

Solutions to the problem sets of the course "QFT 2 : the Standard Model"

Sol. 1 SU(N) in fundamental representation: $U = e^{i\theta^a T^a}$ $\in \text{IR}^{N \times N}$ matrices
with $U U^\dagger = 1$ and $\det U = 1$

(a) $U U^\dagger = e^{i\theta^a T^a} e^{-i\theta^b T^b} = 1$ $\forall_{a, b} \Rightarrow$ for each value of a we have $(T^a)^\dagger = T^a$,
i.e. the generators are hermitian.

$\det U = e^{\text{Tr}(i\theta^a T^a)}$ according to basic linear algebra \Rightarrow in this case

$\det U = e^{\text{Tr}(i\theta^a T^a)} = e^{\text{Tr}(i\theta^a T^a)} = 1, \forall \theta^a \Rightarrow$ for each value of a we have $\text{Tr}(T^a) = 0$,
i.e. the generators are traceless.

(b) Take $N \times N$ matrices: $2N^2$ d.o.f. $\xrightarrow[\text{(a)}]{\text{herm.}} N^2$ d.o.f. $\xrightarrow[\text{(a)}]{\text{Tr}=0} (N^2 - 1)$ d.o.f.

Hence, based on (a) we know that SU(N) has $N^2 - 1$ independent generators.

(c) Fundamental commutation relation for SU(N): $[T^a, T^b] = i f^{abc} T^c$.

Consequence: $[T^a, T^b]^\dagger = T^a T^b - T^b T^a = -i f^{abc} T^c$

$$\Leftrightarrow (i f^{abc} T^c)^\dagger = -i (f^{abc})^* (T^c)^\dagger \quad \text{and} \quad (f^{abc})^* = f^{abc}$$

Hence, the SU(N) structure constants f^{abc} are real.

(d) Rotation subgroup SO(N) \subset SU(N), again in the fundamental representation:

$O = e^{i\theta^a T^a}$ $N \times N$ matrices, with $O O^\dagger = 1$, $\det O = 1$ and $O O^\dagger = 1$.

Because of (a), $(T^a)^\dagger = T^a$ and $\text{Tr}(T^a) = 0$. However, this time we have the additional condition $(T^a)^\dagger = -T^a$ as a result of $O O^\dagger = 1$

$\Rightarrow T^a = (T^a)^\dagger = (T^a)^\ast = -(T^a)^\ast$, i.e. T^a has purely imaginary components.

Again take $N \times N$ matrices: $2N^2$ d.o.f. $\xrightarrow{\text{herm.}} N^2$ d.o.f. $\xrightarrow{T^a \text{ purely imaginary}}$

$\frac{1}{2}(N^2 - N)$ d.o.f., i.e. SO(N) has $\frac{1}{2}N(N-1)$ independent generators.

Diagonal elements vanish in this case

This number coincides with the number of planes of rotation in N dim.

$$(\vec{\theta} \cdot \vec{e}_n) \frac{\vec{\sigma} \cdot \vec{e}_n}{2}$$

Sol. 2 Consider $U(x) = e^{i\theta^j \vec{\sigma}^j \vec{e}_n} \in \text{SU}(2)$, with $\theta \in \text{IR}$, $\vec{e}_n^2 = 1$ and $\vec{\sigma}^1, \vec{\sigma}^2, \vec{\sigma}^3$ the usual Pauli spin matrices.

(a) Since $\{\vec{\sigma}^j, \vec{\sigma}^k\} = 2\delta^{jk} I_2$, we can derive that $(\vec{\sigma} \cdot \vec{e}_n)^2 = e_n^j e_n^k \sigma^j \sigma^k$
 $= \frac{1}{2} e_n^j e_n^k \{\vec{\sigma}^j, \vec{\sigma}^k\} = e_n^j e_n^k \delta^{jk} I_2 = \vec{e}_n^2 I_2 = I_2$. With this identity we can show that

$$U(x) = \sum_{m=0}^{\infty} \left(\frac{i\theta}{2} \right)^{2m} \frac{(\vec{\sigma} \cdot \vec{e}_n)^{2m}}{(2m)!} + \sum_{m=0}^{\infty} \left(\frac{i\theta}{2} \right)^{2m+1} \frac{(\vec{\sigma} \cdot \vec{e}_n)^{2m+1}}{(2m+1)!}$$

$$= I_2 \underbrace{\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} \left(\frac{\theta}{2} \right)^{2m}}_{\cos(\theta/2)} + i(\vec{\sigma} \cdot \vec{e}_n) \underbrace{\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} \left(\frac{\theta}{2} \right)^{2m+1}}_{\sin(\theta/2)}.$$

(b) Property of Pauli spin matrices: $(\sigma^1)^* = \sigma^1$, $(\sigma^2)^* = -\sigma^2$, $(\sigma^3)^* = \sigma^3$

$$\Rightarrow i\sigma^2(\sigma^j)^* = -\sigma^j\sigma^2 \quad (j=1,2,3) \quad ①$$

$$\sum \{ \sigma^2, \sigma^{j+2} \} = 0$$

Consider the conjugate doublet $\tilde{\Phi} = i\sigma^2 \bar{\Phi}^*(x)$, with $\bar{\Phi}(x)$ a doublet under $SU(2)$: $\bar{\Phi}(x) \xrightarrow{SU(2)} \bar{\Phi}'(x) = U(x)\bar{\Phi}(x)$. Then the conjugate doublet transforms as

$$\begin{aligned} \tilde{\Phi}(x) &\xrightarrow{SU(2)} \tilde{\Phi}'(x) = i\sigma^2 (\bar{\Phi}'(x))^* = i\sigma^2 [U(x)\bar{\Phi}(x)]^* = i\sigma^2 e^{-i\theta(x) \frac{\vec{\sigma}}{2} \cdot \vec{\epsilon}_h} \bar{\Phi}^*(x) \\ &\stackrel{①}{=} e^{+i\theta(x) \frac{\vec{\sigma}}{2} \cdot \vec{\epsilon}_h} i\sigma^2 \bar{\Phi}^*(x) = U(x) \tilde{\Phi}(x). \end{aligned}$$

Hence, $\tilde{\Phi}(x)$ transforms just like $\bar{\Phi}(x)$ under $SU(2)$ --- which will be used in the Higgs sector of the Standard Model.