Quantum Field Theory Exercises week 6

Exercise 7: Feynman rules for the scalar Yukawa theory + arrow convention

Take the Lagrangian for the scalar Yukawa theory:

$$\mathcal{L} \;=\; (\partial_\mu\psi^*)(\partial^\mu\psi) \,+\, \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) \,-\, M^2\psi^*\psi \,-\, \frac{1}{2}m^2\phi^2 \,-\, g\psi^*\psi\phi \;,$$

with $\phi \in \mathbb{R}$ and $\psi \in \mathbb{C}$.

(a) Which term in the Lagrangian gives rise to an interaction?

(b) Calculate up to first order in g:

$$\langle 0|\,T\Big(\hat{\psi}_I^\dagger(x_1)\hat{\phi}_I(x_2)\hat{\psi}_I(x_3)\,e^{-i\int d^4x\,\hat{\mathcal{H}}_I(x)}\Big)|0\rangle \;,$$

indicating the Feynman propagator for the ψ field by

$$\langle 0|\,T\big(\hat{\psi}_I(x)\hat{\psi}_I^\dagger(y)\big)|0\rangle \;=\; D_F(x-y;M^2) \;=\; \int\frac{d^4p}{(2\pi)^4}\,\frac{i\,e^{-ip\cdot(x-y)}}{p^2-M^2+i\epsilon} \qquad (\epsilon>0 \text{ infinitesimal})$$

and the one for the ϕ field by $D_F(x-y;m^2)$.

Hint: use that $\langle 0|\,T\big(\hat{\psi}_I(x)\hat{\psi}_I(y)\big)|0\rangle = \langle 0|\,T\big(\hat{\psi}_I^\dagger(x)\hat{\psi}_I^\dagger(y)\big)|0\rangle = 0$ according to exercise 5(b).

(c) Write the answer in part (b) in terms of Feynman diagrams. Use a solid line for the ψ propagators and a dotted line for the ϕ propagators. Use the following extra drawing convention: draw an arrow on ψ propagators, representing the direction of particle-number flow. Hence, the ψ-particles are defined to flow along the arrow, whereas ψ-antiparticles flow against it!

Explain in this context why the operator $\hat{\psi}_I(x)$ corresponds to an arrow that flows into the external/internal point x, whereas $\hat{\psi}_I^\dagger(x)$ corresponds to an arrow that flows out of the external/internal point x.

(d) – Extract the Feynman rules in position space.
– Why do we not have to worry about symmetry factors?

(e) Translate these Feynman rules to the momentum representation.

Exercise 8: More on the arrow convention for particles and antiparticles

In exercises 5 and 7 you have seen various aspects of particle flow (arrows) in the scalar Yukawa theory, pertaining to internal/external points as well as propagators. Use these aspects to explain why the arrows in the associated Feynman diagrams link up to form a continuous flow.

What conservation law is actually causing this phenomenon?

Exercise 9: dealing with matrix elements in the scalar Yukawa theory

Consider the scalar Yukawa theory, for which you have derived some Feynman rules in exercise 7.

(a) Derive the additional momentum-space Feynman rules that are needed in order to calculate amplitudes in this theory. Explain in this context why you should treat ψ-particles and $\bar{\psi}$-particles (i.e. ψ-antiparticles) separately.

Use the same drawing convention as introduced in exercise 7: draw an arrow on ψ and $\bar{\psi}$ lines, representing the direction of particle-number flow. Hence, the ψ-particles are defined to flow along the arrow, whereas $\bar{\psi}$-particles flow against it! Use solid lines to indicate $\psi/\bar{\psi}$-particles and dashed lines for the ϕ-particles.

If you have derived the additional Feynman rules correctly you should have found that $\hat{\psi}_I(x)$ corresponds to an arrow that flows into the interaction vertex, whereas $\hat{\psi}_I^\dagger(x)$ corresponds to an arrow that flows out of the interaction vertex!

(b) Use these Feynman rules to calculate the amplitude for the decay process

$$\phi(k_A) \;\to\; \psi(p_1)\bar{\psi}(p_2)$$

to lowest non-vanishing order, which is usually referred to as the lowest-order amplitude. To this end you may use that at lowest non-vanishing order $i\mathcal{M}$ = sum of all fully connected amputated Feynman diagrams in momentum space.

Compare to page 36 of the lecture notes and judge for yourself whether you like Feynman rules or not.

(c) Calculate the lowest-order amplitudes for the following scattering processes:

$$\psi(k_A)\psi(k_B) \;\to\; \psi(p_1)\psi(p_2)\;,$$

$$\psi(k_A)\bar{\psi}(k_B) \;\to\; \phi(p_1)\phi(p_2)\;,$$

$$\psi(k_A)\bar{\psi}(k_B) \;\to\; \psi(p_1)\bar{\psi}(p_2)\;,$$

where k_A and k_B are the momenta of the initial-state particles and p_1 and p_2 the momenta of the final-state particles. Compare the matrix elements for the first and third process, and indicate how the third matrix element can be obtained directly from the first one. This way of switching from particles in the initial/final state to antiparticles in the final/initial state (and vice versa) is called crossing.

(d) Explain why the lowest-order amplitude vanishes for the scattering process

$$\psi(k_A)\psi(k_B) \;\to\; \phi(p_1)\phi(p_2)\;.$$