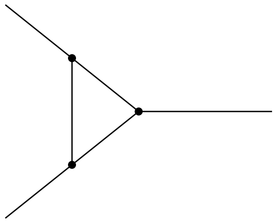
Solution 13:

(a) Only one amputated diagram contributes to the one-loop correction to the ϕ^3 interaction:

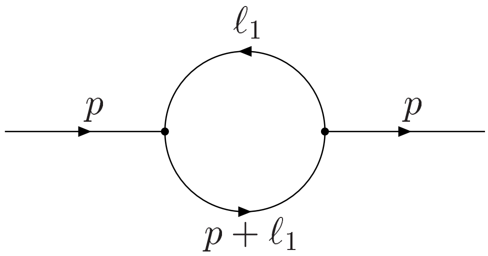


(b) In the diagram of part (a) we have three propagators and one loop momentum. After Wick rotation and Feynman parametrization, the integral will have the form

$$\int_0^{\Lambda^2} d\ell_E^2 \, \frac{\ell_E^2}{(\ell_E^2 + \Delta - i\epsilon)^3} \, .$$

Since there are four powers of ℓ_E in the numerator and six in the denominator, the integral is finite. This can be understood from a dimensional reasoning. For a 4-dimensional theory we have $[\mathcal{L}] = 4$. Since $\mathcal{L}$ contains a term of the form $m^2\phi^2$, that means that $[\phi] = 1$, so $[\phi^3] = 3$ and therefore $[\lambda] = 4 - 3 = 1$. According to the discussion on page 28 of the lecture notes the ϕ^3 interaction is thus weak at high energies, which explains its finiteness.

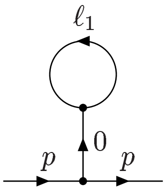
(c+d) First we consider the self-energy diagram



Taking into account the symmetry factor, this diagram yields:

$$\begin{aligned}
\frac{1}{2}\int \frac{d^4\ell_1}{(2\pi)^4} \frac{\lambda^2}{[\ell_1^2 - m^2 + i\epsilon][(\ell_1 + p)^2 - m^2 + i\epsilon]} &= \frac{i\lambda^2}{32\pi^2}\int_0^1 d\alpha \int_0^{\Lambda^2} d\ell_E^2 \, \frac{\ell_E^2}{(\ell_E^2 + \Delta(\alpha, m^2, p^2) - i\epsilon)^2} \\
&= \frac{i\lambda^2}{32\pi^2}\log(\Lambda^2) \, + \text{ finite terms for } \Lambda \to \infty \, .
\end{aligned}$$

Note that for the leading behaviour we do not need to determine the precise form of Δ. For the calculation of the integral, see exercise 12.

The second ("tadpole") diagram  yields:

$$
\begin{aligned}
\frac{1}{2}\int \frac{d^4\ell_1}{(2\pi)^4}\,\frac{-\,\lambda^2/m^2}{[\ell_1^2-m^2+i\epsilon]} &= \frac{i\lambda^2}{32\pi^2 m^2}\int_0^{\Lambda^2} d\ell_E^2\,\frac{\ell_E^2}{\ell_E^2+m^2-i\epsilon} \\
&= \frac{i\lambda^2}{32\pi^2}\frac{\Lambda^2}{m^2} + \text{less divergent terms for } \Lambda\to\infty\ .
\end{aligned}
$$

(e) The full propagator in the ϕ^3-theory reads:

$$
\frac{i}{p^2-m^2-\Sigma(p^2)+i\epsilon} \equiv \frac{iZ}{p^2-m_{ph}^2+i\epsilon} + \text{regular terms at the pole } m_{ph}^2\ ,
$$

with $\Sigma(p^2)$ the self-energy. The physical mass m_{ph} is determined by the location of the pole:

$$
m_{ph}^2 - m^2 - \Sigma(m_{ph}^2) \;=\; 0\ .
$$

We know that the physical mass m_{ph} is an observable and thus cannot depend on Λ. According to part (c+d) the self-energy is divergent for $\Lambda\to\infty$, so the same should hold for the Lagrangian parameter m.

(f) The analysis on page 87 of the lecture notes is for ϕ^4-theory. For ϕ^3-theory we have:

- three lines meet in each vertex, each external line is connected to one vertex and each propagator is connected to two vertices or to the same vertex twice, so $3V = N+2P \Rightarrow P = \frac{3}{2}V - \frac{1}{2}N$.
- By the same reasoning as on page 87, we have $L = P - V + 1 = \frac{1}{2}V - \frac{1}{2}N + 1$ so we only have loops if $V \geq N$.
- The superficial degree of divergence is given by $D = nL - 2P = (\frac{1}{2}n-3)V + (1-n/2)N + n$.

Now as for renormalizability:

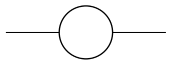
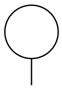
$\boldsymbol{n=4}$ In our four-dimensional world we have $D = 4 - V - N$, so D decreases at higher loop order since the coefficient in front of V is negative. The only divergences occur at $N=1$ and $V=1$ or 3 (tadpole graphs) and $V=N=2$ (the one-loop self-energy graph from part c), bearing in mind that $V = N + 2(L-1)$. So this case is superrenormalizable.

$\boldsymbol{n=6}$ In this case we have $D = 6-2N$, so $D \geq 0$ for $N \leq 3$. That means that divergences occur at all orders, but there is only a finite number of divergent amplitudes. So this theory is renormalizable.

$\boldsymbol{n<6}$ For any theory with $n<6$, a similar argumentation as in the $n=4$ case shows that it is superrenormalizable.

$\boldsymbol{n>6}$ All amplitudes become divergent at sufficiently high loop order, since the coefficient in front of V is positive, so the theory is nonrenormalizable.

(g) For an n dimensional theory, we have $[\mathcal{L}] = n$. Since $\mathcal{L}$ contains a term of the form $m^2\phi^2$, that means that $[\phi] = (n-2)/2$, so $[\phi^3] = 3(n-2)/2$ and therefore $[\lambda] = n - 3(n-2)/2 = 3 - n/2$. That means that λ is dimensionless for $n = 6$, which is the renormalizable case. For $n < 6$, $[\lambda] > 0$, so the ϕ^3 interaction is weak at high energies, which explains why it is superrenormalizable. For $n > 6$, $[\lambda] < 0$, so the ϕ^3 interaction is strong at high energies, which explains why the integrals diverge and the theory is nonrenormalizable.

(h) According to part (f), the superficially divergent 1-particle irreducible one-loop diagrams for four-dimensional spacetime are the self-energy diagram  $\propto \log(\Lambda^2)$ and the tadpole diagram  $\propto \Lambda^2$. This agrees with the discussion in part (d).

Solution 14:

(a) Infinitesimal rotation in terms of rotation angle $\delta\alpha$ and rotation axis $\vec{e}_n$:
$\vec{x} \to \vec{x}\,' \approx \vec{x} + \delta\alpha\, \vec{e}_n \times \vec{x} \equiv \vec{x} + \delta\vec{\alpha} \times \vec{x}\,.$
Rewritten as infinitesimal Lorentz transformation $x'^{\rho} = \Lambda^{\rho}{}_{\sigma}\, x^{\sigma} \approx (g^{\rho}{}_{\sigma} + \omega^{\rho}{}_{\sigma})\, x^{\sigma}$:
$x'^{\,0} = x^0 \;\Rightarrow\; \omega_{0\sigma} = 0$ for $\sigma = 0, 1, 2, 3.$
$x'^{j} \approx x^j + (\delta\vec{\alpha} \times \vec{x}\,)^j = x^j + \epsilon^{jlk}(\delta\alpha)^l x^k \;\Rightarrow\; \omega_{j0} = 0$ and $\omega_{jk} = -\,\epsilon^{jlk}(\delta\alpha)^l = \epsilon^{jkl}(\delta\alpha)^l$.
The non-vanishing components are: $\omega_{12} = -\,\omega_{21} = (\delta\alpha)^3$ and cyclic permutations thereof.

(b) Infinitesimal boost with boost velocity $\delta\vec{v}$: $x^0 \to x'^{\,0} \approx x^0 + \delta\vec{v}\cdot\vec{x}$, $\vec{x} \to \vec{x}\,' \approx \vec{x} + x^0\delta\vec{v}\,.$
Rewritten as infinitesimal Lorentz transformation $x'^{\rho} = \Lambda^{\rho}{}_{\sigma}\, x^{\sigma} \approx (g^{\rho}{}_{\sigma} + \omega^{\rho}{}_{\sigma})\, x^{\sigma}$:
$\omega_{00} = 0\,,\ \omega_{0j} = \delta v^j$ from the temporal part and $\omega_{ij} = 0\,,\ \omega_{j0} = -\,\delta v^j$ from the spatial part.