

Sol. 10 Consider a neutrino that is produced through a CC weak interaction at  $t=0$  in the flavour eigenstate  $|\nu_\alpha\rangle$ . The neutrino is detected a distance  $L$  away as a  $|\nu_\beta\rangle$  flavour eigenstate, again through a CC weak interaction in the detector.

(a) We start by writing the flavour eigenstates in terms of the mass eigenstates (= energy eigenstates). In the language of quantum fields we have seen that

$$\bar{\nu}_\alpha(x) = \sum_j \bar{\nu}_j(x) (U^\dagger)_{j\alpha}$$

mass eigenstates with mass  $m_j$  ( $j=1,2,3$ )
flavour eigenstates ( $\alpha=e,\mu,\tau$ )

(corresponding lepton in CC int.)

$\bar{\nu}_\alpha$  contains creation operators for creating the flavour eigenstates  $|\nu_\alpha\rangle$  out of the vacuum and  $\bar{\nu}_j$  for creating mass eigenstates  $|\nu_j\rangle \Rightarrow$

$$|\nu_\alpha\rangle = \sum_j (U)_{j\alpha}^\dagger |\nu_j\rangle = \sum_j (U_{j\alpha}^*) |\nu_j\rangle \quad \text{and} \quad \langle \nu_\beta | = \sum_k \langle \nu_k | (U^\dagger)_{k\beta} = \sum_k \langle \nu_k | U_{\beta k}$$

(b)  $t=0 : |\nu(t=0)\rangle = |\nu_\alpha\rangle = \sum_j U_{j\alpha}^* |\nu_j\rangle$ , with fixed momentum  $\vec{p}$ .  
 QM evolution:  $|\nu(t>0)\rangle = \sum_j U_{j\alpha}^* |\nu_j(t)\rangle = \sum_j U_{j\alpha}^* e^{-iE_j t} |\nu_j\rangle$

with  $E_j = \sqrt{\vec{p}^2 + m_j^2}$

$\Rightarrow$  probability amplitude for oscillation to flavour state  $|\nu_\beta\rangle$  at  $t>0$ :

$$\langle \nu_\beta | \nu(t) \rangle = \sum_j U_{j\alpha}^* e^{-iE_j t} \langle \nu_\beta | \nu_j \rangle \stackrel{(a)}{=} \sum_{j,k} U_{j\alpha}^* U_{\beta k} e^{-iE_j t} \underbrace{\langle \nu_k | \nu_j \rangle}_{\text{norm.: } \delta_{jk}}$$

$$= \sum_j U_{\beta j} U_{j\alpha}^* e^{-iE_j t}$$

(c) Ultrarelativistic neutrino: ① distance traveled  $L \approx$  time  $t$ ,  
 ②  $m_j \ll E_j \Rightarrow E_j = \sqrt{\vec{p}^2 + m_j^2} \approx |\vec{p}| (1 + \frac{m_j^2}{2|\vec{p}|^2})$

Oscillation probability  $|\langle \nu_\beta | \nu(t) \rangle|^2 \approx |e^{-i|\vec{p}|t} \sum_j U_{\beta j} U_{j\alpha}^* e^{-im_j^2 t / 2|\vec{p}|}|^2$

$$= |\sum_j U_{\beta j} U_{j\alpha}^* e^{-im_j^2 t / 2|\vec{p}|}|^2$$

since all three  $E_j$  differ by a small amount from  $|\vec{p}|$ ,  $|\vec{p}| \approx$  average energy  $E$

(d) Neutrino detected at distance  $L$  from the source (i.e.  $t \approx L$ )

$\Rightarrow$  oscillation probability  $P(\nu_\alpha \rightarrow \nu_\beta) \approx |\sum_j U_{\beta j} U_{j\alpha}^* e^{-im_j^2 L / 2E}|^2$

$$= \sum_j |U_{\beta j}|^2 |U_{\alpha j}|^2 + \sum_{j>k} 2 \operatorname{Re}(U_{\beta j} U_{\alpha j}^* U_{\beta k}^* U_{\alpha k}) e^{i(\frac{\Delta m_{jk}^2}{2} L/2E)}, \text{ with } \Phi_{jk} = \frac{L \Delta m_{jk}^2}{4E}$$

$$2 \operatorname{Re}(U_{\beta j} U_{\alpha j}^* U_{\beta k}^* U_{\alpha k}) \cos(2\Phi_{jk}) + 2 \operatorname{Im}(U_{\beta j} U_{\alpha j}^* U_{\beta k}^* U_{\alpha k}) \sin(2\Phi_{jk})$$

Use that  $\sum_j U_{\beta j} U_{\alpha j}^* = \delta_{\alpha\beta}$  to express this in the generic form

$$P(\nu_\alpha \rightarrow \nu_\beta) \approx \underbrace{\left| \sum_j U_{\beta j} U_{\alpha j}^* \right|^2}_{\delta_{\alpha\beta}} + \sum_{j>k} 2 \operatorname{Re}(U_{\beta j} U_{\alpha j}^* U_{\beta k}^* U_{\alpha k}) [\underbrace{\cos(2\Phi_{jk})}_{-2\sin^2(\Phi_{jk})} - 1] + \sum_{j>k} 2 \operatorname{Im}(U_{\beta j} U_{\alpha j}^* U_{\beta k}^* U_{\alpha k}) \sin(2\Phi_{jk}), \text{ with } \Phi_{jk} = \frac{L \Delta m_{jk}^2}{4E}$$

(e)  $m_1 = m_2 = m_3$ ;  $m_j^2 - m_k^2 = \Delta m_{jk}^2 = 0 \Rightarrow P(\nu_\alpha \rightarrow \nu_\beta) = \delta_{\alpha\beta}$ , no oscillation will occur.

(f) We are only sensitive to differences in squared masses  $\Delta m_{jk}^2$   
 $\Rightarrow$  if we measure two of the  $\Delta m_{jk}^2$  to be non-zero and different, then we can at best say that two of the three neutrinos are massive ... the lightest neutrino is still allowed to be massless!

(g) Interchanging  $j$  and  $k$ :  $\operatorname{Re}(U_{\beta j} U_{\alpha j}^* U_{\beta k}^* U_{\alpha k})$  and  $\sin^2(\Phi_{jk})$  are both symmetric,  $\operatorname{Im}(U_{\beta j} U_{\alpha j}^* U_{\beta k}^* U_{\alpha k})$  and  $\sin(2\Phi_{jk})$  are antisymmetric  
 $\Rightarrow P(\nu_\alpha \rightarrow \nu_\beta)$  is insensitive to the interchange of  $j$  and  $k$ !

Consequence: we cannot say anything about the neutrino-mass hierarchy by means of this type of oscillation experiment!

(h) Oscillation phase  $\Phi_{jk} = \Delta m_{jk}^2 \frac{L}{4E} = 1.27 \Delta m_{jk}^2 (\text{eV}^2) \frac{L(\text{km})}{E(\text{GeV})}$

This phase becomes noticeable if  $\Phi_{j,k} = \mathcal{O}(1)$ :

(A) Atmospheric neutrinos:  $E = 1 \text{ GeV}$ ,  $L = 1.27 \times 10^4 \text{ km} = \text{diameter Earth}$   
 $\Rightarrow \Phi_{jk} = 1.6 \times 10^4 \Delta m_{jk}^2 (\text{eV}^2)$ , allows access to  $\Delta m_{jk}^2 \geq \mathcal{O}(10^{-4} \text{ eV}^2)$ .

Exp.:  $\nu_\mu - \nu_\tau$  oscillations,  $\Delta m_{\text{atm}}^2 = |\Delta m_{32}^2| \approx |\Delta m_{31}^2| = 2 \times 10^{-3} \text{ eV}^2$

(B) Solar neutrinos:  $E = 1 \text{ MeV}$ ,  $L = 1.5 \times 10^8 \text{ km} = 1 \text{ AU}$  (Earth-Sun distance)  
 $\Rightarrow \Phi_{jk} = 1.9 \times 10^{11} \Delta m_{jk}^2 (\text{eV}^2)$ , allows access to  $\Delta m_{jk}^2 \geq \mathcal{O}(10^{-11} \text{ eV}^2)$ .

Exp.:  $\nu_e - \nu_{\mu\tau}$  oscillations,  $\Delta m_{\text{solar}}^2 = |\Delta m_{21}^2| = 8 \times 10^{-5} \text{ eV}^2$

e.g. Homestake, SNO

e.g. Superkamiokande