Quantum Field Theory Exercises week 15

Exercise 19 (continued)
Complete exercise 19.

Exercise 20
Consider the electromagnetic Lagrangian density with *'t Hooft-Feynman* gauge fixing term:

$$\mathcal{L} \;=\; -\,\frac{1}{4}\,F_{\mu\nu}F^{\mu\nu} \,-\, \frac{1}{2}(\partial\cdot A)^2$$

in terms of the electromagnetic field tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Since the Euler-Lagrange equation takes the form of the massless Klein-Gordon equation $\Box A_\mu(x) = 0$, the quantized solution can be written as

$$\hat{A}_\mu(x) \;=\; \hat{A}^{(+)}_\mu(x) + \hat{A}^{(-)}_\mu(x) \;,$$

$$\hat{A}^{(+)}_\mu(x) \;=\; \int \frac{d\vec{p}}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\vec{p}}}}\,\sum_{r=0}^{3} \hat{a}^r_{\vec{p}}\,\epsilon^r_\mu(p)\,e^{-ip\cdot x}\Big|_{p_0=E_{\vec{p}}=|\vec{p}\,|} \;,$$

$$\hat{A}^{(-)}_\mu(x) \;=\; \int \frac{d\vec{p}}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\vec{p}}}}\,\sum_{r=0}^{3} \hat{a}^{r\dagger}_{\vec{p}}\,\epsilon^{r*}_\mu(p)\,e^{ip\cdot x}\Big|_{p_0=E_{\vec{p}}=|\vec{p}\,|} \;.$$

The creation and annihilation operators $\hat{a}^{r\dagger}_{\vec{p}}$ and $\hat{a}^r_{\vec{p}}$ of the massless electromagnetic energy quanta (photons = antiphotons) satisfy the bosonic quantization conditions

$$\big[\hat{a}^r_{\vec{p}}\,,\hat{a}^{r'\,\dagger}_{\vec{p}\,'}\big] \;=\; -\,(2\pi)^3 g^{rr'}\delta(\vec{p}-\vec{p}\,')\hat{1} \qquad \text{and} \qquad \big[\hat{a}^r_{\vec{p}}\,,\hat{a}^{r'}_{\vec{p}\,'}\big] \;=\; \big[\hat{a}^{r\,\dagger}_{\vec{p}}\,,\hat{a}^{r'\,\dagger}_{\vec{p}\,'}\big] \;=\; 0 \;,$$

The polarization vectors $\epsilon^r_\mu(p)$, with $r = 0,\cdots,3$ span Minkowski space:

$$r=0 \quad\to\quad \epsilon^0_0(p) = 1 \;\;,\;\; \vec{\epsilon}^{\,0}(p) = \vec{0} \qquad \text{(scalar polarization)} \qquad \Rightarrow \quad \epsilon^0(p)\cdot p = E_{\vec{p}} \;,$$

$$r=1,2 \quad\to\quad \epsilon^r_0(p) = 0 \;\;,\;\; \vec{\epsilon}^{\,r}(p)\cdot\vec{p} = 0 \quad \text{(transverse polarization)} \qquad \Rightarrow \quad \epsilon^r(p)\cdot p = 0 \;,$$

$$r=3 \quad\to\quad \epsilon^3_0(p) = 0 \;\;,\;\; \vec{\epsilon}^{\,3}(p) = \frac{\vec{p}}{E_{\vec{p}}} \quad \text{(longitudinal polarization)} \qquad \Rightarrow \quad \epsilon^3(p)\cdot p = -E_{\vec{p}} \;,$$

with the normalization condition $\epsilon^r(p)\cdot\epsilon^{r'*}(p) = g^{rr'}$.

The weird minus sign in one of the commutation relations (i.e. for $r = r' = 0$) is a consequence of the wrong sign of the kinetic term for the unphysical scalar A^0-field in the Lagrangian density:

$$\mathcal{L}(x) \;\xrightarrow{\text{kinetic terms}}\; \frac{1}{2}\dot{\vec{A}}^{\,2} - \frac{1}{2}\dot{A}_0^2 \;.$$

The energy spectrum thus seems unbounded from below. However, that is caused by unphysical modes of the theory! So, let's try to decouple the unphysical (non-transverse) asymptotic Fock states from physical ones. To this end we want to select physical states $|\psi\rangle$ in Fock space such that these states implement the Lorenz condition (gauge choice) $\partial\cdot A = 0$ at the quantum level. This selection procedure should, however, be consistent with the canonical quantization condition

$$\big[\hat{A}_\mu(\vec{x},t),\hat{\pi}_\nu(\vec{y},t)\big] \;=\; ig_{\mu\nu}\delta(\vec{x}-\vec{y}\,)\hat{1} \quad \text{with in particular} \quad \hat{\pi}_0(\vec{y},t) = -\,\partial\cdot\hat{A}(\vec{y},t) \;.$$

(a) Why does this exclude the implementation of the Lorenz condition at the operator level, $\partial\cdot\hat{A} = 0$, or restricted to physical states, $\partial\cdot\hat{A}\,|\psi\rangle = 0$?

(b) We resort to the weaker condition $\partial \cdot \hat{A}^{(+)}\,|\psi\rangle = 0$. Show that the Lorenz condition has been implemented as $\langle\psi|\partial \cdot \hat{A}|\psi\rangle = 0$ in this way, i.e. as an expectation value.

(c) Use the definition of the polarization vectors $\epsilon^r_\mu(p)$ to derive that the condition $\partial{\cdot}\hat{A}^{(+)}\,|\psi\rangle = 0$ is equivalent to $(\hat{a}^0_{\vec{p}} - \hat{a}^3_{\vec{p}})\,|\psi\rangle = 0$ for arbitrary photon momentum $\vec{p}$.

To quantify what this implies we write the physical states $|\psi\rangle$ as

$$|\psi\rangle \;\equiv\; |\psi_T\rangle|\phi\rangle \;=\; |\psi_T\rangle\big[\,|\phi_0\rangle + c_1\,|\phi_1\rangle + c_2\,|\phi_2\rangle + \cdots\big] \qquad (c_1, c_2, \cdots \in \mathbb{C})\;.$$

- $|\psi_T\rangle$ is a normalized physical state containing transverse photons only: $\langle\psi_T|\psi_T\rangle = 1$ and $\hat{a}^0_{\vec{p}}|\psi_T\rangle = \hat{a}^3_{\vec{p}}|\psi_T\rangle = 0$.
- $|\phi\rangle$ is a state containing the right mix of scalar and longitudinal photons, without involving any transverse photons: $(\hat{a}^0_{\vec{p}} - \hat{a}^3_{\vec{p}})\,|\phi\rangle = 0$, $\hat{a}^1_{\vec{p}}|\phi\rangle = \hat{a}^2_{\vec{p}}|\phi\rangle = 0$.
- $|\phi_n\rangle$ contains n scalar or longitudinal photons.
- $|\phi_0\rangle$ is the normalized scalar/longitudinal vacuum state: $\langle\phi_0|\phi_0\rangle = 1$.

(d) Consider the number operator for scalar and longitudinal photons:

$$\hat{N}_{SL} \;=\; \int \frac{d\vec{p}}{(2\pi)^3}\,(\hat{a}^{3\dagger}_{\vec{p}}\,\hat{a}^3_{\vec{p}} - \hat{a}^{0\dagger}_{\vec{p}}\,\hat{a}^0_{\vec{p}})\;.$$

- Use the commutation relations for $\hat{a}^{r\dagger}_{\vec{p}}$ and $\hat{a}^r_{\vec{p}}$ to show that the state

$$|1_s\rangle \;\equiv\; \int \frac{d\vec{p}}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\vec{p}}}}\,f(\vec{p})\,\hat{a}^{0\dagger}_{\vec{p}}|0\rangle\;,$$

  with f some complex-valued function, contains one such photon.
- Use $\hat{N}_{SL}$ to prove that $n\langle\phi_n|\phi_n\rangle \;=\; 0 \;\Rightarrow\; \langle\phi_n|\phi_n\rangle \;=\; 0$ if $n \geq 1$.

(e) Use the form of the Hamiltonian:

$$N(\hat{H}) \;=\; \int \frac{d\vec{p}}{(2\pi)^3}\,E_{\vec{p}}\left(\sum_{r=1}^{3}\,\hat{a}^{r\dagger}_{\vec{p}}\,\hat{a}^r_{\vec{p}} - \hat{a}^{0\dagger}_{\vec{p}}\,\hat{a}^0_{\vec{p}}\right)$$

to derive that

$$\langle\psi|N(\hat{H})|\psi\rangle \;=\; \int \frac{d\vec{p}}{(2\pi)^3}\,E_{\vec{p}}\,\sum_{r=1}^{2}\,\langle\psi_T|\hat{a}^{r\dagger}_{\vec{p}}\,\hat{a}^r_{\vec{p}}|\psi_T\rangle \;\geq\; 0\;.$$

Note again the wrong-sign contribution of the unphysical $r = 0$ modes to the Hamiltonian, which reflects the wrong sign of the corresponding kinetic term. In the physical energy expectation values, however, this contribution exactly cancels the contribution from the unphysical $r = 3$ modes!

(f) Show that $\langle\psi|\hat{A}_\mu(x)|\psi\rangle \;=\; \langle\psi_T|\hat{A}^{(tr)}_\mu(x)|\psi_T\rangle \;+\; \partial_\mu\chi(x)$, with $\Box\chi(x) \;=\; 0$. Here $\hat{A}^{(tr)}(x)$ contains transverse modes only and $\partial_\mu\chi(x)$ represents the contribution from scalar and longitudinal modes. Hint: first prove the relation $E_{\vec{p}}\,\big(\,\epsilon^0_\mu(p) + \epsilon^3_\mu(p)\,\big) = p_\mu$.

(g) Why can the physical photon states be chosen as $|\psi\rangle \;=\; |\psi_T\rangle|\phi_0\rangle$, consisting of transverse photons only?