Quantum Field Theory Exercises week 5

Exercise 5: The complex KG theory (a few loose ends)

Consider the quantized free Klein-Gordon theory for a complex scalar field \( \phi(x) \) with mass \( m \).

(a) Show that the Feynman propagator \( D_F(x - y) \equiv \langle 0|T(\hat{\phi}(x)\hat{\phi}(y))|0 \rangle \) can be written as

\[
D_F(x - y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip \cdot (x - y)} \quad (\epsilon > 0 \text{ infinitesimal}).
\]

This proves the statement made on page 24 of the lecture notes just above figure 3.

(b) Argue mathematically that \( \langle 0|T(\hat{\phi}(x)\hat{\phi}(y))|0 \rangle = \langle 0|T(\hat{\phi}(x)\hat{\phi}(y))|0 \rangle = 0. \)

Also give the physical argumentation, based on the creation and annihilation of particles.

(c) Show that for infinitesimal values of \( \Delta t \in \mathbb{R} \) the transformation \( e^{i\hat{H}\Delta t}\hat{\phi}(x)e^{-i\hat{H}\Delta t} \) is equivalent to an infinitesimal time translation of the scalar field \( \hat{\phi}(x) \).

Exercise 6: Time-ordered exponentials

This is a slightly modified version of exercise 2.9 from ‘An introduction to quantum field theory’ by George Sterman. In the lecture notes time ordering has been defined for time-dependent operators. We will now look at a generalization of this: for \( t \geq \tau \) the so-called ‘time-ordered exponential’ of a time-dependent operator \( \hat{A}(t) \) is defined as

\[
\hat{E}(t, \tau) \equiv \hat{1} + \int_{\tau}^{t} dt_1 \hat{A}(t_1) + \int_{\tau}^{t} dt_1 \hat{A}(t_1) \int_{\tau}^{t_1} dt_2 \hat{A}(t_2) + \cdots .
\]

(a) Prove that \( \hat{E}(t, \tau) \) is a solution to the linear differential equation \( \partial \hat{E}(t, \tau)/\partial t = \hat{A}(t)\hat{E}(t, \tau) \) with boundary condition \( \hat{E}(\tau, \tau) = 1. \)

(b) Show that the time-ordered operator

\[
T \left( e \int_{\tau}^{t} dt' \hat{A}(t') \right) \equiv \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\tau}^{t} dt_1 \cdots \int_{\tau}^{t} dt_n T(\hat{A}(t_1) \cdots \hat{A}(t_n))
\]

satisfies the same linear differential equation and boundary condition. This proves the identity

\[
\hat{E}(t, \tau) = T \left( e \int_{\tau}^{t} dt' \hat{A}(t') \right)
\]

that was claimed on page 33 of the lecture notes.

(c) Assume that \( [\hat{A}(t_j), \hat{A}(t_k)] = 0 \) for all times \( t_j \) and \( t_k \). Show that \( \hat{E}(t, \tau) \) reduces to an ordinary exponential in that case.