/ + m)}{p^2 - m^2} \equiv \frac{i}{p\!\!\!/ - m} ,$$

with the same prescription to go around the complex poles as in the Klein-Gordon case. Similarly the Feynman prescription yields the Feynman propagator

$$\begin{aligned}
\big[S_F(x-y)\big]_{ab} &= \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\, \frac{i(\not{p}+mI_4)_{ab}}{p^2-m^2+i\epsilon}\, e^{-ip\cdot(x-y)} \;=\; (i\not{\partial}_x + mI_4)_{ab}\, D_F(x-y) \\
&= \begin{cases} (i\not{\partial}_x + mI_4)_{ab}\, D(x-y) \;=\; \langle 0|\hat{\psi}_a(x)\,\hat{\bar{\psi}}_b(y)|0\rangle & \text{if } x^0 > y^0 \\[2mm] (i\not{\partial}_x + mI_4)_{ab}\, D(y-x) \;=\; -\,\langle 0|\hat{\bar{\psi}}_b(y)\,\hat{\psi}_a(x)|0\rangle & \text{if } x^0 < y^0 \end{cases} \\
&\equiv \langle 0|T(\hat{\psi}_a(x)\,\hat{\bar{\psi}}_b(y))|0\rangle \ .
\end{aligned}$$

Note the extra minus sign in the definition of time ordering for fermionic fields. Just like in the Klein-Gordon case the Feynman propagator $\big[S_F(x-y)\big]_{ab}$ is the time-ordered propagation amplitude, which will play a crucial role in the Feynman rules for fermions.

Lorentz transformations and $\hat{\psi}(x)$: just like in the Klein-Gordon case the 1-particle states are normalized according to $|\vec{p},s\rangle \equiv \sqrt{2E_{\vec{p}}}\,\hat{a}^{s\,\dagger}_{\vec{p}}\,|0\rangle$, with a similar expression holding for 1-antiparticle states. Using this definition we can define the unitary operator that implements (active) Lorentz transformations in the Hilbert space of quantum states:

$$|\overrightarrow{\Lambda p},s\rangle \;\equiv\; \hat{U}(\Lambda)\,|\vec{p},s\rangle \;\;\Rightarrow\;\; \sqrt{2E_{\overrightarrow{\Lambda p}}}\;\hat{a}^{s\,\dagger}_{\overrightarrow{\Lambda p}}\,|0\rangle \;=\; \sqrt{2E_{\vec{p}}}\;\hat{U}(\Lambda)\,\hat{a}^{s\,\dagger}_{\vec{p}}\,\hat{U}^{-1}(\Lambda)\,\overbrace{\hat{U}(\Lambda)\,|0\rangle}^{\equiv\,|0\rangle}$$

$$\Rightarrow\;\; \text{define:}\;\; \hat{U}(\Lambda)\,\hat{a}^{s\,\dagger}_{\vec{p}}\,\hat{U}^{-1}(\Lambda) \;=\; \sqrt{\frac{E_{\overrightarrow{\Lambda p}}}{E_{\vec{p}}}}\;\hat{a}^{s\,\dagger}_{\overrightarrow{\Lambda p}}\ ,$$

provided that we choose the axis of spin quantization to be parallel to the boost/rotation axis. The transformation property of $\hat{b}^{s\,\dagger}_{\vec{p}}$ has an analogous form. As a result:

$$\hat{U}(\Lambda)\hat{\psi}(x)\hat{U}^{-1}(\Lambda) \;=\; \int \frac{\mathrm{d}\vec{p}}{(2\pi)^3}\,\frac{1}{2E_{\vec{p}}}\,\sqrt{2E_{\overrightarrow{\Lambda p}}}\,\sum_{s=1}^{2}\Big(\hat{a}^{s}_{\overrightarrow{\Lambda p}}\,u^s(p)e^{-ip\cdot x} + \hat{b}^{s\,\dagger}_{\overrightarrow{\Lambda p}}\,v^s(p)e^{ip\cdot x}\Big)$$

$$\underline{\underline{p'=\Lambda p}} \quad \int \frac{\mathrm{d}\vec{p}\,'}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\vec{p}\,'}}}\,\sum_{s=1}^{2}\Big(\hat{a}^{s}_{\vec{p}\,'}\,\overbrace{u^s(\Lambda^{-1}p')}^{\Lambda^{-1}_{1/2}u^s(p')}\,e^{-ip'\cdot\Lambda x} + \hat{b}^{s\,\dagger}_{\vec{p}\,'}\,\overbrace{v^s(\Lambda^{-1}p')}^{\Lambda^{-1}_{1/2}v^s(p')}\,e^{ip'\cdot\Lambda x}\Big)$$

$$\Rightarrow\quad \hat{U}(\Lambda)\hat{\psi}(x)\hat{U}^{-1}(\Lambda) \;=\; \Lambda^{-1}_{1/2}\,\hat{\psi}(\Lambda x)\ ,$$

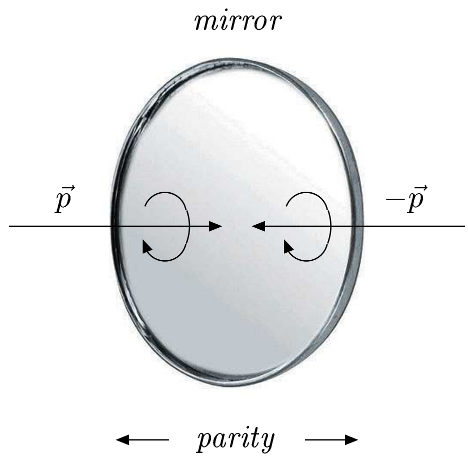
where the second line is obtained by using that $\int \mathrm{d}\vec{p}/(2E_{\vec{p}})$ and $e^{\pm ip\cdot x}$ are all Lorentz invariant. This implies that the transformed field creates/destroys antiparticles/particles at the spacetime point Λx.

3.4 Discrete symmetries

11i *Apart from the symmetry under continuous Lorentz transformations and translations, there are two more spacetime symmetries a free Lagrangian should have in relativistic field theories. These correspond to the discrete Lorentz transformations that complete the Lorentz group:*

- parity (spatial inversion) P, which reverses the handedness of space: $t\,,\vec{x} \stackrel{P}{=} t\,,-\vec{x}\,.$
- time reversal T, which interchanges forward and backward light cones: $t\,,\vec{x} \stackrel{T}{=} -t\,,\vec{x}\,.$

In addition it is also useful to consider a non-spacetime discrete operation called charge conjugation C, which interchanges particles and antiparticles. In particular P and C play a crucial role in constructing the Standard Model of electroweak interactions. In these lecture notes we will explicitly consider parity transformations. The details for the other discrete transformations can be found in the textbook of Peskin & Schroeder.



Parity: this mirror-reflection spacetime transformation is implemented in the Hilbert space of quantum states by a unitary operator (basis transformation) $\hat{P}$. Its action on the creation and annihilation operators is such that a state $|\vec{p}\,,s\rangle$ is transformed into a state $|-\vec{p}\,,s\rangle$, provided that the spin is quantized along an arbitrary fixed axis.[5] This implies

$$\hat{P}\hat{a}^s_{\vec{p}}\hat{P}^\dagger \;=\; \beta_a\hat{a}^s_{-\vec{p}} \qquad \text{and} \qquad \hat{P}\hat{b}^s_{\vec{p}}\hat{P}^\dagger \;=\; \beta_b\hat{b}^s_{-\vec{p}}\;,$$

where $\beta_{a,b}$ are phase factors. Applying $\hat{P}$ twice should have no effect on observables in the Dirac theory. These observables contain as many ψ as $\bar{\psi}$ fields, so the phase factors drop out as long as β_a and β_b are related appropriately (see below). In analogy to the case of continuous Lorentz transformations, the transformation property of the Dirac field under parity then becomes

$$\hat{P}\hat{\psi}(x)\hat{P}^\dagger \;=\; \int\frac{\mathrm{d}\vec{p}}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\vec{p}}}}\,\sum_{s=1}^{2}\Big(\beta_a\hat{a}^s_{-\vec{p}}\,u^s(p)e^{-ip\cdot x}+\beta_b^*\hat{b}^{s\,\dagger}_{-\vec{p}}\,v^s(p)e^{ip\cdot x}\Big)\Big|_{p_0\,=\,E_{\vec{p}}} \;\equiv\; \Lambda^{(P)}_{1/2}\,\hat{\psi}(\tilde{x})\;,$$

with $\tilde{x}^\mu \equiv (x^0,-\vec{x}\,)$. Using $\tilde{p}^\mu \equiv (p^0,-\vec{p}\,) \;\Rightarrow\; \tilde{p}\cdot\bar{\sigma} = p\cdot\sigma$ and $\tilde{p}\cdot\sigma = p\cdot\bar{\sigma}$, we can rewrite the u and v spinors according to

[5] If we would instead use spin quantization along the momentum direction, then also the associated quantum number helicity would be reversed under parity.

$$u^s(p) \;=\; \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar\sigma}\,\xi^s \end{pmatrix} \;=\; \begin{pmatrix} \sqrt{\tilde p\cdot\bar\sigma}\,\xi^s \\ \sqrt{\tilde p\cdot\sigma}\,\xi^s \end{pmatrix} \;=\; \gamma^0 \begin{pmatrix} \sqrt{\tilde p\cdot\sigma}\,\xi^s \\ \sqrt{\tilde p\cdot\bar\sigma}\,\xi^s \end{pmatrix} \;=\; \gamma^0 u^s(\tilde p)\ ,$$

$$v^s(p) \;=\; \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta^s \\ -\sqrt{p\cdot\bar\sigma}\,\eta^s \end{pmatrix} \;=\; \begin{pmatrix} \sqrt{\tilde p\cdot\bar\sigma}\,\eta^s \\ -\sqrt{\tilde p\cdot\sigma}\,\eta^s \end{pmatrix} \;=\; -\gamma^0 \begin{pmatrix} \sqrt{\tilde p\cdot\sigma}\,\eta^s \\ -\sqrt{\tilde p\cdot\bar\sigma}\,\eta^s \end{pmatrix} \;=\; -\gamma^0 v^s(\tilde p)\ .$$

Bearing in mind that the integral over $\vec p$ and the energy $E_{\vec p}$ are unaffected by the transition from $\vec p$ to $\vec{\tilde p} = -\vec p$, this leads to

$$\hat P\hat\psi(x)\hat P^\dagger = \int\frac{\mathrm{d}\vec{\tilde p}}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\vec{\tilde p}}}}\,\sum_{s=1}^{2}\Big(\beta_a\hat a^s_{\vec{\tilde p}}\gamma^0 u^s(\tilde p)e^{-i\tilde p\cdot\tilde x} - \beta_b^*\hat b^{s\,\dagger}_{\vec{\tilde p}}\gamma^0 v^s(\tilde p)e^{i\tilde p\cdot\tilde x}\Big)\Big|_{\tilde p_0 = E_{\vec{\tilde p}}} = \Lambda^{(P)}_{1/2}\hat\psi(\tilde x)$$

$$\Rightarrow\quad \beta_b^* \;=\; -\beta_a \qquad\text{and}\qquad \Lambda^{(P)}_{1/2} \;=\; \beta_a\gamma^0\ .$$

The transformation property of $\hat{\bar\psi}(x)$ then follows:

$$\hat P\hat{\bar\psi}(x)\hat P^\dagger \;=\; \hat P\hat\psi^\dagger(x)\gamma^0\hat P^\dagger \;\xlongequal{[\hat P,\gamma^0]\,=\,0}\; \big(\hat P\hat\psi(x)\hat P^\dagger\big)^\dagger\gamma^0 \;=\; \beta_a^*\hat{\bar\psi}(\tilde x)\gamma^0 \;=\; \hat{\bar\psi}(\tilde x)\Lambda^{(P)^{-1}}_{1/2}\ ,$$

since $\hat P$ acts on the Hilbert space of quantum states and not on spinor space. The effective transformation properties of the γ-matrices then read

$$\Lambda^{(P)^{-1}}_{1/2}\gamma^\mu\Lambda^{(P)}_{1/2} \;\xlongequal{\beta_a^*\beta_a\,=\,1}\; \gamma^0\gamma^\mu\gamma^0 \;=\; \gamma_\mu\ ,$$

which we can write as

$$\Lambda^{(P)^{-1}}_{1/2}\gamma^\mu\Lambda^{(P)}_{1/2} \;=\; (\Lambda^P)^\mu{}_\nu\gamma^\nu \quad\text{with}\quad (\Lambda^P)^\mu{}_\nu \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \text{spatial inversion}$$

in analogy with the continuous Lorentz transformations. Furthermore

$$\Lambda^{(P)^{-1}}_{1/2}\gamma^5\Lambda^{(P)}_{1/2} \;=\; \gamma^0\gamma^5\gamma^0 \;=\; -\gamma^5 \;=\; \det(\Lambda^P)\gamma^5\ .$$

Now we have all ingredients for deriving the transformation properties of the normal-ordered currents that are the basic building blocks for observables:

$$\underline{\textit{scalar current}}:\ N\big(\hat j_S(x)\big) \;\xrightarrow{\ \mathrm{P}\ }\; N\big(\hat j_S(\tilde x)\big)\ ,$$

$$\underline{\textit{vector current}}:\ N\big(\hat j_V^\mu(x)\big) \;\xrightarrow{\ \mathrm{P}\ }\; N\big(\hat j^V_\mu(\tilde x)\big)\ ,$$

$$\underline{\textit{tensor current}}:\ N\big(\hat j_T^{\mu\nu}(x)\big) \;\xrightarrow{\ \mathrm{P}\ }\; N\big(\hat j^T_{\mu\nu}(\tilde x)\big)\ ,$$

$$\underline{\textit{axial vector current}}:\ N\big(\hat j_A^\mu(x)\big) \;\xrightarrow{\ \mathrm{P}\ }\; -N\big(\hat j^A_\mu(\tilde x)\big)\ ,$$

$$\underline{\textit{pseudo scalar current}}:\ N\big(\hat j_P(x)\big) \;\xrightarrow{\ \mathrm{P}\ }\; -N\big(\hat j_P(\tilde x)\big)\ .$$

These transformation properties actually follow from the fact that left/right-handed fields are transformed into right/left-handed fields under partity. Note that the phase factor β_a does not occur in any of these transformation properties. Therefore we might just as well set $\beta_a = -\beta_b = 1$, resulting in the following textbook statement:

11i *in the Dirac theory particles and antiparticles have opposite intrinsic parity.*

Since $\partial_\mu \xrightarrow{\text{P}} \partial^\mu$, the free Dirac Lagrangian is evidently invariant under parity.

Charge conjugation and time reversal: after similar sets of steps it can be derived how the normal-ordered currents transform under charge conjugation and time reversal. These topics will not be discussed in these lecture notes. The interested reader is referred to p. 67–71 in the textbook of Peskin & Schroeder.

Interacting relativistic field theories: the free Dirac Lagrangian is invariant under all three discrete symmetries. For interacting theories involving Dirac fields, however, the following holds:

- electromagnetic, strong and gravitational interactions are P- and C-invariant;
- weak interactions violate P- and C-invariance (maximally) in the Standard Model, but preserve the combined CP-invariance;
- rare processes involving K-mesons violate CP-invariance: within the Standard Model this leads to the requirement that there should be at least three families of fermions;
- all interacting relativistic field theories should be CPT-invariant in order to have a theory that preserves causality and that has a Lorentz-scalar hermitian Lagrangian.

4 Interacting Dirac fields and Feynman diagrams

The next lecture covers § 4.7 of Peskin & Schroeder.

> 12 *We have already seen in detail how the Feynman rules come about in scalar theories. Next we move on to theories that involve Dirac fermions. In that case the interaction Hamiltonian will contain an even number of spinor fields in order to have a Lorentz invariant action.*

4.1 Wick's theorem for fermions

The first thing we have to do is to generalize Wick's theorem. We start with the propagator using explicit spinor indices a and b:

$$
\begin{aligned}
\langle 0|T\big(\hat\psi_a(x)\bar{\hat\psi}_b(y)\big)|0\rangle \;&=\; \big[S_F(x-y)\big]_{ab} \;=\; \int\frac{\mathrm{d}^4p}{(2\pi)^4}\,\frac{i(\not{p}+m\,I_4)_{ab}}{p^2-m^2+i\epsilon}\,e^{-ip\cdot(x-y)} \\
&=\; \begin{cases} +\,\langle 0|\hat\psi_a(x)\,\bar{\hat\psi}_b(y)|0\rangle & \text{if } \ x^0>y^0 \\[2ex] -\,\langle 0|\bar{\hat\psi}_b(y)\,\hat\psi_a(x)|0\rangle & \text{if } \ x^0<y^0 \end{cases} \;=\; -\,\langle 0|T\big(\bar{\hat\psi}_b(y)\,\hat\psi_a(x)\big)|0\rangle,
\end{aligned}
$$

which involves time-ordered fields. For Wick's theorem we will have to generalize the definition of time ordering to cases with more fields. Define time ordering to pick up one minus sign for each interchange of fermionic operators: e.g. for $x_3^0>x_1^0>x_4^0>x_2^0$

$$
T\big(\hat\psi_{a_1}(x_1)\hat\psi_{a_2}(x_2)\hat\psi_{a_3}(x_3)\hat\psi_{a_4}(x_4)\big) \;=\; (-1)^3\,\hat\psi_{a_3}(x_3)\hat\psi_{a_1}(x_1)\hat\psi_{a_4}(x_4)\hat\psi_{a_2}(x_2)\ .
$$

Similarly the definition of normal ordering is generalized for more than two fermionic operators according to

$$
N(\hat a^s_{\vec p}\,\hat a^r_{\vec q}\,\hat a^{t\,\dagger}_{\vec l}) \;=\; (-1)^2\,\hat a^{t\,\dagger}_{\vec l}\,\hat a^s_{\vec p}\,\hat a^r_{\vec q} \;=\; (-1)^3\,\hat a^{t\,\dagger}_{\vec l}\,\hat a^r_{\vec q}\,\hat a^s_{\vec p}\ ,
$$

where again each interchange of fermionic operators gives rise to a minus sign.

> 12a *In the proof of Wick's theorem the order of the creation and annihilation operators will matter this time.*

Based on these generalizations of time ordering and normal ordering, we can extend the definition of contractions (see Ex. 18):

$$
T\big(\hat\psi_a(x)\,\bar{\hat\psi}_b(y)\big) \;=\; N\big(\hat\psi_a(x)\,\bar{\hat\psi}_b(y)\big) \;+\; \overbracket{\hat\psi_a(x)\,\bar{\hat\psi}_b}(y)\ ,
$$

with

$$
\overbracket{\hat\psi_a(x)\,\bar{\hat\psi}_b}(y) \;=\; -\,\overbracket{\bar{\hat\psi}_b(y)\,\hat\psi_a}(x) \;\equiv\; \begin{cases} +\,\big\{\hat\psi^+_a(x),\bar{\hat\psi}^-_b(y)\big\} & \text{if } \ x^0>y^0 \\[2ex] -\,\big\{\bar{\hat\psi}^+_b(y),\hat\psi^-_a(x)\big\} & \text{if } \ x^0<y^0 \end{cases} \;=\; \big[S_F(x-y)\big]_{ab}\hat 1
$$

where $\hat{\psi}^+$ and $\hat{\psi}^-$ correspond to the positive and negative frequency parts. Furthermore

$$\contraction{}{\hat{\psi}}{_a(x)}{\hat{\psi}}\hat{\psi}_a(x)\,\hat{\psi}_b(y) \;=\; \contraction{}{\hat{\bar{\psi}}}{_a(x)}{\hat{\bar{\psi}}}\hat{\bar{\psi}}_a(x)\,\hat{\bar{\psi}}_b(y) \;=\; 0 \ ,$$

since these fields anticommute.

Wick's theorem for fermionic fields: let's again skip the subscript I that we would normally use to indicate (free) interaction picture fields. Wick's theorem then states

$$T\big(\hat{\psi}_{a_1}(x_1)\cdots\hat{\psi}_{a_n}(x_n)\big) \;=\; N\big(\hat{\psi}_{a_1}(x_1)\cdots\hat{\psi}_{a_n}(x_n) + \text{all possible contractions}\big) \ ,$$

as before, with for example

$$\begin{aligned} N\big(\contraction{}{\hat{\psi}}{_{a_1}(x_1)\hat{\psi}_{a_2}(x_2)}{\hat{\bar{\psi}}}\hat{\psi}_{a_1}(x_1)\hat{\psi}_{a_2}(x_2)\hat{\bar{\psi}}_{a_3}(x_3)\hat{\bar{\psi}}_{a_4}(x_4)\big) \;&\equiv\; -\,\contraction{}{\hat{\psi}}{_{a_1}(x_1)}{\hat{\bar{\psi}}}\hat{\psi}_{a_1}(x_1)\,\hat{\bar{\psi}}_{a_3}(x_3)\,N\big(\hat{\psi}_{a_2}(x_2)\,\hat{\bar{\psi}}_{a_4}(x_4)\big) \\ &=\; -\,N\big(\hat{\psi}_{a_2}(x_2)\,\hat{\bar{\psi}}_{a_4}(x_4)\big)\big[S_F(x_1-x_3)\big]_{a_1a_3} \ . \end{aligned}$$

The proof of this version of Wick's theorem (see Ex. 18) proceeds in a way similar to the one that was worked out for scalar fields.

4.2 Feynman rules for the Yukawa theory

In order to assess the consequences of the fermionic version of Wick's theorem we consider the Yukawa theory for the interactions between fermions and scalars. The Lagrangian of the Yukawa theory is given by

$$\mathcal{L}_{\text{Yukawa}} \;=\; \bar{\psi}(i\gamma^\mu\partial_\mu - m_\psi)\psi \,+\, \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) \,-\, \frac{1}{2}m_\phi^2\phi^2 \,-\, g\bar{\psi}\psi\phi \;\equiv\; \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{KG}} + \mathcal{L}_{int} \ ,$$

with ϕ a real scalar field and ψ a Dirac field. This gives rise to the following interaction term in the Hamilton operator of the Yukawa theory: $\hat{H}_{int} = g\int \mathrm{d}\vec{x}\; \hat{\bar{\psi}}(x)\hat{\psi}(x)\hat{\phi}(x)$.

4.2.1 Implications of Fermi statistics

In order to study the consequences of fermionic minus signs we consider the ψ-fermion scattering reaction

$$\psi(k_A, s_A)\,\psi(k_B, s_B) \;\to\; \psi(p_1, r_1)\,\psi(p_2, r_2)$$

at lowest order in perturbation theory, with the momenta and spin states of the particles indicated between parentheses. In chapter 2 we have seen that the corresponding T-matrix element is given by (skipping spin labels)

$$\langle \vec{p}_1\vec{p}_2|i\hat{T}|\vec{k}_A\vec{k}_B\rangle \;=\; \Big({}_0\langle\vec{p}_1\vec{p}_2|T\Big(e^{-i\int d^4x\,\hat{\mathcal{H}}_I(x)}\Big)|\vec{k}_A\vec{k}_B\rangle_0\Big)_{\substack{\text{fully connected}\\ \text{and amputated}}} \times \text{factor}$$

in terms of free-particle plane-wave states and interaction-picture (free) fields. The lowest-order contribution to the ψ-scattering reaction then reads

$$ {}_0\langle \vec{p}_1\vec{p}_2|T\Big(\frac{(-ig)^2}{2!}\int \mathrm{d}^4x\ \hat{\bar{\psi}}_I(x)\hat{\psi}_I(x)\hat{\phi}_I(x)\int \mathrm{d}^4y\ \hat{\bar{\psi}}_I(y)\hat{\psi}_I(y)\hat{\phi}_I(y)\Big)|\vec{k}_A\vec{k}_B\rangle_0\ . $$

In order to perform the corresponding calculation we have to define

$$ \hat{\psi}_I(x)|\vec{k},s\rangle_0\ \equiv\ \hat{\psi}_I^+(x)|\vec{k},s\rangle_0\ = \int \frac{\mathrm{d}\vec{q}}{(2\pi)^3}\,\frac{e^{-iq\cdot x}}{\sqrt{2E_{\vec{q}}}}\,\sum_r \hat{a}^r_{\vec{q}}\,u^r(q)\,\sqrt{2E_{\vec{k}}}\,\hat{a}^{s\,\dagger}_{\vec{k}}|0\rangle\ =\ e^{-ik\cdot x}u^s(k)\,|0\rangle\ , $$

with similar expressions for other initial and final states.

> 12b *Since $\hat{\psi}_I$ contains $\hat{a}$ and $\hat{b}^\dagger$ operators, it can be contracted with a fermion state on the right (initial state) or an antifermion state on the left (final state). The opposite holds for $\hat{\bar{\psi}}_I$, since it contains $\hat{b}$ and $\hat{a}^\dagger$ operators.*

Minus signs from interchanging fermions: for the lowest-order ψ-fermion-scattering T-matrix elements we obtain (with the numbers indicating the order of the contractions)

$$ \begin{array}{c} 2:+ \qquad\qquad\qquad 3:+ \\ \dfrac{(-ig)^2}{2!}\,2!\ {}_0\langle \vec{p}_1\vec{p}_2|\int \mathrm{d}^4x\ \hat{\bar{\psi}}_I(x)\hat{\psi}_I(x)\hat{\phi}_I(x)\int \mathrm{d}^4y\ \hat{\bar{\psi}}_I(y)\hat{\psi}_I(y)\hat{\phi}_I(y)|\vec{k}_A\vec{k}_B\rangle_0 \\ 1:- \qquad\qquad\qquad 4:+ \end{array} $$

$$ = -\int \frac{\mathrm{d}^4q}{(2\pi)^4}\,\frac{-ig^2}{q^2-m_\phi^2+i\epsilon}\,(2\pi)^4\delta^{(4)}(k_A-p_1-q)\,(2\pi)^4\delta^{(4)}(k_B-p_2+q)\ \times $$

$$ \times\ \big[\bar{u}^{s_1}(p_1)\,u^{s_A}(k_A)\big]\big[\bar{u}^{s_2}(p_2)\,u^{s_B}(k_B)\big] $$

$$ = \frac{ig^2}{(k_A-p_1)^2-m_\phi^2+i\epsilon}\ \big[\bar{u}^{s_1}(p_1)\,u^{s_A}(k_A)\big]\big[\bar{u}^{s_2}(p_2)\,u^{s_B}(k_B)\big]\,(2\pi)^4\delta^{(4)}(k_A+k_B-p_1-p_2) $$

$$ \equiv\ (2\pi)^4\delta^{(4)}(k_A+k_B-p_1-p_2)\,i\mathcal{M}_1 $$

and

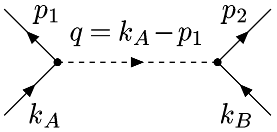
$$ \begin{array}{c} 1:- \qquad\qquad\qquad 3:+ \\ \dfrac{(-ig)^2}{2!}\,2!\ {}_0\langle \vec{p}_1\vec{p}_2|\int \mathrm{d}^4x\ \hat{\bar{\psi}}_I(x)\hat{\psi}_I(x)\hat{\phi}_I(x)\int \mathrm{d}^4y\ \hat{\bar{\psi}}_I(y)\hat{\psi}_I(y)\hat{\phi}_I(y)|\vec{k}_A\vec{k}_B\rangle_0 \\ 2:- \qquad\qquad\qquad 4:+ \end{array} $$

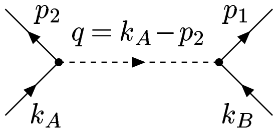
$$ = \frac{-ig^2}{(k_A-p_2)^2-m_\phi^2+i\epsilon}\ \big[\bar{u}^{s_2}(p_2)\,u^{s_A}(k_A)\big]\big[\bar{u}^{s_1}(p_1)\,u^{s_B}(k_B)\big]\,(2\pi)^4\delta^{(4)}(k_A+k_B-p_1-p_2) $$

$$ \equiv\ (2\pi)^4\delta^{(4)}(k_A+k_B-p_1-p_2)\,i\mathcal{M}_2\ , $$

which have opposite signs. Note that we have used here a definition for the two-fermion initial and final states: $|\vec{k}_A,s_A;\vec{k}_B,s_B\rangle \propto \hat{a}^{s_A\,\dagger}_{\vec{k}_A}\hat{a}^{s_B\,\dagger}_{\vec{k}_B}|0\rangle$ and $\langle \vec{p}_1,r_1;\vec{p}_2,r_2| \propto \langle 0|\hat{a}^{r_1}_{\vec{p}_1}\hat{a}^{r_2}_{\vec{p}_2}$.

These T-matrix elements correspond to the following Feynman diagrams:

$i\mathcal{M}_1 \;=$ 

and $i\mathcal{M}_2 \;=$ 

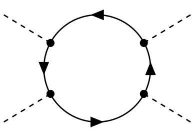
.

In these Feynman diagrams solid lines are used to indicate the fermions and dashed ones to indicate the scalar particles.

> (12b) *Due to Fermi statistics the second contribution has a relative minus sign with respect to the first one, since it involves the interchange of two fermions.*

The overall sign depends on the definitions of the multiparticle states, for instance one might define $\langle \vec{p}_1 \vec{p}_2 | \propto \langle 0 | \hat{a}_{\vec{p}_2} \hat{a}_{\vec{p}_1}$ instead of $\propto \langle 0 | \hat{a}_{\vec{p}_1} \hat{a}_{\vec{p}_2}$.

Minus signs from closed fermion loops: the Feynman rules for fermions result in a factor -1 for closed fermion loops, as well as a trace of a product of Dirac matrices. We find for instance that



$$\propto \sum_{a,b,c,d} \bar{\hat{\psi}}_{a_I}(x_1)\hat{\psi}_{a_I}(x_1)\,\bar{\hat{\psi}}_{b_I}(x_2)\hat{\psi}_{b_I}(x_2)\,\bar{\hat{\psi}}_{c_I}(x_3)\hat{\psi}_{c_I}(x_3)\,\bar{\hat{\psi}}_{d_I}(x_4)\hat{\psi}_{d_I}(x_4)$$

$$= -\sum_{a,b,c,d} \hat{\psi}_{d_I}(x_4)\bar{\hat{\psi}}_{a_I}(x_1)\,\hat{\psi}_{a_I}(x_1)\bar{\hat{\psi}}_{b_I}(x_2)\,\hat{\psi}_{b_I}(x_2)\bar{\hat{\psi}}_{c_I}(x_3)\,\hat{\psi}_{c_I}(x_3)\bar{\hat{\psi}}_{d_I}(x_4)$$

$$= -\,\mathrm{Tr}\big(S_F(x_4 - x_1)S_F(x_1 - x_2)S_F(x_2 - x_3)S_F(x_3 - x_4)\big)\ .$$

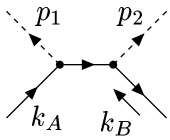
In order to figure out how the matrices in the propagators should be contracted, we have used explicit (repeated) spinor labels during intermediate steps.

> (12b) *This sign difference between fermionic and bosonic loops has important implications for the high-energy behaviour of the fundamental interactions: strong interactions are asymptotically free, electromagnetic interactions are not.*

4.2.2 Drawing convention

> (12c) *In analogy with the conventions for the scalar Yukawa theory, we also draw arrows on the fermion lines in the actual Yukawa theory. These arrows represent the direction of particle-number flow: particles flow along the arrow, antiparticles flow against it. In this convention $\hat{\psi}$ corresponds to an arrow flowing into a vertex, whereas $\bar{\hat{\psi}}$ corresponds to an arrow flowing out of a vertex. Since every interaction vertex features both $\hat{\psi}$ and $\bar{\hat{\psi}}$, the arrows link up to form a continuous flow. But this time there is more to it!*

Consider for example a Feynman diagram like



$$= \; -g^2 \sum_{a,b} {}_0\langle \vec{p}_1 \vec{p}_2 | \int \mathrm{d}^4x \; \hat{\bar{\psi}}_{a_I}(x) \hat{\psi}_{a_I}(x) \hat{\phi}_I(x) \int \mathrm{d}^4y \; \hat{\bar{\psi}}_{b_I}(y) \hat{\psi}_{b_I}(y) \hat{\phi}_I(y) | \vec{k}_A \vec{k}_B \rangle_0$$

$$= \; -ig^2 \int \mathrm{d}^4x \int \mathrm{d}^4y \; e^{i(p_2 - k_B)\cdot x} \sum_{a,b} \bar{v}_a(k_B) \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \, \frac{(\not{q} + m_\psi I_4)_{ab}}{q^2 - m_\psi^2 + i\epsilon} \; e^{-iq\cdot(x-y)} \, u_b(k_A) \, e^{i(p_1 - k_A)\cdot y}$$

$$= \; -ig^2 \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \, \frac{\bar{v}(k_B)(\not{q} + m_\psi) u(k_A)}{q^2 - m_\psi^2 + i\epsilon} \; (2\pi)^4 \delta^{(4)}(p_2 - q - k_B) \, (2\pi)^4 \delta^{(4)}(p_1 + q - k_A)$$

$$= \; \frac{-ig^2}{(k_A - p_1)^2 - m_\psi^2 + i\epsilon} \; \bar{v}(k_B)(\not{k}_A - \not{p}_1 + m_\psi) u(k_A) \, (2\pi)^4 \delta^{(4)}(k_A + k_B - p_1 - p_2) \; .$$

In order to figure out how the spinors and matrices should be contracted, we have used explicit (repeated) spinor labels during intermediate steps.

> 12c *Here we see the importance of introducing the arrow convention: the spinor indices are in this way always contracted along the fermion line, with the arrow indicating the reversed order. Phrased differently, you should insert γ-matrices and spinors while going against the arrow of particle-number flow!*

4.2.3 Källén–Lehmann spectral representation for fermions

The non-perturbative analysis of the 2-point Green's function follows the same steps as in the scalar case with just a few obvious modifications:

$$\int \mathrm{d}^4x \; e^{ip\cdot x} \langle \Omega | T\big(\hat{\psi}(x) \hat{\bar{\psi}}(0)\big) | \Omega \rangle \; = \; \frac{iZ_2 \sum_s u^s(p) \bar{u}^s(p)}{p^2 - m_{ph}^2 + i\epsilon} + \text{remainder containing other states}$$

$$= \; \frac{iZ_2 \, (\not{p} + m_{ph})}{p^2 - m_{ph}^2 + i\epsilon} \; + \; \cdots \; .$$

Like before, Z_2 represents the probability for the fermionic quantum field to create or annihilate an exact "1-dressed particle" eigenstate of $\hat{H}$ from the ground state, with m_{ph} denoting its observable physical mass:

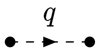
$$\langle \Omega | \hat{\psi}(0) | \vec{k}\,, s \rangle \; = \; u^s(k) \sqrt{Z_2} \quad , \quad \langle \Omega | \hat{\bar{\psi}}(0) | \vec{k}\,, s \rangle \; = \; \bar{v}^s(k) \sqrt{Z_2} \; ,$$

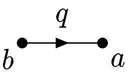
$$\langle \vec{p}, r | \hat{\bar{\psi}}(0) | \Omega \rangle \; = \; \bar{u}^r(p) \sqrt{Z_2} \quad , \quad \langle \vec{p}, r | \hat{\psi}(0) | \Omega \rangle \; = \; v^r(p) \sqrt{Z_2} \; .$$

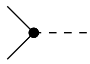
More details will be worked out in the next chapter.

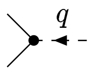
4.2.4 Momentum-space Feynman rules for the Yukawa theory

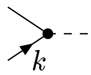
The Feynman rules that we have obtained for the Yukawa theory in the previous sections can be summarized by the following list:

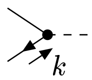
1. For each scalar propagator  insert $\dfrac{i}{q^2 - m_\phi^2 + i\epsilon}$.

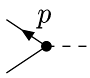
   For each fermion propagator  insert $\dfrac{i(\not{q} + m_\psi I_4)_{ab}}{q^2 - m_\psi^2 + i\epsilon}$.

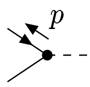
2. For each vertex  insert $-ig$.

3. For each external scalar line  insert $\sqrt{Z}$.

   For each incoming fermion line  insert $u^s(k)\sqrt{Z_2}$, originating from $\hat{\psi}$.

   For each incoming antifermion line  insert $\bar{v}^s(k)\sqrt{Z_2}$, originating from $\hat{\bar{\psi}}$.

   For each outgoing fermion line  insert $\bar{u}^r(p)\sqrt{Z_2}$, originating from $\hat{\bar{\psi}}$.

   For each outgoing antifermion line  insert $v^r(p)\sqrt{Z_2}$, originating from $\hat{\psi}$.

4. Impose energy-momentum conservation at each vertex.

5. Integrate over each undetermined loop momentum l_j: $\displaystyle\int \frac{d^4 l_j}{(2\pi)^4}$.

6. Figure out the relative signs of the diagrams, caused by interchanging fermions.

7. Insert γ-matrices and spinors while going against the arrow of particle-number flow.

8. Each fermion loop receives a minus sign and involves a trace over spinor space.

The following observations can be made. First of all, no symmetry factors are needed in the Yukawa theory since all fields in the interaction are different. Secondly, as can be seen from the propagator, the sign (direction) of the momentum matters for fermions. Finally, each distinct type of particle in the theory will have its own wave-function renormalization factor, i.e. $\sqrt{Z}$ for the scalar particles and $\sqrt{Z_2}$ for the fermions.

4.3 How to calculate squared amplitudes

12d *The final expressions for amplitudes that involve external fermions typically feature subexpressions starting with a $\bar{u}$ or $\bar{v}$ spinor, followed by a chain of contracted matrices in spinor space, and closed by a u or v spinor. How should we calculate squared amplitudes of that form?*

In order to answer this question we select a typical term that features in $|\mathcal{M}|^2$:

$$\big[\bar{u}(p)\Gamma_1\, u(k)\big]\big[\bar{u}(p)\Gamma_2\, u(k)\big]^* \ ,$$

where the first factor originates from $\mathcal{M}$ and the second one from $\mathcal{M}^*$. Here $\Gamma_{1,2}$ denote arbitrary 4×4 matrices in spinor space.

Step 1: expressing things in terms of traces in spinor space.

We can make use of the identity

$$\big[\bar{u}(p)\Gamma_2\, u(k)\big]^* \;=\; \bar{u}(k)\gamma^0\Gamma_2^\dagger\gamma^0 u(p)$$

to rewrite the expression given above in terms of traces in spinor space:

$$\big[\bar{u}(p)\Gamma_1\, u(k)\big]\big[\bar{u}(p)\Gamma_2\, u(k)\big]^* \;=\; \sum_{a,b,c,d}\big[\bar{u}_a(p)\,\Gamma_{1_{ab}}u_b(k)\big]\big[\bar{u}_c(k)(\gamma^0\Gamma_2^\dagger\gamma^0)_{cd}\,u_d(p)\big] \;=$$

$$\sum_{a,b,c,d}\big[u(p)\bar{u}(p)\big]_{da}\Gamma_{1_{ab}}\big[u(k)\bar{u}(k)\big]_{bc}(\gamma^0\Gamma_2^\dagger\gamma^0)_{cd} \;=\; \mathrm{Tr}\Big(\big[u(p)\bar{u}(p)\big]\Gamma_1\big[u(k)\bar{u}(k)\big]\gamma^0\Gamma_2^\dagger\gamma^0\Big)\ .$$

Looking at the trace in the last line, the various combinations of Dirac spinors occurring between square brackets are in fact 4×4 matrices in spinor space.

Step 2: employing polarization sums.

- If we are not able to produce polarized beams or to measure the polarization of the final-state particles, then we have to average over the initial-state polarizations and to sum over the final-state polarizations:

$$\begin{aligned}
&\mathrm{Tr}\Big(\big[u(p)\bar{u}(p)\big]\Gamma_1\big[u(k)\bar{u}(k)\big]\gamma^0\Gamma_2^\dagger\gamma^0\Big)\\
\to\;&\frac{1}{2}\,\mathrm{Tr}\Big(\big[\sum_r u^r(p)\bar{u}^r(p)\big]\Gamma_1\big[\sum_s u^s(k)\bar{u}^s(k)\big]\gamma^0\Gamma_2^\dagger\gamma^0\Big)\\
=\;&\frac{1}{2}\,\mathrm{Tr}\Big([\not{p}+m_\psi]\,\Gamma_1[\not{k}+m_\psi]\gamma^0\Gamma_2^\dagger\gamma^0\Big)\ ,
\end{aligned}$$

  if we assume that one of the fermions is an initial-state fermion and the other one a final-state fermion. The final trace can be worked out using the trace technology developed in Ex. 16.

- If we are able to polarize the beams or measure polarization, then we can use a similar trick provided that we project the Dirac spinors on the correct polarization states:

$$u^s(k) \;\to\; P u^s(k) \quad \Rightarrow \quad \bar{u}^s(k) \;\to\; \bar{u}^s(k)\gamma^0 P^\dagger \gamma^0 \;\equiv\; \bar{u}^s(k)\bar{P}$$

$$u^r(p) \;\to\; P' u^r(p) \quad \Rightarrow \quad \bar{u}^r(p) \;\to\; \bar{u}^r(p)\gamma^0 P'^\dagger \gamma^0 \;\equiv\; \bar{u}^r(p)\bar{P}' \; .$$

  This allows us to perform the spin sums as before, but this time without plugging in the spin average factor $1/2$. The spin projection matrices $P, \bar{P}$ and $P', \bar{P}'$ will select the correct states! This time we get

$$\mathrm{Tr}\Big(P'[\not{p} + m_\psi]\,\bar{P}'\Gamma_1 P[\not{k} + m_\psi]\,\bar{P}\gamma^0\Gamma_2^\dagger\gamma^0 \Big) \; .$$

  Once we know the spin projection matrices, the resulting traces can be calculated.

- If we are interested in polarized cross sections at high energies, then m_ψ can be neglected with respect to the energy of the fermions. On top of that it will in that case make sense to consider helicity states as our polarization states of choice. After all, helicity eigenstates and chirality eigenstates coincide and are not mixed by Lorentz transformations if $m_\psi = 0$. In that case the spin projections become

$$u^s(k) \;\to\; P_{R/L} u^s(k) \quad , \quad \bar{u}^s(k) \;\to\; \bar{u}^s(k) P_{L/R} \qquad \text{for helicity } +/- \text{ fermions} \; ,$$

$$v^s(k) \;\to\; P_{R/L} v^s(k) \quad , \quad \bar{v}^s(k) \;\to\; \bar{v}^s(k) P_{L/R} \qquad \text{for helicity } -/+ \text{ antifermions} \; ,$$

  where it is used that $\bar{P}_{L/R} = P_{R/L}$. The indicated chiralities reflect the fact that particles and antiparticles have an opposite definition for their polarization states.

  12d *To summarize: calculating cross sections involving fermions simply boils down to working out a collection of traces, irrespective of the fact whether one is able to polarize the beams and/or measure final-state polarization.*

Trace technology: the most important trace identities have been worked out in Ex. 16. The relevant part that we need later on is summarized by

$$\mathrm{Tr}(\gamma^{\mu_1}\cdots\gamma^{\mu_{2n+1}}) \;=\; \mathrm{Tr}(\text{odd number of } \gamma\text{-matrices}) \;=\; 0 \; ,$$

$$\mathrm{Tr}(\gamma^{\mu_1}\cdots\gamma^{\mu_{2n+1}}\gamma^5) \;=\; 0 \; ,$$

$$\mathrm{Tr}(I_4) \;=\; 4 \quad , \quad \mathrm{Tr}(\gamma^\mu\gamma^\nu) \;=\; 4g^{\mu\nu} \quad , \quad \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) \;=\; 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) \; ,$$

$$\mathrm{Tr}(\gamma^5) \;=\; \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^5) \;=\; 0 \quad , \quad \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5) \;=\; -\,4i\epsilon^{\mu\nu\rho\sigma} \; ,$$

with $\epsilon^{\mu\nu\alpha\beta}\epsilon_{\mu\nu\rho\sigma} = 2\,\delta^\alpha_{\;\;\sigma}\delta^\beta_{\;\;\rho} - 2\,\delta^\alpha_{\;\;\rho}\delta^\beta_{\;\;\sigma}$.

5 Quantum Electrodynamics (QED)

During the last two lectures material will be covered that is not treated in this form in the textbook of Peskin & Schroeder.

> 13 *In this last chapter we will have a look at electromagnetic interactions of matter particles. This will be used as motivation for the gauge principle, which introduces the concept of gauge bosons as fundamental force carriers.*

5.1 Electromagnetism

We start with the derivation of Maxwell's equations in vacuum in covariant form. For an electromagnetic field in vacuum with charge density $\rho_c(t,\vec{x}\,) \equiv \rho_c(x) \in \mathbb{R}$ and current density $\vec{j}_c(t,\vec{x}\,) \equiv \vec{j}_c(x) \in \mathbb{R}^3$ the Maxwell equations read:

$$\vec{\nabla}\cdot\vec{\mathcal{B}}(x) \;=\; 0 \quad , \quad \vec{\nabla}\times\vec{\mathcal{E}}(x) \;=\; -\frac{\partial}{\partial t}\,\vec{\mathcal{B}}(x) \ ,$$

$$\vec{\nabla}\cdot\vec{\mathcal{E}}(x) \;=\; \rho_c(x) \ ,$$

$$\vec{\nabla}\times\vec{\mathcal{B}}(x) \;=\; \frac{\partial}{\partial t}\,\vec{\mathcal{E}}(x) \,+\, \vec{j}_c(x) \ ,$$

where $\vec{\mathcal{E}}(t,\vec{x}\,) \equiv \vec{\mathcal{E}}(x) \in \mathbb{R}^3$ and $\vec{\mathcal{B}}(t,\vec{x}\,) \equiv \vec{\mathcal{B}}(x) \in \mathbb{R}^3$ are the electric and magnetic fields. Next we introduce the electromagnetic 4-vector potential

$$A^\mu(x) \;\equiv\; \big(\phi(x)\,,\vec{A}(x)\big) \ ,$$

such that

$$\vec{\mathcal{E}}(x) \;=\; -\,\vec{\nabla}\phi(x) \,-\, \frac{\partial}{\partial t}\,\vec{A}(x) \quad , \quad \vec{\mathcal{B}}(x) \;=\; \vec{\nabla}\times\vec{A}(x) \ .$$

In this way the two Maxwell equations on the first line are satisfied automatically, since $\vec{\nabla}\cdot\big(\vec{\nabla}\times\vec{A}(x)\big) = 0$ and $\vec{\nabla}\times\big(\vec{\nabla}\phi(x)\big) = \vec{0}$. The other two Maxwell equations can be rewritten as

$$\begin{aligned}\rho_c(x) \;&=\; -\frac{\partial}{\partial t}\big(\vec{\nabla}\cdot\vec{A}(x)\big) \,-\, \vec{\nabla}\cdot\big(\vec{\nabla}\phi(x)\big) \\ &=\; \Big(\frac{\partial^2}{\partial t^2} - \vec{\nabla}^{\,2}\Big)\phi(x) \,-\, \frac{\partial}{\partial t}\Big(\vec{\nabla}\cdot\vec{A}(x) \,+\, \frac{\partial}{\partial t}\,\phi(x)\Big)\end{aligned}$$

and

$$\begin{aligned}\vec{j}_c(x) \;&=\; \vec{\nabla}\times\big(\vec{\nabla}\times\vec{A}(x)\big) \,+\, \frac{\partial^2}{\partial t^2}\,\vec{A}(x) \,+\, \frac{\partial}{\partial t}\,\vec{\nabla}\phi(x) \\ &=\; \Big(\frac{\partial^2}{\partial t^2} - \vec{\nabla}^{\,2}\Big)\vec{A}(x) \,+\, \vec{\nabla}\Big(\vec{\nabla}\cdot\vec{A}(x) \,+\, \frac{\partial}{\partial t}\,\phi(x)\Big) \ ,\end{aligned}$$

using the identity

$$\vec{\nabla} \times \big(\vec{\nabla} \times \vec{A}(x)\big) \stackrel{\text{general}}{=\!=\!=} \vec{\nabla}\big(\vec{\nabla} \cdot \vec{A}(x)\big) - \vec{\nabla}^{\,2} \vec{A}(x) \, .$$

Defining the electromagnetic 4-current density

$$j_c^\mu(x) \equiv \big(\rho_c(x)\,, \vec{j}_c(x)\big)$$

Maxwell's equations can be cast in the form of the covariant electromagnetic wave equation

$$\Box\, A^\nu(x) - \partial^\nu\big(\partial_\mu A^\mu(x)\big) = j_c^\nu(x) \, .$$

Gauge freedom: the vector potential $A^\mu(x)$ is not fixed completely by its relation to the electric and magnetic fields. For an arbitrary, sufficiently differentiable scalar function $\chi(x)$ that vanishes sufficiently quickly as $|\vec{x}\,| \to \infty$, the transformed vector potential

$$A^\mu(x) \to A'^{\,\mu}(x) = A^\mu(x) + \partial^\mu \chi(x)$$

gives rise to the same electric and magnetic fields and therefore describes the same physics.

13a *The associated freedom to choose the vector potential is called the gauge freedom.*

Since the current density $j_c^\nu(x)$ is a physical observable, the field-derivative combination $\Box\, A^\nu(x) - \partial^\nu\big(\partial_\mu A^\mu(x)\big)$ should be gauge invariant, i.e. independent of the choice of gauge.

Proof: introduce the electromagnetic field tensor (see Ex. 2)

$$F^{\mu\nu}(x) = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x) = -\,F^{\nu\mu}(x) = \begin{pmatrix} 0 & -\,\mathcal{E}^1(x) & -\,\mathcal{E}^2(x) & -\,\mathcal{E}^3(x) \\ \mathcal{E}^1(x) & 0 & -\,\mathcal{B}^3(x) & \mathcal{B}^2(x) \\ \mathcal{E}^2(x) & \mathcal{B}^3(x) & 0 & -\,\mathcal{B}^1(x) \\ \mathcal{E}^3(x) & -\,\mathcal{B}^2(x) & \mathcal{B}^1(x) & 0 \end{pmatrix} ,$$

then the electromagnetic wave equation can be rewritten as

$$j_c^\nu(x) = \partial_\mu \partial^\mu A^\nu(x) - \partial^\nu \partial_\mu A^\mu(x) = \partial_\mu\big(\partial^\mu A^\nu(x) - \partial^\nu A^\mu(x)\big) = \partial_\mu F^{\mu\nu}(x) \, .$$

Since the electromagnetic field tensor is gauge invariant, i.e.

$$F'^{\,\mu\nu}(x) = \partial^\mu A'^{\,\nu}(x) - \partial^\nu A'^{\,\mu}(x) = \partial^\mu\big(A^\nu(x) + \partial^\nu \chi(x)\big) - \partial^\nu\big(A^\mu(x) + \partial^\mu \chi(x)\big) = F^{\mu\nu}(x) \, ,$$

the same holds for $\partial_\mu F^{\mu\nu}(x) = j_c^\nu(x)$.

Local charge conservation: from the electromagnetic wave equation one can derive that

$$\partial_\nu j_c^\nu(x) = \partial_\nu \partial_\mu F^{\mu\nu}(x) = 0 \, ,$$

since $F^{\mu\nu}(x) = -\,F^{\nu\mu}(x)$. Hence,

13a *the current density $j_c^\nu(x)$ is a conserved current and the electric charge $\int_V d\vec{x}\; j_c^0(x) = \int_V d\vec{x}\; \rho_c(x)$ is conserved locally.*

Electromagnetic Lagrangian: the Lagrangian density belonging to the electromagnetic wave equation is given by

$$\mathcal{L}_{\text{e.m.}}(x) \;=\; -\,\frac{1}{4}\,F_{\mu\nu}(x)F^{\mu\nu}(x) \;-\; j_c^{\mu}(x)A_{\mu}(x)\;.$$

<u>Proof</u>: first we consider

$$\frac{\partial\mathcal{L}_{\text{e.m.}}}{\partial(\partial_\mu A_\nu)} \;=\; -\,\frac{1}{4}\Big(\frac{\partial}{\partial(\partial_\mu A_\nu)}\,F_{\rho\sigma}\Big)F^{\rho\sigma} \;-\; \frac{1}{4}\,F_{\rho\sigma}\Big(\frac{\partial}{\partial(\partial_\mu A_\nu)}\,F^{\rho\sigma}\Big)$$

$$\stackrel{\text{Ex. 2}}{=\!=} \;-\,\frac{1}{4}\,(g^{\mu}{}_{\rho}\,g^{\nu}{}_{\sigma} - g^{\nu}{}_{\rho}\,g^{\mu}{}_{\sigma})F^{\rho\sigma} \;-\; \frac{1}{4}\,F_{\rho\sigma}(g^{\mu\rho}g^{\nu\sigma} - g^{\nu\rho}g^{\mu\sigma}) \;=\; -\,F^{\mu\nu}\,.$$

As a result, the Euler-Lagrange equation for the field $A_\nu(x)$ indeed reads

$$-\,\partial_\mu F^{\mu\nu}(x) + j_c^{\nu}(x) \;=\; 0 \quad\Rightarrow\quad \partial_\mu F^{\mu\nu}(x) \;=\; j_c^{\nu}(x)\;.$$

5.2 QED and the gauge principle

For Dirac fermions (<u>matter particles</u>) with charge q the electromagnetic current density is given by $j^{\nu}_{c,\text{Dirac}}(x) \;=\; q\,\bar{\psi}(x)\gamma^\nu\psi(x)$, since

- this current is indeed conserved (cf. page 96);
- after normal ordering the 0^{th} component can indeed be identified with the total charge density (cf. page 104):

$$\int\!\mathrm{d}\vec{x}\;N\big(\hat{j}^{\,0}_{c,\text{Dirac}}(x)\big) \;=\; q\int\!\frac{\mathrm{d}\vec{p}}{(2\pi)^3}\,\sum_{s=1}^{2}\,(\hat{a}_{\vec{p}}^{\,s\,\dagger}\hat{a}_{\vec{p}}^{\,s} \;-\; \hat{b}_{\vec{p}}^{\,s\,\dagger}\hat{b}_{\vec{p}}^{\,s})\;,$$

  counting particles with charge q and antiparticles with charge $-q$.

Minimal substitution and QED: the Lagrangian density of Dirac fermions with charge q in an electromagnetic field is obtained by applying the

13b <u>*minimal substitution prescription*</u> $p^\mu \to p^\mu - q\,A^\mu \xrightarrow{QM} i\partial^\mu \to i\partial^\mu - q\,A^\mu$

to the Lagrangian density $\mathcal{L}_{\text{Dirac}}(x)$ of the free Dirac theory and by subsequently adding the kinetic pure electromagnetic term $\mathcal{L}_{\text{Maxwell}}(x) = -\frac{1}{4}\,F_{\mu\nu}(x)F^{\mu\nu}(x)$.

13b *This results in the Lagrangian density for Quantum Electrodynamics (QED):*

$$\mathcal{L}_{\text{QED}}(x) \;=\; \bar{\psi}(x)(i\not{\partial} - m)\psi(x) \;-\; \frac{1}{4}\,F_{\mu\nu}(x)F^{\mu\nu}(x) \;-\; q\,\bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x)\;,$$

containing the aforementioned interaction term

$$\mathcal{L}_{\text{int}}(x) \;=\; -\,q\,\bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x) \;=\; -\,j^{\mu}_{c,\text{Dirac}}(x)A_\mu(x)\;.$$

As in the case of the Yukawa interactions, also the local electromagnetic interactions between the matter particles are mediated by force carriers. This was to be expected, bearing in mind that charged objects are observed to interact while being at non-zero disctance! Since $[\psi] = [\bar{\psi}] = 3/2$ and $[A_\mu] = 1$, the electric charge $q \in \mathbb{R}$ is a dimensionless coupling constant. We will see later that this dimensionless coupling constant indeed implies that QED is a renormalizable theory.

QED from a symmetry principle: gauge invariance and the gauge principle.

Alternatively we could start from the free Dirac Lagrangian

$$\mathcal{L}_{\mathrm{Dirac}}(x) \;=\; i\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x) \;-\; m\bar{\psi}(x)\psi(x) \;,$$

which is invariant under the global gauge transformation (abelian $U(1)$ transformation)

$$\psi(x) \to \psi'(x) = e^{i\alpha}\psi(x) \quad , \quad \bar{\psi}(x) \to \bar{\psi}'(x) = e^{-i\alpha}\bar{\psi}(x) \qquad (\alpha \in \mathbb{R} \text{ independent of } x^\mu)\;.$$

According to Noether's theorem this global gauge symmetry can be associated with a conserved current and charge. In non-relativistic quantum mechanics this global gauge invariance of a free-fermion system simply underlines the unobservability of the absolute phase of a wave function: only relative phases are observable through interference.

13c *The gauge principle: in the context of relativistic gauge theories, which should be local, it is now postulated that this gauge invariance should also hold locally.*

Consider to this end the local gauge transformation

$$\psi(x) \to \psi'(x) = e^{i\alpha(x)}\psi(x) \quad , \quad \bar{\psi}(x) \to \bar{\psi}'(x) = e^{-i\alpha(x)}\bar{\psi}(x) \qquad (\alpha(x) \text{ a real scalar field})\;.$$

The requirement of local gauge invariance[6] has profound consequences, since the kinetic term transforms as

$$i\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x) \;\to\; i\bar{\psi}(x)e^{-i\alpha(x)}\gamma^\mu\partial_\mu\big[e^{i\alpha(x)}\psi(x)\big] \;=\; i\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x) - \bar{\psi}(x)\gamma^\mu\psi(x)\big[\partial_\mu\alpha(x)\big]$$

and therefore is not invariant under local gauge transformations. The last term, which involves the covariant vector field $\partial_\mu\alpha(x)$, explicitly spoils the invariance. So, we need to replace the ordinary derivative ∂_μ by a gauge covariant derivative (or short: covariant derivative) D_μ such that

$$D_\mu\psi(x) \;\to\; D'_\mu\psi'(x) \;=\; e^{i\alpha(x)}D_\mu\psi(x)\;,$$

causing $D_\mu\psi(x)$ and $\psi(x)$ to transform similarly under local gauge transformations! This can be achieved by

$$D_\mu \;\equiv\; \partial_\mu + i\mathrm{g}A_\mu(x)\;, \quad \text{with} \quad A_\mu(x) \;\to\; A'_\mu(x) \;=\; A_\mu(x) - \frac{1}{\mathrm{g}}\,\partial_\mu\alpha(x)\;,$$

[6]See the bachelor thesis of Pim van Oirschot for more details and extra motivation

where g is a gauge coupling and $A_\mu(x)$ a gauge field. In view of the Lorentz transformation property of $\partial_\mu\alpha(x)$, this gauge field should be a covariant vector field. Its transformation property resembles a gauge transformation for the electromagnetic vector potential with $\chi(x) = -\alpha(x)/\mathrm{g}$. This observed gauge-freedom redundancy in the electromagnetic description is exploited here to reveal the more profound local gauge invariance of QED!

Proof:

$$
\begin{aligned}
D'_\mu\psi'(x) &= \Big(\partial_\mu + i\mathrm{g}\big[A_\mu(x) - \frac{1}{\mathrm{g}}\partial_\mu\alpha(x)\big]\Big)e^{i\alpha(x)}\psi(x) \\
&= e^{i\alpha(x)}\big[\partial_\mu\psi(x) + i\psi(x)\partial_\mu\alpha(x) + i\mathrm{g}A_\mu(x)\psi(x) - i\psi(x)\partial_\mu\alpha(x)\big] = e^{i\alpha(x)}D_\mu\psi(x)\,.
\end{aligned}
$$

This means that the Lagrangian

$$
i\bar{\psi}(x)\gamma^\mu D_\mu\psi(x) - m\bar{\psi}(x)\psi(x) = i\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x) - m\bar{\psi}(x)\psi(x) - \mathrm{g}\,\bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x)
$$

is locally gauge invariant. It contains the gauge interaction

$$
\mathcal{L}_{\text{int}}(x) = -\mathrm{g}\,\bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x)\,,
$$

which involves a gauge field that is coupled to a conserved current. Finally we can add the gauge-invariant kinetic term $\mathcal{L}_{\text{Maxwell}}(x) = -\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x)$ for a free gauge field, where the field tensor $F_{\mu\nu}(x)$ is defined as

$$
\begin{aligned}
i\mathrm{g}F_{\mu\nu}(x) \equiv \big[D_\mu\,, D_\nu\big] &= \big[\partial_\mu + i\mathrm{g}A_\mu(x)\big]\big[\partial_\nu + i\mathrm{g}A_\nu(x)\big] - \big[\partial_\nu + i\mathrm{g}A_\nu(x)\big]\big[\partial_\mu + i\mathrm{g}A_\mu(x)\big] \\
&= i\mathrm{g}\big[\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)\big]\,.
\end{aligned}
$$

In conclusion, for $\mathrm{g} = |e|$ we find the same Lagrangian $\mathcal{L}_{\text{QED}}$ as obtained by minimal substitution for a particle with charge $+|e|$. For a general charge $q = Q|e|$ one has to modify the gauge transformation according to $e^{i\alpha(x)} \to e^{iQ\alpha(x)}$ and the covariant derivative according to $D_\mu \to \partial_\mu + iQ|e|A_\mu(x) = \partial_\mu + iqA_\mu(x)$. Such a rescaling leaves the transformation property of the gauge field unaffected, but changes the interaction strength from $|e|$ to q.

Massless gauge fields: a massive gauge field would correspond to an extra mass term $+\frac{1}{2}M_A^2\,A_\mu(x)A^\mu(x)$ in the Lagrangian, which is obviously not gauge invariant. A theory that is manifestly invariant under local gauge transformations requires the gauge bosons described by $A_\mu(x)$ to be massless, i.e. $M_A = 0$. So, in order to give mass to gauge bosons an additional mechanism is required in the context of gauge theories.

13c *Going beyond QED: motivated by the success of describing QED through the gauge principle, this postulate will later on be extended to other types of* *gauge transformations in order to describe other fundamental interactions in nature, i.e. the strong and weak interactions. The associated extended gauge interactions will describe the fundamental interactions between matter fermions as being mediated by gauge bosons, just like we have just worked out for the electromagnetic interactions that are mediated by photons. In order to find the right group structure for the extended gauge transformations, we will be guided by experimental observations of particle interactions and charge conservation laws!*

5.2.1 Quantization of the free electromagnetic theory

The gauge freedom of the electromagnetic vector potential complicates the usual quantization procedure. The reason for this lies in the following observations.

The electromagnetic gauge freedom revisited:

- The gauge freedom for non-constant $\chi(x)$ reflects the redundancy in our description of electromagnetism: the gauge-transformed fields describe the same physics and are therefore to be identified. This can be traced back to the electromagnetic wave equation

$$\Box\, A^{\nu}(x) - \partial^{\nu}\big(\partial_{\mu}A^{\mu}(x)\big) \;=\; (g^{\nu}{}_{\mu}\Box - \partial^{\nu}\partial_{\mu})A^{\mu}(x) \;=\; j_c^{\nu}(x)\ ,$$

  where the differential operator $(g^{\nu}{}_{\mu}\Box - \partial^{\nu}\partial_{\mu})$ is not invertible in the Green's function sense as $(g^{\nu}{}_{\mu}\Box - \partial^{\nu}\partial_{\mu})\partial^{\mu}\chi(x) = 0$ for arbitrary $\chi(x)$. Given an initial field configuration $A^{\mu}(t_0,\vec{x}\,)$ we cannot unambiguously determine $A^{\mu}(t,\vec{x}\,)$, since $A^{\mu}(x)$ and $A^{\mu}(x) + \partial^{\mu}\chi(x)$ are not distinguishable.

  13a *Hence, $A^{\mu}(x)$ is actually not a physical object as it contains redundant information! All fields that are linked by a gauge transformation form an equivalence class and are therefore to be identified: the physics is uniquely described by selecting a representative of each equivalence class. Different configurations of these representatives are called different gauges. By fixing the gauge the redundancy is removed and an unambiguous electromagnetic evolution is obtained. We can choose freely here, but some choices will prove more handy for certain problems than others.*

- By choosing an appropriate $\chi(x)$ it is possible to cast $A_{\mu}(x)$ in such a form that the Coulomb condition $\vec{\nabla}\cdot\vec{A}^{\rm trans}(x) = A_0^{\rm trans}(x) = 0$ is satisfied. In this form we see immediately that $A_{\mu}^{\rm trans}(x)$ has in fact only two physical (transverse) degrees of freedom! These are the degrees of freedom that should be quantized in the corresponding quantum field theory ... however, the Coulomb condition is not Lorentz invariant and therefore leads to Feynman rules that are rather unpleasant.

- Lorentz invariance is manifest, resulting in simple Feynman rules, if we choose $\chi(x)$ such that the Lorenz condition $\partial \cdot A(x) = 0$ is satisfied. In this form we do not see straightaway that $A^{\mu}(x)$ has two physical degrees of freedom. One would expect three physical degrees of freedom in view of the Lorenz condition $\partial \cdot A(x) = 0$, but there is still more gauge freedom left as a result of the gauge transformation $A_{\mu}(x) \to A'_{\mu}(x) = A_{\mu}(x) + \partial_{\mu}\chi'(x)$ with $\Box\chi'(x) = 0$.

Quantized free electromagnetic field: the quantized electromagnetic theory should reproduce the classical Maxwell theory in the classical limit. Due to the correspondence principle this implies that the above-given gauge-fixing conditions are to be implemented as expectation values for physical (asymptotic) Fock states $|\psi\rangle$. As a direct consequence of implementing the Lorenz condition $\langle\psi|\partial \cdot \hat{A}(x)|\psi\rangle = 0$, all relevant components of the electromagnetic potential satisfy the massless KG equation $\Box A_{\mu}(x) = 0$. In the Coulomb gauge we can therefore quantize as in the massless scalar case:

$$\hat{A}^{\text{trans}}_{\mu}(x) \overset{\hat{A}^{\dagger}_{\mu}(x)=\hat{A}_{\mu}(x)}{=\!=\!=} \int \frac{d\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_{r=1}^{2} \left(\hat{a}^{r}_{\vec{p}}\, \epsilon^{r}_{\mu}(p)\, e^{-ip\cdot x} + \hat{a}^{r\,\dagger}_{\vec{p}}\, \epsilon^{r\,*}_{\mu}(p)\, e^{ip\cdot x} \right)\Big|_{p_0 = E_{\vec{p}} = |\vec{p}\,|} ,$$

in terms of the two physical transverse polarization vectors

$$\epsilon^{1}_{0}(p) = \epsilon^{2}_{0}(p) = 0 \quad , \quad \vec{\epsilon}^{\,1}(p) \cdot \vec{p} = \vec{\epsilon}^{\,2}(p) \cdot \vec{p} = 0$$

with normalization condition $\epsilon^{r}(p) \cdot \epsilon^{r'*}(p) = -\,\delta_{rr'}$. The creation and annihilation operators $\hat{a}^{r\dagger}_{\vec{p}}$ and $\hat{a}^{r}_{\vec{p}}$ of the

13d *massless electromagnetic spin-1 energy quanta (photons = antiphotons)*

satisfy the bosonic quantization conditions

$$\left[\hat{a}^{r}_{\vec{p}}\,, \hat{a}^{r'\,\dagger}_{\vec{p}\,'}\right] = (2\pi)^3 \delta_{rr'} \delta(\vec{p} - \vec{p}\,')\hat{1} \qquad \text{and} \qquad \left[\hat{a}^{r}_{\vec{p}}\,, \hat{a}^{r'}_{\vec{p}\,'}\right] = \left[\hat{a}^{r\,\dagger}_{\vec{p}}\,, \hat{a}^{r'\,\dagger}_{\vec{p}\,'}\right] = 0 \ .$$

If we replace the Coulomb condition by the Lorenz condition, the two versions of the electromagnetic field are linked by the identity $\langle\psi|\hat{A}_{\mu}(x)|\psi\rangle = \langle\psi|\hat{A}^{\text{trans}}_{\mu}(x)|\psi\rangle + \partial_{\mu}\chi(x)$, with $\Box\chi(x) = 0$. This identity reflects the remaining gauge arbitrariness of the classical electromagnetic field $\langle\psi|\hat{A}_{\mu}(x)|\psi\rangle$ in the Lorenz gauge.

Feynman propagator and polarization sum: for performing Feynman-diagram calculations we need one more ingredient, the photon propagator. The amplitude for the propagation of photons from y to x reads

$$\langle 0|\hat{A}^{\text{trans}}_{\mu}(x)\hat{A}^{\text{trans}}_{\nu}(y)|0\rangle = \int \frac{d\vec{p}\,d\vec{p}\,'}{(2\pi)^6} \frac{e^{-ip\cdot x + ip'\cdot y}}{2\sqrt{E_{\vec{p}}\,E_{\vec{p}\,'}}} \sum_{r,r'=1}^{2} \epsilon^{r}_{\mu}(p)\epsilon^{r'\,*}_{\nu}(p')\,\langle 0|\hat{a}^{r}_{\vec{p}}\hat{a}^{r'\,\dagger}_{\vec{p}\,'}|0\rangle\Big|_{p_0 = |\vec{p}\,|\,,\,p'_0 = |\vec{p}\,'|}$$

$$= \int \frac{d\vec{p}}{(2\pi)^3} \frac{e^{-ip\cdot(x-y)}}{2E_{\vec{p}}} \sum_{r=1}^{2} \epsilon^{r}_{\mu}(p)\epsilon^{r\,*}_{\nu}(p)\Big|_{p_0 = |\vec{p}\,|} .$$

This expression for the propagation amplitude is rather awkward, since it involves the so-called polarization sum for external (physical) photons:

$$\sum_{r=1}^{2} \epsilon^r_\mu(p)\epsilon^{r\,*}_\nu(p) \;=\; -\,g_{\mu\nu} \;-\; \frac{p_\mu p_\nu}{(n\cdot p)^2} \;+\; \frac{p_\mu n_\nu + n_\mu p_\nu}{n\cdot p}\ ,$$

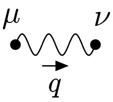
expressed in terms of the temporal unit vector $n_\mu \equiv (1,\vec{0}\,)$. Such a complicated expression is unavoidable for external photons and for the propagator in the Coulomb gauge, but we can exploit the gauge freedom in the Lorenz gauge to remove all terms $\propto\, p_\mu\,,\, p_\nu$. In this so-called 't Hooft-Feynman gauge the photon propagator reduces to

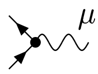
$$\langle 0|T\big(\hat{A}_\mu(x)\hat{A}_\nu(y)\big)|0\rangle \;=\; \int \frac{\mathrm{d}^4p}{(2\pi)^4}\,\frac{-\,ig_{\mu\nu}}{p^2+i\epsilon}\;e^{-ip\cdot(x-y)} \;=\; -\,g_{\mu\nu}\,D_F(x-y;m^2=0)\ .$$

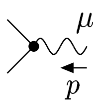
14a *The propagator for internal (virtual) photons has become extremely simple and manifestly Lorentz covariant in the 't Hooft-Feynman gauge!*

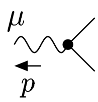
5.3 Feynman rules for QED (§ 4.8 in the book)

In order to obtain the full set of momentum-space Feynman rules for QED we simply have to supplement the Feynman rules for fermions, which were given in the context of the Yukawa theory, by the following four photonic Feynman rules:

1. For each photon propagator  insert $\dfrac{-\,ig_{\mu\nu}}{q^2+i\epsilon}$.

2. For each QED vertex  insert $-\,iq\gamma^\mu$.

3. For each incoming photon line  $=\ \hat{A}_\mu(x)|\vec{p},r\rangle_0$ insert $\epsilon^r_\mu(p)\sqrt{Z_3}\quad (r\!=\!1,2)$.

   For each outgoing photon line  $=\ {}_0\langle\vec{p},r|\hat{A}_\mu(x)$ insert $\epsilon^{r\,*}_\mu(p)\sqrt{Z_3}\quad (r\!=\!1,2)$.

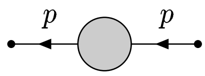
The following remarks are in order. First of all, the polarization vectors featuring in the last two Feynman rules are transverse (physical) ones and $\sqrt{Z_3}$ is the wave-function renormalization factor for photons. Secondly, the sign (direction) of the momentum in the photon propagator does not matter, like in the scalar case. Finally, the γ-matrix occurring in the QED vertex will be wedged between Dirac spinors, with the Dirac indices contracted as usual along the fermion line against the arrow.

14a *Remark: since* $\langle\psi|\hat{A}_\mu(x)|\psi\rangle = \langle\psi|\hat{A}^{trans}_\mu(x)|\psi\rangle + \partial_\mu\chi(x)$ *with* $\Box\chi(x)=0$, *we can always add to* $\epsilon^r_\mu(p)$ *a term* $\propto p_\mu$ *with* $p^2=0$ *without changing the physics outcome (see § 5.5).*

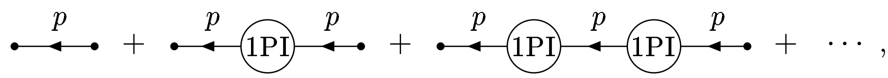
5.4 Full fermion propagator (§ 7.1 in the book)

To all orders in perturbation theory the full fermion propagator in QED is given by the Dyson series

$$\int \mathrm{d}^4x\; e^{ip\cdot x}\, \langle\Omega|T\big(\hat\psi(x)\hat{\bar\psi}(0)\big)|\Omega\rangle \quad \equiv$$

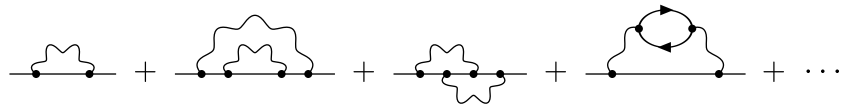


$$=$$



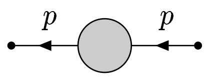
where

$$\equiv\; -\,i\Sigma(\not p) \;=$$



is the collection of all 1-particle irreducible fermion self-energy diagrams. This Dyson series can again be summed up as a geometric series:

$$\int \mathrm{d}^4x\; e^{ip\cdot x}\, \langle\Omega|T\big(\hat\psi(x)\hat{\bar\psi}(0)\big)|\Omega\rangle \quad =$$



$$= \quad \frac{i(\not p+m)}{p^2-m^2+i\epsilon} + \frac{i(\not p+m)}{p^2-m^2+i\epsilon}\Big(-\,i\Sigma(\not p)\Big)\frac{i(\not p+m)}{p^2-m^2+i\epsilon} + \cdots$$

$$= \quad \frac{i}{\not p - m - \Sigma(\not p)} \;\equiv\; S(p)\ ,$$

using that $\Sigma(\not p) = \Sigma_S(p^2)\,m + \Sigma_V(p^2)\,\not p$ commutes with $\not p$ and the mass parameter m in the Lagrangian. The full propagator has a simple pole located at the physical mass m_{ph}, which is shifted away from m by the fermion self-energy:

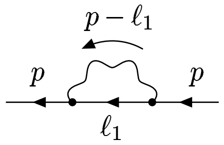
$$\Big[\not p - m - \Sigma(\not p)\Big]\bigg|_{\not p = m_{ph}} = \;0 \quad\Rightarrow\quad m_{ph} - m - \Sigma(\not p = m_{ph}) \;=\; 0\ .$$

Close to this pole the denominator of the full propagator can be expanded according to

$$\not p - m - \Sigma(\not p) \;\approx\; (\not p - m_{ph})\big[1-\Sigma'(\not p = m_{ph})\big] \;+\; \mathcal{O}\big([\not p - m_{ph}]^2\big) \quad \text{for} \quad \not p \approx m_{ph}\ ,$$

where $\Sigma'(\not p)$ stands for the derivative of the fermion self-energy with respect to $\not p$. Just like in the Källén−Lehmann spectral representation, the full propagator has a single-particle pole of the form $iZ_2(\not p + m_{ph})/(p^2 - m_{ph}^2 + i\epsilon)$ with $Z_2 = 1/\big[1-\Sigma'(\not p = m_{ph})\big]$ (see p. 114).

The fermion self-energy: in order to find out whether the fermion self-energy is more difficult to calculate we consider the 1-loop contribution in QED. Indicating the photon mass by λ we then obtain



$$= -i\Sigma_2(\not{p}) = (-iq)^2 \int \frac{\mathrm{d}^4\ell_1}{(2\pi)^4}\, \gamma^\mu \frac{i(\not{\ell}_1+m)}{\ell_1^2-m^2+i\epsilon}\, \gamma^\nu \frac{-ig_{\mu\nu}}{(p-\ell_1)^2-\lambda^2+i\epsilon}\bigg|_{\lambda\downarrow 0}$$

$$= -q^2 \int \frac{\mathrm{d}^4\ell_1}{(2\pi)^4}\, \frac{4m-2\not{\ell}_1}{\big[\ell_1^2-m^2+i\epsilon\big]\big[(p-\ell_1)^2-\lambda^2+i\epsilon\big]}$$

$$\overset{\text{p. 69},\ \ell_1=\ell+\alpha_2 p}{=\!=\!=\!=\!=} -q^2 \int_0^1 \mathrm{d}\alpha_2 \int \frac{\mathrm{d}^4\ell}{(2\pi)^4}\, \frac{4m-2\not{\ell}-2\alpha_2\not{p}}{(\ell^2-\Delta+i\epsilon)^2}$$

$$= -q^2 \int_0^1 \mathrm{d}\alpha_2\, (4m-2\alpha_2\not{p}) \int \frac{\mathrm{d}^4\ell}{(2\pi)^4}\, \frac{1}{(\ell^2-\Delta+i\epsilon)^2}\,,$$

with

$$\Delta = \alpha_2\lambda^2 + (1-\alpha_2)m^2 - \alpha_2(1-\alpha_2)p^2$$

just like in the scalar case. In the second line of this expression we have used that

$$\gamma^\mu(\not{\ell}_1+m)\gamma_\mu = (m-\not{\ell}_1)\gamma^\mu\gamma_\mu + 2\not{\ell}_1 = 4m-2\not{\ell}_1\ .$$

The threshold for the creation of a fermion−photon 2-particle state is here situated at $p^2=(m+\lambda)^2$, which approaches m^2 in the limit $\lambda\downarrow 0$ for massless photons. The rest of the calculation, including the regularization of the UV divergence, goes like in the scalar case worked out in § 2.9.2. Note that the fermion mass receives a UV-divergent shift $\Sigma_2(\not{p}=m_{ph}) \propto m_{ph}\log(\Lambda^2/m_{ph}^2)$.

> 14b *Fermion masses are naturally protected against high-scale quantum corrections: if there would be no coupling between left- and right-handed Dirac fields in the Dirac Lagrangian (i.e. $m=0$), then no such coupling can be induced by the perturbative vector-current QED corrections! Fermion masses are protected by the invariance under chiral transformations of the massless theory.*

5.5 The Ward−Takahashi identity in QED (§ 7.4 in the book)

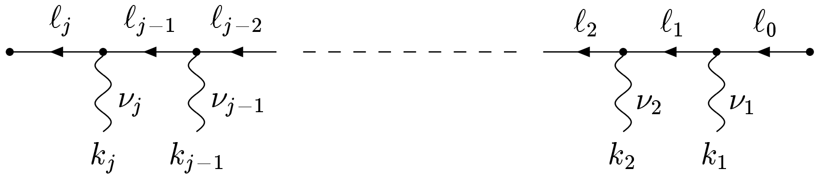
> 14c *Question: how does the gauge invariance of QED manifest itself in Green's functions and scattering amplitudes?*

In order to answer this question, we consider a QED diagram to which we want to attach an additional photon with momentum k. Upon contraction of this photon line with the corresponding momentum k a special identity can be derived that is related to the $U(1)$ gauge symmetry.

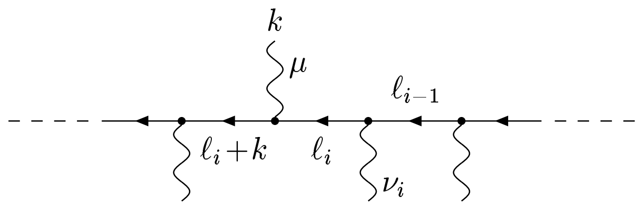
Step 1: how can the photon be attached to an arbitrary diagram involving (anti)fermions and photons?

- The photon cannot be attached to a photon, since it has charge 0.
- The photon can be attached to a fermion line that connects two external points or to a fermion loop.

Step 2: consider an arbitrary fermion line with j photons attached to it and all photon momenta defined to be incoming. Graphically this can be represented by



where $\ell_i = \ell_0 + \sum_{n=1}^{i} k_n$. This line can either flow between external points or close into a loop (which means that $l_0 = l_j$) and the photons can either be on-shell or virtual. There will be $j+1$ places to insert the extra photon with momentum k, for example between photons i and $i+1$:

$k_\mu \, *$ 

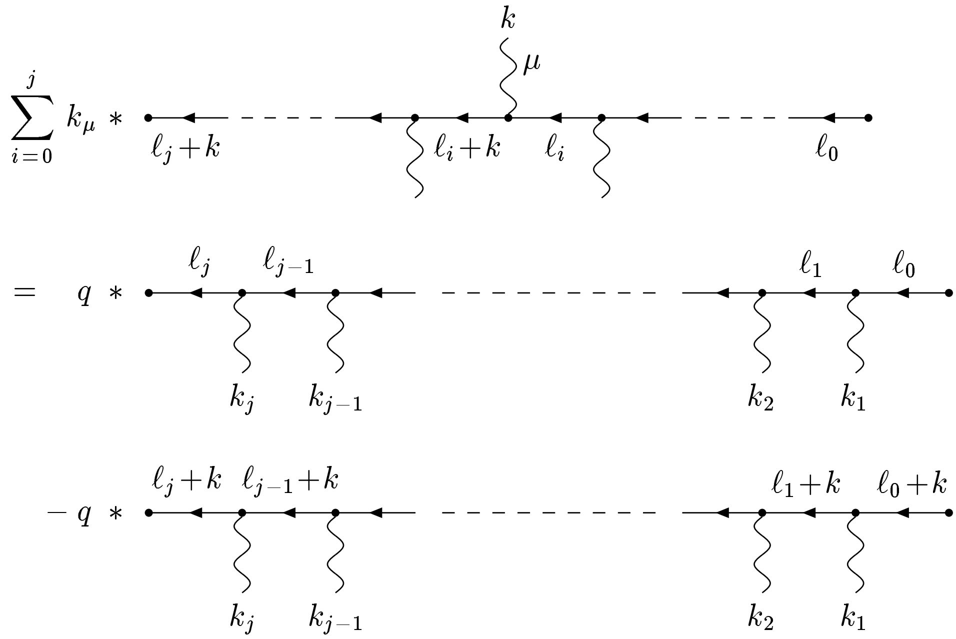
$$= \cdots \Big[\frac{i}{\not{\ell}_i + \not{k} - m} (-iq\not{k}) \frac{i}{\not{\ell}_i - m} (-iq\gamma^{\nu_i}) \frac{i}{\not{\ell}_{i-1} - m} \Big] \cdots$$

$$= \cdots \Big[q \Big(\frac{i}{\not{\ell}_i - m} - \frac{i}{\not{\ell}_i + \not{k} - m} \Big) (-iq\gamma^{\nu_i}) \frac{i}{\not{\ell}_{i-1} - m} \Big] \cdots ,$$

where we have used that $\not{k} = \not{\ell}_i + \not{k} - m - (\not{\ell}_i - m)$ in the last step. Insertion between photons $i-1$ and i gives in a similar way:

$$\cdots \Big[\frac{i}{\not{\ell}_i + \not{k} - m} (-iq\gamma^{\nu_i}) \, q \Big(\frac{i}{\not{\ell}_{i-1} - m} - \frac{i}{\not{\ell}_{i-1} + \not{k} - m} \Big) \Big] \cdots .$$

Note that the second term of the i^{th} insertion cancels the first term of the $(i-1)^{\text{th}}$ insertion. Finally we have to sum over all possible insertions along the fermion line. This causes all terms to cancel pairwise except for two unpaired terms at the very end of the chain:



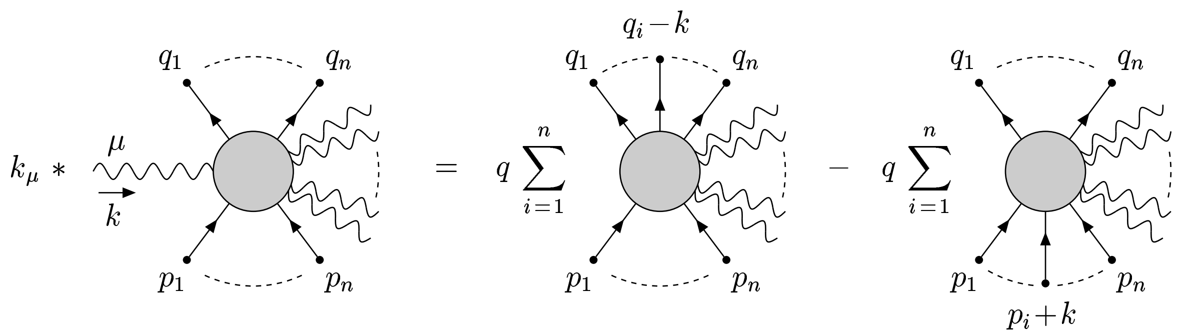
Consequence 1: if the fermion line is part of an on-shell matrix element and connects two of the external states, then the corresponding amputation procedure removes both terms on the right-hand-side. This is caused by the fact that one of the endpoints gives rise to a shifted 1-particle pole, i.e. $1/(\ell_j^2-m^2)$ instead of $1/[(\ell_j+k)^2-m^2]$ or $1/[(\ell_0+k)^2-m^2]$ instead of $1/(\ell_0^2-m^2)$.

Consequence 2: if the fermion line closes in itself to form a loop (i.e. $\ell_0=\ell_j+k$), then the two terms on the right-hand-side give rise to the integrals

$$-\,q^{j+1}\int\frac{\mathrm{d}^4\ell_0}{(2\pi)^4}\Bigg[\mathrm{Tr}\Big(\frac{1}{\not\ell_0-m}\,\gamma^{\nu_j}\,\frac{1}{\not\ell_{j-1}-m}\,\gamma^{\nu_{j-1}}\ \cdots\ \frac{1}{\not\ell_1-m}\,\gamma^{\nu_1}\Big)$$
$$-\,\mathrm{Tr}\Big(\frac{1}{\not\ell_0+\not k-m}\,\gamma^{\nu_j}\,\frac{1}{\not\ell_{j-1}+\not k-m}\,\gamma^{\nu_{j-1}}\ \cdots\ \frac{1}{\not\ell_1+\not k-m}\,\gamma^{\nu_1}\Big)\Bigg]\ =\ 0\ ,$$

if we are allowed to change the integration variable from ℓ_0 to ℓ_0+k in the first term!

Consequence 3: the Ward–Takahashi identity for Green's functions reads

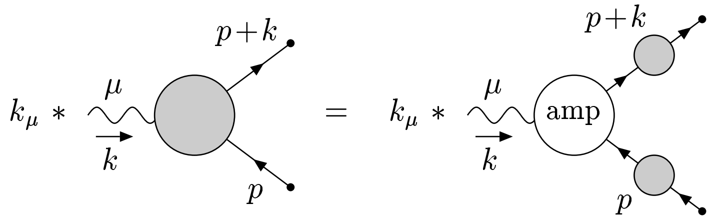


where the blobs represent all possible diagrams and photon insertions. In formula language this can be written compactly as

$$
\begin{aligned}
k_\mu G^\mu(k;p_1,\cdots,p_n;q_1,\cdots,q_n) \;=\; q\sum_i \Big(& G(p_1,\cdots,p_n;q_1,\cdots,q_{i-1},q_i-k,q_{i+1},\cdots,q_n) \\
&- G(p_1,\cdots,p_{i-1},p_i+k,p_{i+1},\cdots,p_n;q_1,\cdots,q_n)\Big) \ .
\end{aligned}
$$

14c *This is the diagrammatic identity that imposes the $U(1)$ gauge symmetry and associated electric charge conservation on quantum mechanical amplitudes!*

Example of a Ward–Takahashi identity:



$$
= \; S(p+k)\Big(-iq\,k_\mu\Gamma^\mu(p+k,p)\Big)S(p) \;\overset{\text{Ward–Takahashi}}{=\!=\!=}\; q\,S(p) - q\,S(p+k)
$$

$$
\Rightarrow \; -ik_\mu\Gamma^\mu(p+k,p) \;=\; S^{-1}(p+k) - S^{-1}(p) \;=\; -i\left[\not{k} + \Sigma(\not{p}) - \Sigma(\not{p}+\not{k})\right] \ .
$$

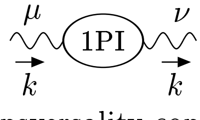
Here $S(p)$ is the full fermion propagator, $\Sigma(\not{p})$ the corresponding 1-particle irreducible self-energy and $-iq\,\Gamma^\mu(p+k,p)$ the sum of all amputated 3-point diagrams contributing to the QED vertex. Hence, $\Gamma^\mu(p+k,p)$ is given by γ^μ at lowest order in perturbation theory, which is indeed in agreement with the Ward–Takahashi identity.

5.6 The photon propagator (§ 7.5 in the book)

The Ward–Takahashi identity has important implications for the properties of the photon propagator.

Transversality: the 1-particle irreducible photon self-energy

$$
i\Pi^{\mu\nu}(k) \;\equiv
$$



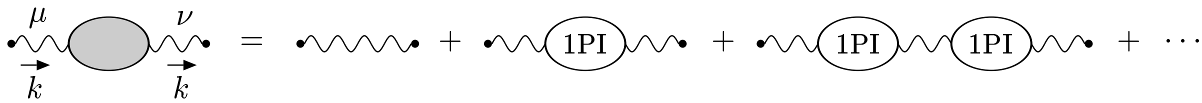
satisfies the Ward–Takahashi identity (transversality condition)

$$
k_\mu\Pi^{\mu\nu}(k) \;=\; 0 \ .
$$

In view of Lorentz covariance $\Pi^{\mu\nu}(k)$ can be decomposed into only two possible terms, a term $\propto g^{\mu\nu}$ and a term $\propto k^\mu k^\nu$. Therefore the Ward–Takahashi identity translates into the condition

$$
\Pi^{\mu\nu}(k) \;=\; (k^2 g^{\mu\nu} - k^\mu k^\nu)\Pi(k^2) \ ,
$$

with $\Pi(k^2)$ regular at $k^2 = 0$ since a pole at $k^2 = 0$ would imply the existence of a single-massless-particle intermediate state. As a result, the full photon propagator is of the form

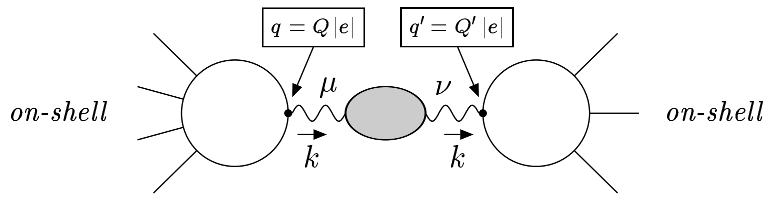


$$= \frac{-ig_{\mu\nu}}{k^2+i\epsilon} + \frac{-ig_{\mu\rho}}{k^2+i\epsilon}\left[i(k^2 g^{\rho\sigma} - k^\rho k^\sigma)\,\Pi(k^2)\right]\frac{-ig_{\sigma\nu}}{k^2+i\epsilon} + \cdots$$

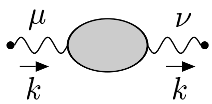
$$= -\frac{i}{k^2+i\epsilon}\,(g_{\mu\nu} - k_\mu k_\nu/k^2)\left[1+\Pi(k^2)+\cdots\right] - \frac{i}{k^2+i\epsilon}\,\frac{k_\mu k_\nu}{k^2}$$

$$= \frac{-i(g_{\mu\nu} - k_\mu k_\nu/k^2)}{(k^2+i\epsilon)\left[1-\Pi(k^2)\right]} - \frac{i}{k^2+i\epsilon}\left(\frac{k_\mu k_\nu}{k^2}\right).$$

Mass of the photon: consider an arbitrary internal photon line



The k_μ and k_ν terms in the full propagator yield a vanishing contribution due to the Ward–Takahashi identity for on-shell amplitudes. Hence,



$$\xrightarrow{\text{effectively}} \quad \frac{-ig_{\mu\nu}}{(k^2+i\epsilon)\left[1-\Pi(k^2)\right]},$$

which has a pole at $k^2 = 0$ with residue $Z_3 \equiv \left[1-\Pi(0)\right]^{-1}$. As a result of the Ward–Takahashi identity, which in turn is a consequence of the gauge symmetry, $m_{\text{photon}} = 0$ to all orders in perturbation theory:

14b + *14c* *the local $U(1)$ gauge symmetry protects the photon from becoming massive through quantum corrections.*

Observable charge: consider the same amplitude as before for

- low $|k^2|$ $\Rightarrow$ $e \to e\sqrt{Z_3}$, which is the finite physically observable charge obtained from the singular quantities e and Z_3;
- high $|k^2|$ $\Rightarrow$ $$\frac{-ig_{\mu\nu}e^2}{k^2} \to \frac{-ig_{\mu\nu}e^2}{k^2\left[1-\Pi(k^2)\right]} = \frac{-ig_{\mu\nu}Z_3 e^2}{k^2\Big(1 - Z_3\left[\Pi(k^2)-\Pi(0)\right]\Big)}$$

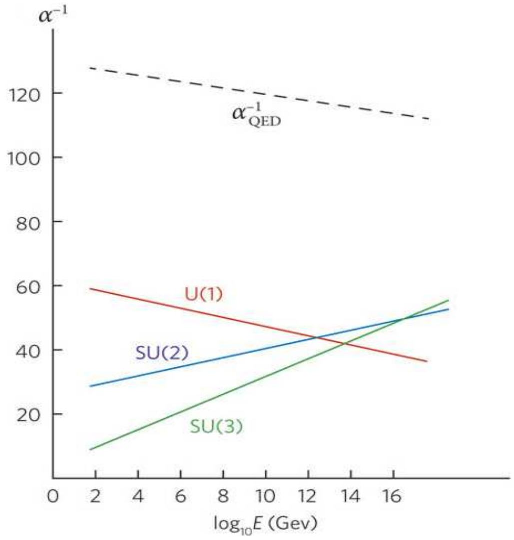
$$\Rightarrow \frac{e^2}{4\pi} = \alpha \to \alpha(k^2) = \frac{Z_3\,\alpha}{1 - Z_3\left[\Pi(k^2)-\Pi(0)\right]},$$

where the factor Z_3 in front of $\big[\Pi(k^2)-\Pi(0)\big]$ turns e^2 inside the photon self-energy into the finite combination $Z_3 e^2$.

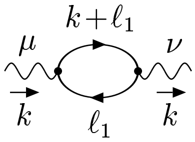
(14d) *The electromagnetic fine structure constant becomes a running coupling, i.e. a coupling that changes with invariant mass. In fact it becomes larger with increasing invariant mass, causing the exchanged (virtual) photon to propagate more easily through spacetime.*

The physical picture behind this is that virtual fermion-antifermion pairs that are created from the vacuum partially screen the charges of the interacting particles (vacuum polarization), resulting in a lower effective charge. For larger $|k^2|$ more of the polarization cloud is penetrated and hence more of the actual charge can be felt.

All couplings in the Standard Model of electroweak interactions are in fact running couplings. As can be seen in the plot, the behaviour of the hypercharge coupling, indicated by U(1), resembles the one for QED. However, due to bosonic loop effects the couplings of the weak interactions, indicated by SU(2), and strong interactions, indicated by SU(3), actually become weaker for increasing invariant mass.



UV divergences: at 1-loop order the photon self-energy in QED is given by



$$i\,\Pi_2^{\mu\nu}(k) \equiv \text{[diagram]} = (-1)\,q^2\int\frac{\mathrm{d}^4\ell_1}{(2\pi)^4}\,\frac{\mathrm{Tr}\big(\gamma^\mu[\not{\ell}_1+m]\gamma^\nu[\not{\ell}_1+\not{k}+m]\big)}{[(\ell_1+k)^2-m^2+i\epsilon][\ell_1^2-m^2+i\epsilon]}$$

$$= -4q^2\int\frac{\mathrm{d}^4\ell_1}{(2\pi)^4}\,\frac{\ell_1^\mu(\ell_1+k)^\nu+\ell_1^\nu(\ell_1+k)^\mu+g^{\mu\nu}[m^2-\ell_1^2-\ell_1\cdot k]}{[(\ell_1+k)^2-m^2+i\epsilon][\ell_1^2-m^2+i\epsilon]}$$

$$\overset{\text{p. 69}}{=} -4q^2\int\frac{\mathrm{d}^4\ell_1}{(2\pi)^4}\int_0^1\mathrm{d}\alpha_2\,\frac{2\,\ell_1^\mu\ell_1^\nu+\ell_1^\mu k^\nu+\ell_1^\nu k^\mu+g^{\mu\nu}[m^2-\ell_1^2-\ell_1\cdot k]}{(\ell_1^2-m^2+2\alpha_2\ell_1\cdot k+\alpha_2k^2+i\epsilon)^2}$$

$$\overset{\ell=\ell_1+\alpha_2k}{=} -4q^2\int_0^1\mathrm{d}\alpha_2\int\frac{\mathrm{d}^4\ell}{(2\pi)^4}\,\frac{2\,\ell^\mu\ell^\nu+g^{\mu\nu}(\Delta-\ell^2)+(k^2g^{\mu\nu}-k^\mu k^\nu)2\alpha_2(1-\alpha_2)}{(\ell^2-\Delta+i\epsilon)^2}$$

$$\overset{\text{p. 70}}{=} -4q^2\,\frac{i}{16\pi^2}\int_0^1\mathrm{d}\alpha_2\int_0^\infty\mathrm{d}\ell_E^2\,\ell_E^2\,\frac{g^{\mu\nu}(\Delta+\ell_E^2/2)+(k^2g^{\mu\nu}-k^\mu k^\nu)2\alpha_2(1-\alpha_2)}{(\ell_E^2+\Delta-i\epsilon)^2}\,,$$

where $\Delta = m^2 - \alpha_2(1-\alpha_2)k^2$. The resulting integral is clearly divergent.

Transversality lost: if we were to regularize (quantify) the UV divergence in the usual way by means of a cutoff Λ, then $\Pi_2^{\mu\nu}(k)$ would contain a leading singularity that is proportional to $g^{\mu\nu}\int_0^{\Lambda^2} \mathrm{d}\ell_E^2 = \Lambda^2 g^{\mu\nu}$. This has disastrous consequences, since it violates the transversality requirement and gives the photon an infinite mass. After all, a $\Lambda^2 g^{\mu\nu}$ term in $\Pi^{\mu\nu}(k)$ gives rise to a Λ^2/k^2 contribution to $\Pi(k^2)$ and therefore shifts the pole of $k^2\big[1-\Pi(k^2)\big]$ away from $k^2 = 0$.

Question: what has happened here?

In fact the fermion-loop Ward – Takahashi identity on p. 129 has been invalidated, since we are actually not allowed to shift the integration variable without consequences when using the cutoff method.

14e *We need another regularization scheme that preserves the fundamental $U(1)$ symmetry, otherwise the results cannot be trusted. Dimensional regularization ('t Hooft – Veltman, 1972): compute Feynman diagrams as analytic functions of the dimensionality of spacetime. Use to this end a d-dimensional Minkowski space consisting of one time dimension and $d-1$ spatial dimensions.*

- *For sufficiently small d any loop integral will converge in the UV domain and the fermion-loop Ward – Takahashi identity is retained for all d.*
- *The final expressions for observables are then obtained as $d \to 4$ limits.*

Examples of integrals calculated with dimensional regularization (DREG):

$$\int \frac{\mathrm{d}^4\ell_E}{(2\pi)^4}\,\frac{1}{(\ell_E^2+\Delta)^2} \xrightarrow{\text{DREG}} \int \frac{\mathrm{d}^d\ell_E}{(2\pi)^d}\,\frac{1}{(\ell_E^2+\Delta)^2} \overset{\text{p. }70,71}{=\!=} \frac{1}{(2\pi)^d}\,\frac{2\pi^{d/2}}{\Gamma(d/2)}\,\frac{1}{2}\int_0^\infty \mathrm{d}\ell_E^2\,\frac{(\ell_E^2)^{d/2-1}}{(\ell_E^2+\Delta)^2}$$

$$\overset{z=\Delta/(\Delta+\ell_E^2)}{=\!=} \frac{\Delta^{d/2-2}}{(4\pi)^{d/2}\,\Gamma(d/2)}\int_0^1 \mathrm{d}z\; z^{1-d/2}(1-z)^{d/2-1} = \frac{\Delta^{d/2-2}}{(4\pi)^{d/2}}\,\Gamma(2-d/2)\,.$$

Here we have used the integral identity

$$\int_0^1 \mathrm{d}z\; z^{b-1}(1-z)^{c-1} = \frac{\Gamma(b)\Gamma(c)}{\Gamma(b+c)}$$

in terms of the gamma function $\Gamma(z)$, which satisfies

$$\Gamma(1/2) = \sqrt{\pi} \quad , \quad \Gamma(1) = 1 \qquad \text{and} \qquad \Gamma(z+1) = z\Gamma(z)\,.$$

This time $\Gamma(2-d/2)$ represents the UV singularities, since the gamma function $\Gamma(z)$ has poles at $z = 0, -1, -2, \cdots$ and therefore $\Gamma(2-d/2)$ has poles at $d = \underline{4}, 6, 8, \cdots$:

$$\Gamma(2-d/2) \overset{d\approx 4}{\approx} -\frac{2}{d-4} - \gamma_E + \mathcal{O}(d-4) \quad \text{with} \quad \gamma_E = 0.5772 = \text{Euler's constant} .$$

In a similar way one finds

$$\int \frac{\mathrm{d}^4\ell_E}{(2\pi)^4} \frac{\ell_E^2}{(\ell_E^2+\Delta)^2} \xrightarrow{\text{DREG}} \frac{\Delta^{d/2-1}}{(4\pi)^{d/2}\,\Gamma(d/2)} \int_0^1 \mathrm{d}z\, z^{-d/2}(1-z)^{d/2} = \frac{d\Delta}{2-d} \int \frac{\mathrm{d}^4\ell_E}{(2\pi)^4} \frac{1}{(\ell_E^2+\Delta)^2} .$$

Transversality restored: returning to the integrand on page 132, we see that the non-transverse term indeed vanishes:

$$2\,\ell^\mu\ell^\nu + g^{\mu\nu}(\Delta - \ell^2) \xrightarrow{\text{DREG}} g^{\mu\nu}\big(\Delta - \ell^2[1-2/d]\big) \xrightarrow{\text{Wick}} g^{\mu\nu}\big(\Delta + \ell_E^2[1-2/d]\big) \xrightarrow{\text{integrals}} 0 ,$$

as required by gauge symmetry. So, dimensional regularization is a viable way of dealing with UV divergences in the context of gauge symmetries. This regularization method was used successfully by 't Hooft and Veltman to prove the renormalizability of the Standard Model of electroweak interactions, for which they were awarded the Nobel Prize in 1999.