Quantum Field Theory Exercises week 12

Exercise 17: Euler-Lagrange equations and some Noether currents in the Dirac theory

(a) Work out the Euler-Lagrange equations for the free Dirac theory in terms of the independent field components $\psi_a^*(x)$ and $\psi_b(x)$ for $a,b = 1,\cdots,4$ being 4-dimensional Dirac spinor labels (which should not be confused with possible spin-1/2 quantum numbers). Compare your results with the vectorial expressions that are derived on p. 96 in the lecture notes.

(b) Consider the global $U(1)$ gauge transformation

$$\psi(x) \;\to\; \psi'(x) \;=\; \mathrm{e}^{i\alpha}\psi(x) \qquad (\alpha \in \mathrm{I\!R}\ \text{constant})\ .$$

Argue that there is symmetry under this transformation and derive the corresponding conserved Noether current. Do this first by using the independent field components ψ_a^*, ψ_b and subsequently try to figure out how to obtain the same result without resorting to explicit indices (i.e. by using vectorial notation).

(c) Now take $m = 0$ and consider the global chiral transformation

$$\psi(x) \;\to\; \psi'(x) \;=\; \mathrm{e}^{i\alpha\gamma^5}\psi(x) \qquad (\alpha \in \mathrm{I\!R}\ \text{constant})\ .$$

- Use exponential series expansions to derive the following identity for γ-matrices:

$$\gamma^\mu \mathrm{e}^{i\alpha\gamma^5} \;=\; \mathrm{e}^{-i\alpha\gamma^5}\gamma^\mu \qquad (\mu = 0,\cdots,3)\ .$$

- Then show that there is symmetry under this transformation.
- Derive the corresponding conserved Noether current.

Exercise 18: Polarisation sum in the Dirac theory

Let's try to reason on general grounds that

$$\sum_{s=1}^{2} u^s(p)\bar{u}^s(p) \;\propto\; \gamma^\mu p_\mu + mI_4 \;=\; \not{p} + mI_4 \qquad \text{for} \qquad p^2 = m^2\ ,$$

where the label s represents an arbitrary choice of spin-1/2 quantum numbers. The u and $\bar{u}$ spinors are 4-dimensional column and row vectors in spinor space, respectively.

(a) The polarisation sum on the left-hand side is a 4×4 matrix in spinor space. Decompose it in terms of the 4×4 basis matrices given on page 92 of the lecture notes with appropriate coefficients. Use the fact that the polarisation sum has no open Minkowski indices and contains no preferred spin vectors to first argue that

$$\sum_{s=1}^{2} u^s(p)\bar{u}^s(p) \;=\; c_1 I_4 + c_2\not{p} + c_4\not{p}\gamma^5 + c_5\gamma^5\ .$$

(b) Subsequently use that $0 = (\not{p} - m)u^s(p) = \bar{u}^s(p)(\not{p} - m)$ to arrive at the above-stated proportionality.

Exercise 19: Parity in the Dirac theory

Apart from the symmetry under continuous Lorentz transformations and translations, in relativistic field theories a free Lagrangian should have a symmetry under the discrete spacetime transformation of parity (spatial inversion):

$$x^\mu \xrightarrow{\mathrm{P}} (\Lambda^P)^\mu{}_\nu x^\nu \qquad \text{with} \qquad (\Lambda^P)^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} .$$

In spinor space the corresponding transformation matrix is given by

$$\Lambda_{1/2}^{(P)} = \exp(i\varphi)\gamma^0 \qquad (\varphi \in \mathbb{R}) .$$

(a) Show that indeed $\Lambda_{1/2}^{(P)\,-1}\gamma^\mu \Lambda_{1/2}^{(P)} = (\Lambda^P)^\mu{}_\nu \gamma^\nu$, in analogy with what has been shown for the continuous Lorentz transformations.

(b) Under a parity transformation

$$\psi(x) \xrightarrow{\mathrm{P}} \Lambda_{1/2}^{(P)} \psi(\tilde{x}) \qquad \text{with} \qquad \tilde{x}^\mu \equiv (x^0, -\vec{x}\,) .$$

Derive from this that

$$\bar{\psi}(x) \xrightarrow{\mathrm{P}} \bar{\psi}(\tilde{x})\,\Lambda_{1/2}^{(P)\,-1}.$$

(c) Under continuous Lorentz transformations the currents $\bar{\psi}(x)\, I_4\, \psi(x)$ and $\bar{\psi}(x)\, i\gamma^5 \psi(x)$ transform like scalars, whereas the currents $\bar{\psi}(x)\gamma^\mu \psi(x)$ and $\bar{\psi}(x)\gamma^\mu\gamma^5 \psi(x)$ transform like vectors. Determine how the two currents in each set differ in their behaviour under a parity transformation, which will allow us to differentiate between a scalar and a so-called pseudo scalar current (first set) and between a vector and a so-called axial vector current (second set).

(d) Use the u and v spinors given in the lecture notes to prove that

$$\gamma^0 u^s(p) = u^s(\tilde{p}) \qquad \text{and} \qquad \gamma^0 v^s(p) = -\,v^s(\tilde{p}) ,$$

with $\tilde{p}^\mu \equiv (p^0, -\vec{p}\,)$. This gives rise to the textbook statement

in the Dirac theory particles and antiparticles have opposite intrinsic parity.