Quantum Field Theory Exercises week 5

Exercise 5: The complex KG theory (a few loose ends)

Consider the quantized free Klein-Gordon theory for a complex scalar field $\phi(x)$ with mass m.

(a) Show that the Feynman propagator $D_F(x-y) \equiv \langle 0|T\big(\hat{\phi}(x)\hat{\phi}^\dagger(y)\big)|0\rangle$ can be written as

$$D_F(x-y) \;=\; \int \frac{d^4p}{(2\pi)^4}\,\frac{i}{p^2-m^2+i\epsilon}\,e^{-ip\cdot(x-y)} \qquad\qquad (\epsilon>0 \text{ infinitesimal})\ .$$

This proves the statement made at the bottom of page 25 of the lecture notes.

(b) – Argue mathematically that $\langle 0|T\big(\hat{\phi}(x)\hat{\phi}(y)\big)|0\rangle = \langle 0|T\big(\hat{\phi}^\dagger(x)\hat{\phi}^\dagger(y)\big)|0\rangle = 0$.
– Also give the physical argumentation, based on the creation and annihilation of particles.

(c) Show that for infinitesimal values of $\Delta t \in \mathbb{R}$ the transformation $e^{i\hat{H}\Delta t}\hat{\phi}(x)e^{-i\hat{H}\Delta t}$ is equivalent to an infinitesimal time translation of the scalar field $\hat{\phi}(x)$.

Exercise 6: Time-ordered exponentials

This is a slightly modified version of exercise 2.9 from ‘An introduction to quantum field theory’ by George Sterman. In the lecture notes time ordering has been defined for time-dependent operators. We will now look at a generalization of this: for $t \geq \tau$ the so-called ‘time-ordered exponential’ of a time-dependent operator $\hat{A}(t)$ is defined as

$$\hat{E}(t,\tau) \;\equiv\; \hat{1} + \int_\tau^t dt_1\,\hat{A}(t_1) + \int_\tau^t dt_1\,\hat{A}(t_1)\int_\tau^{t_1} dt_2\,\hat{A}(t_2) \,+\, \cdots\ .$$

(a) Prove that $\hat{E}(t,\tau)$ is a solution to the linear differential equation $\partial\hat{E}(t,\tau)/\partial t = \hat{A}(t)\hat{E}(t,\tau)$ with boundary condition $\hat{E}(\tau,\tau) = \hat{1}$.

(b) Show that the time-ordered operator

$$T\Big(e^{\int_\tau^t dt'\,\hat{A}(t')}\Big) \;\equiv\; \sum_{n=0}^{\infty}\frac{1}{n!}\int_\tau^t dt_1\cdots\int_\tau^t dt_n\,T\big(\hat{A}(t_1)\cdots\hat{A}(t_n)\big)$$

satisfies the same linear differential equation and boundary condition. This proves the identity

$$\hat{E}(t,\tau) \;=\; T\Big(e^{\int_\tau^t dt'\,\hat{A}(t')}\Big)$$

that was claimed on page 34 of the lecture notes.

(c) Assume that $\big[\hat{A}(t_j),\hat{A}(t_k)\big] = 0$ for all times t_j and t_k. Show that $\hat{E}(t,\tau)$ reduces to an ordinary exponential in that case.