Quantum Field Theory Exercises week 16

Exercise 20
Consider the electromagnetic Lagrangian density with *'t Hooft-Feynman* gauge fixing term:

$$\mathcal{L} \;=\; -\frac{1}{4}\,F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial\cdot A)^2$$

in terms of the electromagnetic field tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Since the Euler-Lagrange equation takes the form of the massless Klein-Gordon equation $\Box A_\mu(x) = 0$, the quantized solution can be written as

$$\hat{A}_\mu(x) \;=\; \hat{A}^{(+)}_\mu(x) + \hat{A}^{(-)}_\mu(x)\;,$$

$$\hat{A}^{(+)}_\mu(x) \;=\int \frac{d\vec{p}}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\vec{p}}}}\sum_{r=0}^{3}\hat{a}^r_{\vec{p}}\,\epsilon^r_\mu(p)\,e^{-ip\cdot x}\Big|_{p_0=E_{\vec{p}}=|\vec{p}\,|}\;,$$

$$\hat{A}^{(-)}_\mu(x) \;=\int \frac{d\vec{p}}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\vec{p}}}}\sum_{r=0}^{3}\hat{a}^{r\dagger}_{\vec{p}}\,\epsilon^{r*}_\mu(p)\,e^{ip\cdot x}\Big|_{p_0=E_{\vec{p}}=|\vec{p}\,|}\;.$$

The polarization vectors $\epsilon^r_\mu(p)$, with $r = 0,\cdots,3$ span Minkowski space:

$$r=0 \;\to\; \epsilon^0_0(p)=1\;,\;\; \vec{\epsilon}^{\,0}(p)=\vec{0} \qquad \text{(scalar polarization)} \qquad \Rightarrow\; \epsilon^0(p)\cdot p = E_{\vec{p}}\;,$$

$$r=1,2 \;\to\; \epsilon^r_0(p)=0\;,\;\; \vec{\epsilon}^{\,r}(p)\cdot\vec{p}=0 \qquad \text{(transverse polarization)} \qquad \Rightarrow\; \epsilon^r(p)\cdot p = 0\;,$$

$$r=3 \;\to\; \epsilon^3_0(p)=0\;,\;\; \vec{\epsilon}^{\,3}(p)=\frac{\vec{p}}{E_{\vec{p}}} \qquad \text{(longitudinal polarization)} \qquad \Rightarrow\; \epsilon^3(p)\cdot p = -E_{\vec{p}}\;,$$

with the normalization condition $\epsilon^r(p)\cdot\epsilon^{r'*}(p) = g^{rr'}$. We want to select physical states $|\psi\rangle$ in Fock space such that these states implement the Lorenz condition. This selection procedure should, however, be consistent with the canonical quantization condition

$$\big[\hat{A}_\mu(\vec{x},t),\hat{\pi}_\nu(\vec{y},t)\big] \;=\; ig_{\mu\nu}\delta(\vec{x}-\vec{y}\,)\hat{1} \quad \text{with in particular} \quad \hat{\pi}_0(\vec{y},t) = -\,\partial\cdot\hat{A}(\vec{y},t)\;.$$

(a) Why does this exclude the implementation of the Lorenz condition at the operator level, $\partial\cdot\hat{A}=0$, or restricted to physical states, $\partial\cdot\hat{A}|\psi\rangle = 0$?

(b) We resort to the weaker condition $\partial\cdot\hat{A}^{(+)}|\psi\rangle = 0$. Show that the Lorenz condition has been implemented as $\langle\psi|\partial\cdot\hat{A}|\psi\rangle = 0$ in this way, i.e. as an expectation value.

(c) Use the definition of the polarization vectors $\epsilon^r_\mu(p)$ to derive that the condition $\partial\cdot\hat{A}^{(+)}|\psi\rangle = 0$ is equivalent to $(\hat{a}^0_{\vec{p}} - \hat{a}^3_{\vec{p}})|\psi\rangle = 0$ for arbitrary photon momentum $\vec{p}$.

To quantify what this implies we write the physical states $|\psi\rangle$ as

$$|\psi\rangle \;\equiv\; |\psi_T\rangle|\phi\rangle \;=\; |\psi_T\rangle\big[\,|\phi_0\rangle + c_1|\phi_1\rangle + c_2|\phi_2\rangle + \cdots\big] \qquad (c_1,c_2,\cdots\in\mathbb{C})\;.$$

- $|\psi_T\rangle$ is a normalized physical state containing transverse photons only: $\langle\psi_T|\psi_T\rangle = 1$ and $\hat{a}^0_{\vec{p}}|\psi_T\rangle = \hat{a}^3_{\vec{p}}|\psi_T\rangle = 0$.
- $|\phi\rangle$ is a state containing the right mix of scalar and longitudinal photons, without involving any transverse photons: $(\hat{a}^0_{\vec{p}} - \hat{a}^3_{\vec{p}})|\phi\rangle = 0$, $\hat{a}^1_{\vec{p}}|\phi\rangle = \hat{a}^2_{\vec{p}}|\phi\rangle = 0$.
- $|\phi_n\rangle$ contains n scalar or longitudinal photons.
- $|\phi_0\rangle$ is the normalized scalar/longitudinal vacuum state: $\langle\phi_0|\phi_0\rangle = 1$.

(d) Consider the number operator for scalar and longitudinal photons:

$$\hat{N}_{SL} = \int \frac{d\vec{p}}{(2\pi)^3} \left(\hat{a}^{3\dagger}_{\vec{p}}\,\hat{a}^{3}_{\vec{p}} - \hat{a}^{0\dagger}_{\vec{p}}\,\hat{a}^{0}_{\vec{p}}\right) .$$

– Use the commutation relations for $\hat{a}^{r\dagger}_{\vec{p}}$ and $\hat{a}^{r}_{\vec{p}}$ to show that the state

$$|1_s\rangle \equiv \int \frac{d\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} f(\vec{p})\,\hat{a}^{0\dagger}_{\vec{p}}|0\rangle ,$$

with f some complex-valued function, contains one such photon.

– Use $\hat{N}_{SL}$ to prove that $n\langle\phi_n|\phi_n\rangle = 0 \Rightarrow \langle\phi_n|\phi_n\rangle = 0$ if $n \geq 1$.

(e) Use the form of the Hamiltonian:

$$N(\hat{H}) = \int \frac{d\vec{p}}{(2\pi)^3} E_{\vec{p}} \left(\sum_{r=1}^{3} \hat{a}^{r\dagger}_{\vec{p}}\,\hat{a}^{r}_{\vec{p}} - \hat{a}^{0\dagger}_{\vec{p}}\,\hat{a}^{0}_{\vec{p}} \right)$$

to derive that

$$\langle\psi|N(\hat{H})|\psi\rangle = \int \frac{d\vec{p}}{(2\pi)^3} E_{\vec{p}} \sum_{r=1}^{2} \langle\psi_T|\hat{a}^{r\dagger}_{\vec{p}}\hat{a}^{r}_{\vec{p}}|\psi_T\rangle \geq 0 .$$

(f) Show that $\langle\psi|\hat{A}_\mu(x)|\psi\rangle = \langle\psi_T|\hat{A}^{(tr)}_\mu(x)|\psi_T\rangle + \partial_\mu\chi(x)$, with $\Box\chi(x) = 0$. Here $\hat{A}^{(tr)}(x)$ contains transverse modes only and $\partial_\mu\chi(x)$ represents the contribution from scalar and longitudinal modes. Hint: first prove the relation $E_{\vec{p}}\left(\epsilon^0_\mu(p) + \epsilon^3_\mu(p)\right) = p_\mu$.

(g) Why can the physical photon states be chosen as $|\psi\rangle = |\psi_T\rangle|\phi_0\rangle$, consisting of transverse photons only?