Quantum Field Theory Exercises week 7

Exercise 9: dealing with matrix elements in the scalar Yukawa theory

Consider the scalar Yukawa theory, for which you have derived some Feynman rules in exercise 7.

(a) Derive the additional momentum-space Feynman rules that are needed in order to calculate amplitudes in this theory. Explain in this context why you should treat $\psi$-particles and $\bar{\psi}$-particles (i.e. $\psi$-antiparticles) separately.

Use the same drawing convention as introduced in exercise 7: draw an arrow on $\psi$ and $\bar{\psi}$ lines, representing the direction of particle-number flow. Hence, the $\psi$-particles are defined to flow along the arrow, whereas $\bar{\psi}$-particles flow against it! Use solid lines to indicate $\psi/\bar{\psi}$-particles and dashed lines for the $\phi$-particles.

If you have derived the additional Feynman rules correctly you should have found that $\hat{\psi}_I(x)$ corresponds to an arrow that flows into the interaction vertex, whereas $\hat{\bar{\psi}}_I(x)$ corresponds to an arrow that flows out of the interaction vertex!

(b) Use these Feynman rules to calculate the amplitude for the decay process

$$\phi(p) \to \psi(q_1)\bar{\psi}(q_2)$$

to lowest non-vanishing order, which is usually referred to as the lowest-order amplitude. To this end you may use that at lowest non-vanishing order $i\mathcal{M} = \text{sum of all fully connected amputated Feynman diagrams in momentum space}$.

Compare to page 35 of the lecture notes and judge for yourself whether you like Feynman rules or not.

(c) Calculate the lowest-order amplitudes for the following scattering processes:

$$\psi(k_A)\psi(k_B) \to \psi(p_1)\psi(p_2),$$

$$\psi(k_A)\bar{\psi}(k_B) \to \phi(p_1)\phi(p_2),$$

$$\psi(k_A)\bar{\psi}(k_B) \to \psi(p_1)\bar{\psi}(p_2),$$

where $k_A$ and $k_B$ are the momenta of the initial-state particles and $p_1$ and $p_2$ the momenta of the final-state particles.

(d) Explain why the lowest-order amplitude vanishes for the scattering process

$$\psi(k_A)\psi(k_B) \to \phi(p_1)\phi(p_2).$$
Exercise 10: Non-relativistic interactions mediated by spin-1 and spin-2 force carriers
(Coulomb potential and Newton’s gravitational potential)

Solve part a of the exercise given below.

Part a: consider scalar QED, i.e. the theory that describes the interactions between complex scalar
fields and real electromagnetic fields. The relevant momentum-space Feynman rules for propagators
and interactions in this theory are given by

\[
\begin{align*}
\begin{array}{rcl}
\text{scalar propagator} & : & \frac{i}{p^2 - m^2 + i\epsilon} \\
\text{em propagator} & : & \frac{-ig_{\mu\nu}}{k^2 + i\epsilon} \\
\text{em interaction} & : & -ie(p + p')^\mu \\
\text{scalar interaction} & : & 2ie^2g^{\mu\nu}
\end{array}
\end{align*}
\]

using dashed lines to indicate the massive scalar bosons and wiggly ones to indicate the massless
electromagnetic bosons (photons). The momenta of the particles are defined along the arrow of
particle-number flow and the indices \(\mu\) and \(\nu\) are Minkowski indices.

Just like in the case of the scalar Yukawa theory, we consider the lowest-order scattering reactions
\(\psi\psi \rightarrow \psi\psi\) and \(\psi\bar{\psi} \rightarrow \psi\bar{\psi}\) for scalar particles (\(\psi\)) and antiparticles (\(\bar{\psi}\)). Show that in scalar QED
the interaction between \(\psi\)-particles is repulsive, whereas the interaction between \(\psi\)-particles and
\(\bar{\psi}\)-antiparticles is attractive.

Hint: don’t perform a complete calculation, but make as much use as possible of the calculation
that is presented on pages 51 and 52 of the lecture notes and remember that the \(\psi\)-particles are
defined to flow along the arrow, whereas \(\bar{\psi}\)-particles flow against it.

Part b: consider the following Feynman rules for the graviton propagator (double wiggly line) and
the interaction between complex scalar fields and real graviton fields:

\[
\begin{align*}
\begin{array}{rcl}
\text{graviton propagator} & : & \frac{i}{k^2 + i\epsilon} \\
\text{graviton interaction} & : & i\sqrt{8\pi G} \left[ p^\mu p'^\nu + p'^\mu p^\nu + g^{\mu\nu}(m^2 - p \cdot p') \right]
\end{array}
\end{align*}
\]

where \(G\) is Newton’s constant. This interaction originates from the coupling of the symmetric
graviton field (tensor field) to the symmetrized energy-momentum tensor \(T^{\mu\nu}\) for the scalar matter.

Optional: consider the free complex KG theory. Derive the corresponding energy-momentum
tensor \(T^{\mu\nu}(x)\) and show that \(\langle \vec{p}' | T^{\mu\nu}(x) | \vec{p} \rangle\) for \(\vec{p} \neq \vec{p}'\) is proportional to the Feynman rule
given above, where \(| \vec{p} \rangle\) denotes an incoming \(\psi\)-particle state and \(\langle \vec{p}' | \) an outgoing \(\psi\)-particle state.

Repeat the analysis of part a to show that the non-relativistic gravitational interaction between
the scalar (anti)particles is always attractive and of the expected form.

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