**Exercise 19** Yukawa theory: $\mathcal{L} = \bar\psi(i\not\partial - m_\psi)\psi + \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m_\phi^2\phi^2 \overbrace{- g\bar\psi\psi\phi}^{\mathcal{L}_{int}}$,
with $\psi$ Dirac field, $\bar\psi$ its adjoint and $\phi \in \mathbb{R}$ scalar field

**(a)** $\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\right) - \frac{\partial\mathcal{L}}{\partial\psi} = \partial_\mu(i\bar\psi\gamma^\mu) + m_\psi\bar\psi + g\bar\psi\phi = \bar\psi(i\overleftarrow{\not\partial} + m_\psi + g\phi) = 0$,

$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar\psi)}\right) - \frac{\partial\mathcal{L}}{\partial\bar\psi} = \partial_\mu(0) - (i\not\partial - m_\psi)\psi + g\psi\phi = -(i\not\partial - m_\psi - g\phi)\psi = 0$,

$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = \partial_\mu(\partial^\mu\phi) + m_\phi^2\phi + g\bar\psi\psi = (\Box + m_\phi^2)\phi + \overbrace{g\bar\psi\psi}^{\text{scalar current}} = 0$.

**(b)** $[S] = [\int d^4x\,\mathcal{L}] = 0 \Rightarrow [\mathcal{L}] = 4$, $[m_\psi] = [m_\phi] = [\partial_\mu] = 1$ $\Bigg\} \Rightarrow$ $[\phi] = \frac{4-2}{2} = 1$, $[\psi] = [\bar\psi] = \frac{4-1}{2} = 3/2$,
$[g] = 4 - 2\cdot\frac{3}{2} - 1 = 0$ ← dimensionless !

**(c)** $\hat\psi_I(x)$ (free field) contains $\hat a_{\vec p}$, $\hat b^\dagger_{\vec p}$ and $\hat{\bar\psi}_I(x)$ contains $\hat a^\dagger_{\vec p}$, $\hat b_{\vec p}$
$\Rightarrow \langle 0|T(\hat\psi_{a_I}(x)\hat\psi_{b_I}(y))|0\rangle = 0$, since the operators contained in the two fields anticommute and therefore annihilate the vacuum $|0\rangle$ or $\langle 0|$. A similar argument holds for $\langle 0|T(\hat{\bar\psi}_{a_I}(x)\hat{\bar\psi}_{b_I}(y))|0\rangle = 0$.

**(d)** $\langle 0|T(\hat\psi_{a_I}(x_1)\hat{\bar\psi}_{b_I}(x_2)\hat\phi_I(x_3)\,e^{-ig\int d^4z\,\hat{\bar\psi}_{c_I}(z)\hat\psi_{c_I}(z)\hat\phi_I(z)})|0\rangle$

$\stackrel{\mathcal{O}(g)}{=}$ $\cancel{\langle 0|T(\hat\psi_{a_I}(x_1)\hat{\bar\psi}_{b_I}(x_2)\hat\phi_I(x_3))|0\rangle}$ → 0, cannot be fully contracted

$-ig\int d^4z\,\langle 0|T(\hat\psi_{a_I}(x_1)\hat{\bar\psi}_{b_I}(x_2)\hat\phi_I(x_3)\hat{\bar\psi}_{c_I}(z)\hat\psi_{c_I}(z)\hat\phi_I(z))|0\rangle$

$\stackrel{\text{Wick}}{=}$ $-ig\int d^4z\,\langle 0|\hat\psi_{a_I}(x_1)\hat{\bar\psi}_{b_I}(x_2)\hat\phi_I(x_3)\hat{\bar\psi}_{c_I}(z)\hat\psi_{c_I}(z)\hat\phi_I(z)|0\rangle$ ①: − ②: +

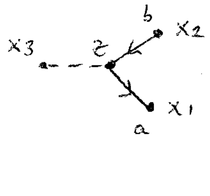
used: $\hat\psi\hat\psi = 0$, $\hat{\bar\psi}\hat{\bar\psi} = 0$ !

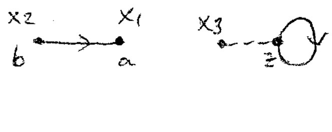
$-ig\int d^4z\,\langle 0|\hat\psi_{a_I}(x_1)\hat{\bar\psi}_{b_I}(x_2)\hat\phi_I(x_3)\hat{\bar\psi}_{c_I}(z)\hat\psi_{c_I}(z)\hat\phi_I(z)|0\rangle$ ①: + ②: +

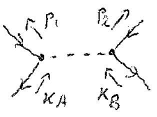
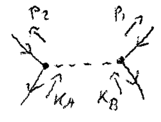
$= ig\int d^4z\,D_F(x_3-z)[S_F(x_1-z)]_{ac}[-S_F(z-x_2)]_{cb} - ig[S_F(x_1-x_2)]_{ab}\int d^4z\,D_F(x_3-z)[-S_F(z-z)]_{cc}$

← $\hat{\bar\psi}\hat\psi = -\hat\psi\hat{\bar\psi}$ !

$= -ig\int d^4z\,D_F(x_3-z)[S_F(x_1-z)S_F(z-x_2)]_{ab} + ig[S_F(x_1-x_2)]_{ab}\int d^4z\,D_F(x_3-z)\,\mathrm{Tr}(S_F(z-z))$



**(e) + (f)** $i\mathcal{M}(\bar\psi(k_A,s_A)\,\bar\psi(k_B,s_B) \to \bar\psi(p_1,r_1)\,\bar\psi(p_2,r_2)) =$   $+$  

$$= \overbrace{-(-ig)^2 i}^{ig^2} \left[ \frac{(\bar v^{s_A}(k_A), v^{r_1}(p_1))\,(\bar v^{s_B}(k_B), v^{r_2}(p_2))}{(k_A-p_1)^2 - m_\phi^2 + i\epsilon} - (p_1,r_1) \leftrightarrow (p_2,r_2) \right]$$

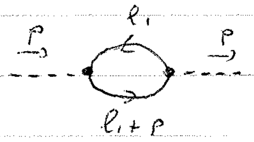
First diagram: $-g^2\ {}_0\langle \vec p_1 \vec p_2 | \int d^4x\, \hat{\bar\psi}(x)\hat\psi(x)\hat\phi(x) \int d^4y\, \hat{\bar\psi}(y)\hat\psi(y)\hat\phi(y) | \vec k_A \vec k_B\rangle_0$

2: − 1: + 4: + 3: +

Second diagram: $-g^2\ {}_0\langle \vec p_1 \vec p_2 | \int d^4x\, \hat{\bar\psi}(x)\hat\psi(x)\hat\phi(x) \int d^4y\, \hat{\bar\psi}(y)\hat\psi(y)\hat\phi(y) | \vec k_A \vec k_B\rangle_0$

1: + 2: + 4: + 3: +

The two diagrams have a relative − sign, caused by the interchange of two fermions (<u>Fermi statistics</u>).

**(g)** $-i\Sigma_\phi(p^2) \overset{\text{1-loop}}{=}$   $= -(-ig)^2 i^2 \int \frac{d^4 l_1}{(2\pi)^4} \frac{\mathrm{Tr}\,([\not l_1 + m_\psi][\not l_1 + \not p + m_\psi])}{[l_1^2 - m_\psi^2 + i\epsilon][(l_1+p)^2 - m_\psi^2 + i\epsilon]}$

fermion loop !

fermion loop !

$$\mathrm{Tr}\,(\not l_1(\not l_1 + \not p) + m_\psi(2\not l_1 + \not p) + m_\psi^2) \underset{(a),(d)}{\overset{\text{Ex. 16}}{=}} 4\,(l_1\cdot(l_1+p) + m_\psi^2)$$

$$\Rightarrow -i\Sigma_\phi(p^2) \overset{\text{1-loop}}{=} -4g^2 \int \frac{d^4 l_1}{(2\pi)^4} \frac{m_\psi^2 + l_1^2 + l_1\cdot p}{[l_1^2 - m_\psi^2 + i\epsilon][(l_1+p)^2 - m_\psi^2 + i\epsilon]}.$$

$-g^2/4\pi^4$

**(h)** Consider an arbitrary loop diagram in the Yukawa theory with $N_F$ external fermions, $N_B$ external bosons, $P_F$ fermion propagators, $P_B$ boson propagators, $V$ vertices, $L$ loop momenta

*) 2 fermions and 1 boson meet in each vertex and each propagator is connected to two vertices‡: $2V = N_F + 2P_F$, $V = N_B + 2P_B \Rightarrow P_F = V - \frac{1}{2}N_F$, $P_B = \frac{1}{2}(V - N_B)$.

*) as usual $L = \underbrace{(P_F + P_B)}_{P} - V + 1 = \frac{1}{2}(V - N_F - N_B) + 1 \geq 1$ if $V \geq N_B + N_F$.

even

Naive power counting: each loop momentum yields $\Lambda^4$, each boson prop. $\Lambda^{-2}$ and each fermion prop. $\Lambda^{-1}$ in 4 spacetime dimensions

$\Rightarrow$ superficial degree of divergence: $D = 4L - 2P_B - P_F = 4 - \frac{3}{2}N_F - N_B$.

This means that divergences occur at all orders, but there is only a finite number of divergent amplitudes [for $(N_F, N_B) = (0, \leq 4)$, $(2, \leq 1)$]

$\Rightarrow$ <u>the Yukawa theory is renormalizable in 4 spacetime dimensions</u> !

‡ each external line is connected to one vertex