Opg. 24) Matrices in de Dirac-verg.: β en $\vec{\alpha}$, met eigenschappen

$$\beta^2 = I\,,\quad \beta = \beta^\dagger\,,\quad \{\alpha^j,\beta\} = 0\,,\quad \{\alpha^j,\alpha^k\} = 2\delta^{jk} I\,,\quad \alpha^j = (\alpha^j)^\dagger \qquad (j,k=1,2,3) \qquad (D)$$

(i) $\beta = \begin{pmatrix} I & \emptyset \\ \emptyset & -I \end{pmatrix}$ en $\vec{\alpha} = \begin{pmatrix} \emptyset & \vec{\sigma} \\ \vec{\sigma} & \emptyset \end{pmatrix}$ voldoen hieraan, want $\beta^2 = I$ en $\beta = \beta^\dagger$ gelden triviaal, $\vec{\alpha} = \vec{\alpha}^\dagger$ volgt uit $\vec{\sigma} = \vec{\sigma}^\dagger$.

$$\{\vec{\alpha},\beta\} = \begin{pmatrix} \emptyset & -[\vec{\sigma},I] \\ \underbrace{[\vec{\sigma},I]}_{0} & \emptyset \end{pmatrix} = 0 \quad \text{en} \quad \{\alpha^j,\alpha^k\} = \begin{pmatrix} \{\sigma^j,\sigma^k\} & \emptyset \\ \emptyset & \underbrace{\{\sigma^j,\sigma^k\}}_{2\delta^{jk} I} \end{pmatrix} = 2\delta^{jk} I\,.$$

(ii) Voer de Dirac γ-matrices γ^μ in, met $\gamma^0 \equiv \beta$ en $\vec{\gamma} \equiv \beta\vec{\alpha}$. Dan geldt:

$$\{\gamma^\mu,\gamma^\nu\} = \begin{cases} \mu=\nu=0: & 2\beta^2 \overset{(D)}{=} 2I \\ \mu=0,\ \nu=j: & \{\beta,\beta\alpha^j\} = \beta^2\alpha^j + \beta\alpha^j\beta = \beta\{\beta,\alpha^j\} \overset{(D)}{=} 0 \\ \mu=j,\ \nu=k: & \{\beta\alpha^j,\beta\alpha^k\} = \beta\alpha^j\beta\alpha^k + \beta\alpha^k\beta\alpha^j \overset{(D)}{=} -\beta^2\{\alpha^j,\alpha^k\} \overset{(D)}{=} -2\delta^{jk} I \end{cases} = 2g^{\mu\nu} I\,,$$

$$(\gamma^\mu)^\dagger = \begin{cases} \mu=0: & \beta^\dagger \overset{(D)}{=} \beta = \gamma^0 \\ \mu=j: & (\beta\alpha^j)^\dagger \overset{(D)}{=} \alpha^j\beta \overset{(D)}{=} -\beta\alpha^j = -\gamma^j \end{cases}$$

(iii) Hieruit volgt dat $(\gamma^0)^2 = \frac{1}{2}\{\gamma^0,\gamma^0\} \overset{(ii)}{=} I$, $(\gamma^j)^2 = \frac{1}{2}\{\gamma^j,\gamma^j\} \overset{(ii)}{=} -I$,

$$\gamma^0\gamma^\mu\gamma^0 = \begin{cases} \mu=0: & (\gamma^0)^2\gamma^0 = \gamma^0 \\ \mu=j: & \gamma^0\gamma^j\gamma^0 \overset{(ii)}{=} -(\gamma^0)^2\gamma^j = -\gamma^j \end{cases} \overset{(ii)}{=} (\gamma^\mu)^\dagger \quad \text{en}$$

$$\mathrm{Tr}(\gamma^\mu) = \begin{cases} \mu=0: & \mathrm{Tr}(\gamma^0) = -\mathrm{Tr}([\gamma^j]^2\gamma^0) \overset{(ii)}{=} \mathrm{Tr}(\gamma^j\gamma^0\gamma^j) \overset{\text{cycl.}}{=} \mathrm{Tr}([\gamma^j]^2\gamma^0) = 0 \\ \mu=j: & \mathrm{Tr}(\gamma^j) = \mathrm{Tr}([\gamma^0]^2\gamma^j) \overset{(ii)}{=} -\mathrm{Tr}(\gamma^0\gamma^j\gamma^0) \overset{\text{cycl.}}{=} -\mathrm{Tr}([\gamma^0]^2\gamma^j) = 0 \end{cases}$$

(iv) "slash"-notatie: $\not{a} \equiv \gamma^\mu a_\mu \Rightarrow \not{a}^2 = (\gamma^\mu a_\mu)(\gamma^\nu a_\nu) = \frac{1}{2} a_\mu a_\nu \{\gamma^\mu,\gamma^\nu\}$

$$\overset{(ii)}{=} a_\mu a_\nu g^{\mu\nu} I = a^2 I\,.$$

(v) Dirac-operator: $(i\hbar\not{\partial} - mc)$

Klein-Gordon operator: $(\Box + m^2c^2/\hbar^2)$

Verband tussen beide operatoren: $(i\hbar\not{\partial} + mc)(i\hbar\not{\partial} - mc)$

$$= -\hbar^2\not{\partial}^2 - m^2c^2 I \overset{(iv)}{=} (-\hbar^2\Box - m^2c^2) I = -\hbar^2 I\,(\Box + m^2c^2/\hbar^2)\,.$$

Opg. 25) Dirac-vergelijking en Lorentz-transformaties (rotaties + boosts): uit het relativiteitsprincipe volgt dat als $x'^\mu \overset{\text{inf.}}{\approx} (g^\mu{}_\nu + d\omega^\mu{}_\nu)\, x^\nu$, dan moet gelden dat $\psi'(x') \approx (I + S(d\omega))\,\psi(x)$ met
$[\gamma^\rho, S(d\omega)] = d\omega^\rho{}_\nu \gamma^\nu \quad (\rho = 0,\dots,3)$ ① en $S(d\omega) = -S(-d\omega)$ ②.

Om de oplossing voor de 4x4 matrix $S(d\omega)$ te vinden ontbinden we t.o.v. de basis

(303) : $S(d\omega) = c\,I + c_\mu \gamma^\mu + c_{\mu\nu}\sigma^{\mu\nu} + \bar{c}_\mu \gamma^\mu\gamma^5 + \bar{c}\gamma^5$.

\# basismatrices: ① ④ ⑥ ④ ①

Vervolgens gaan we de complexe coëfficiënten $c, c_\mu, c_{\mu\nu}, \bar{c}_\mu, \bar{c}$ voor $\mu,\nu = 0,\dots,3$ bepalen. Opmerking: <u>$d\omega_{\mu\nu}$, $\sigma_{\mu\nu}$ en $c_{\mu\nu}$ zijn antisymmetrisch onder $\mu \leftrightarrow \nu$.</u>

(i) $\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} I$ geldt per definitie voor de γ-matrices

$\Rightarrow \gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu$ als $\mu \neq \nu$, $\gamma^\mu\gamma^\nu = \frac{1}{2}\{\gamma^\mu,\gamma^\nu\} = g^{\mu\nu} I$ als $\mu = \nu$,

$\gamma^\mu(i\gamma^0\gamma^1\gamma^2\gamma^3) \equiv \gamma^\mu\gamma^5 = (-1)^3\gamma^5\gamma^\mu = -\gamma^5\gamma^\mu$ (oftewel: $\{\gamma^\mu, \gamma^5\} = 0$, $\mu = 0,\dots,3$),

$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu,\gamma^\nu] = \frac{i}{2}\gamma^\mu\gamma^\nu - \frac{i}{2}\gamma^\nu\gamma^\mu = i\gamma^\mu\gamma^\nu$ als $\mu \neq \nu$, $\sigma^{\mu\nu} = 0$ als $\mu = \nu$.

(ii) ① met $\rho = 0$ en $d\omega^0{}_0 \overset{\text{anti-}}{\underset{\text{symm.}}{=}} 0 \overset{(i)}{\Longrightarrow} \overset{[\gamma^0, I]}{c * 0} + \overset{[\gamma^0,\gamma^0]}{c_0 * 0} + \sum_{k=1}^3 c_k * \overset{[\gamma^0,\gamma^k]}{\overbrace{2\gamma^0\gamma^k}} + \sum_{j,k=1}^3 c_{jk} * \overset{[\gamma^0, i\gamma^j\gamma^k]}{0}$

$+ \sum_{k=1}^3 \underbrace{(c_{0k} * \overset{[\gamma^0, i\gamma^0\gamma^k]}{\overbrace{2i\gamma^k}} - c_{k0} * 2i\gamma^k)}_{\text{antisymm.: } 4i c_{0k}\gamma^k} + \bar{c}_0 * \overset{[\gamma^0,\gamma^0\gamma^5]}{\overbrace{2\gamma^5}} + \sum_{k=1}^3 \bar{c}_k * \overset{[\gamma^0,\gamma^k\gamma^5]}{0} + \bar{c} * \overset{[\gamma^0,\gamma^5]}{\overbrace{2\gamma^0\gamma^5}} \overset{①}{=} \sum_{k=1}^3 d\omega^0{}_k \gamma^k = \sum_{k=1}^3 d\omega_{0k}\gamma^k$

$\overset{\text{basis}}{\Longrightarrow} c_k = \bar{c}_0 = \bar{c} = 0$ en $c_{0k} = -\frac{i}{4} d\omega_{0k}$ $(k = 1,2,3)$.

(iii) ① met $\rho = j$ (1,2,3) en $d\omega^j{}_j = 0 \overset{(i),(ii)}{\Longrightarrow} \overset{[\gamma^j, I]}{c * 0} + c_0 * \overset{[\gamma^j,\gamma^0]}{\overbrace{2\gamma^j\gamma^0}} + \sum_{k,l \neq j} c_{kl} * \overset{[\gamma^j, i\gamma^k\gamma^l]}{0}$

$+ \sum_{k\neq j} \underbrace{(c_{kj} * \overset{[\gamma^j, i\gamma^k\gamma^j]}{\overbrace{2i\gamma^k}} - c_{jk} * 2i\gamma^k)}_{\text{antisymm.: } -4i c_{jk}\gamma^k} + \underbrace{c_{0j} * \overset{[\gamma^j, i\gamma^0\gamma^j]}{\overbrace{2i\gamma^0}} - c_{j0} * 2i\gamma^0}_{\text{antisymm.: } -4i c_{j0}\gamma^0} + \sum_{k\neq j} \bar{c}_k * \overset{[\gamma^j,\gamma^k\gamma^5]}{0} + \bar{c}_j * \overset{[\gamma^j,\gamma^j\gamma^5]}{\overbrace{(-2\gamma^5)}}$

$\overset{①}{=} d\omega^j{}_0\gamma^0 + \sum_{k\neq j} d\omega^j{}_k\gamma^k = -d\omega_{j0}\gamma^0 - \sum_{k\neq j} d\omega_{jk}\gamma^k \overset{\text{basis}}{\Longrightarrow} c_0 = \bar{c}_j = 0$ en $c_{jk} = -\frac{i}{4} d\omega_{jk}$ $(j,k = 1,2,3)$

(iv) We hebben nu gevonden dat $c_{\mu\nu} = -\frac{i}{4} d\omega_{\mu\nu}$, maar de coëfficiënt c is nog steeds onbepaald. Echter, uit conditie ② volgt een extra voorwaarde:

$S(d\omega) = c\,I - \frac{i}{4} d\omega_{\mu\nu}\sigma^{\mu\nu} = -S(-d\omega) = -c\,I - \frac{i}{4} d\omega_{\mu\nu}\sigma^{\mu\nu} \Rightarrow c = 0$, zodat

$\sigma^{\mu\nu}$ genereert inf. Lor. transf. $\rightarrow$ $\boxed{S(d\omega) = -\frac{i}{4} d\omega_{\mu\nu}\sigma^{\mu\nu}}$