Exercise 16: trace technology for gamma-matrices

The γ-matrices γ^μ for μ = 0, 1, 2, 3 and γ^5 = iγ^0γ^1γ^2γ^3 have the following properties:

1. \{γ^μ, γ^ν\} = 2g^μν I_4 ⇒ \(γ^0)^2 = -(γ^1)^2 = -(γ^2)^2 = -(γ^3)^2 = (γ^5)^2 = I_4\),
2. \{γ^μ, γ^5\} = 0 ,
3. \(γ^μ)^γ = γ^0γ^μγ^0\) and \((γ^5)^γ = γ^5\).

Use these properties to answer a few questions about trace identities.

(a) – Show that for an odd number of γ-matrices \(\text{Tr}(γ^{μ_1} · · · γ^{μ_{2n+1}}) = 0\).
   Hint: multiply the argument of the trace by \((γ^5)^2 = I_4\).
   – Why does that automatically imply that \(\text{Tr}(γ^{μ_1} · · · γ^{μ_{2n+1}}γ^5) = 0\)?

(b) Reason that similar tricks can be applied to prove that \(\text{Tr}(γ^5) = \text{Tr}(γ^μγ^νγ^5) = 0\).

(c) Show that \(\text{Tr}(γ^{μ_1} · · · γ^{μ_{2n}})\) can be expressed as a sum where each term is of the form \(\text{Tr}(2n−2 \text{ γ-matrices}) \times \text{ a metric tensor}\).

(d) Use this method to derive the following trace identities:
   \(\text{Tr}(γ^μγ^ν) = 4g^μν\),
   \(\text{Tr}(γ^μγ^νγ^ργ^σ) = 4(g^μνg^ρσ - g^μρg^νσ + g^μσg^νρ)\).

(e) – Argue that \(\text{Tr}(γ^μγ^νγ^ργ^σγ^5) = 0\) if \((μνρσ)\) ≠ permutation of \((0123)\).
   – Determine \(\text{Tr}(γ^0γ^1γ^2γ^3γ^5)\) and argue that \(\text{Tr}(γ^μγ^νγ^ργ^σγ^5) = -4ǫ^{μνρσ}\) with
   \[ε^{μνρσ} = \begin{cases} +1 & \text{if } (μνρσ) = \text{even permutation of } (0123) \\ -1 & \text{if } (μνρσ) = \text{odd permutation of } (0123) \\ 0 & \text{else.} \end{cases}\]

Exercise 17: Euler-Lagrange equations and some Noether currents in the Dirac theory

(a) Work out the Euler-Lagrange equations for the free Dirac theory in terms of the independent field components \(ψ^*_a(x)\) and \(ψ_b(x)\) for \(a, b = 1, · · · , 4\). Check in this way the correctness of the vectorial expressions that are given in the lecture notes.

(b) Consider the global \(U(1)\) gauge transformation
   \[ψ(x) → ψ'(x) = e^{iα}ψ(x) \quad (α ∈ \mathbb{R}\, \text{constant})\] .
   Argue that there is symmetry under this transformation and derive the corresponding conserved Noether current. Do this first by using the independent field components \(ψ^*_a, ψ_b\) and subsequently try to figure out how to obtain the same result without resorting to explicit indices (i.e. by using vectorial notation).
(c) Now take $m = 0$ and consider the global chiral transformation

$$\psi(x) \rightarrow \psi'(x) = e^{i\alpha \gamma^5} \psi(x) \quad (\alpha \in \mathbb{R} \text{ constant}) .$$

- Use exponential series expansions to derive the following identity for $\gamma$-matrices:

$$\gamma^\mu e^{i\alpha \gamma^5} = e^{-i\alpha \gamma^5} \gamma^\mu \quad (\mu = 0, \cdots, 3) .$$

- Then show that there is symmetry under this transformation.
- Derive the corresponding conserved Noether current.