Opgaven bij het college Kwantummechanica 3

Opg. 1) $|n_1,n_2,\dots\rangle = \frac{(\hat a_1^\dagger)^{n_1}}{\sqrt{n_1!}}\frac{(\hat a_2^\dagger)^{n_2}}{\sqrt{n_2!}}\cdots|\Psi^{(0)}\rangle \equiv \prod_j \frac{(\hat a_j^\dagger)^{n_j}}{\sqrt{n_j!}}|\Psi^{(0)}\rangle$ } basistoestanden die de Fock-ruimte opspannen ($n_j = 0,1,\dots$)

$|\Psi^{(0)}\rangle \equiv |0,0,\dots\rangle$, $\langle\Psi^{(0)}|\Psi^{(0)}\rangle = 1$ en $\hat a_j|\Psi^{(0)}\rangle = 0$

Bosonische commutatierelaties: $[\hat a_j^\dagger,\hat a_k^\dagger] = [\hat a_j,\hat a_k]=0$ en $[\hat a_j,\hat a_k^\dagger]=\delta_{jk}\hat 1$ (B)

(i) Overgang naar andere 1-deeltjesrepr.: creatie/annihilatie-operatoren $\hat b_r^\dagger$ en $\hat b_r$

Relaties: $\hat b_r^\dagger = \sum_j \hat a_j^\dagger\langle q_j|p_r\rangle$ en $\hat b_r = \sum_j \hat a_j\langle p_r|q_j\rangle$

$\Rightarrow [\hat b_r^\dagger,\hat b_v^\dagger] = \sum_{j,k}\langle q_j|p_r\rangle\langle q_k|p_v\rangle[\hat a_j^\dagger,\hat a_k^\dagger] \overset{(B)}{=} 0 \;\underline{\text{evenzo}}\; [\hat b_r,\hat b_v]$,

$[\hat b_r,\hat b_v^\dagger] = \sum_{j,k}\langle p_r|q_j\rangle\langle q_k|p_v\rangle[\hat a_j,\hat a_k^\dagger] \overset{(B)}{=} \sum_j\langle p_r|q_j\rangle\langle q_j|p_v\rangle\hat 1 \overset{(19)}{=} \delta_{rv}\hat 1$

De bosonische commutatierelaties zijn dus vorminvariant!

(ii) Te bewijzen m.b.v. volledige inductie: $[\hat a_j,(\hat a_j^\dagger)^{n_j}] = n_j(\hat a_j^\dagger)^{n_j-1}$ ①.

Geldt triviaal voor $n_j=0,1$. Stel nu dat ① geldt voor $n_j=n$, dan

$[\hat a_j,(\hat a_j^\dagger)^{n+1}] = [\hat a_j,(\hat a_j^\dagger)^n]\hat a_j^\dagger + (\hat a_j^\dagger)^n[\hat a_j,\hat a_j^\dagger] = n(\hat a_j^\dagger)^{n-1}\hat a_j^\dagger + (\hat a_j^\dagger)^n\hat 1 = (n+1)(\hat a_j^\dagger)^n$ □

(iii) Alle creatie-operatoren $\hat a_1^\dagger,\hat a_2^\dagger,\dots$ commuteren en kunnen dus vrijelijk verwisseld worden in de toestanden $|n_1,n_2,\dots\rangle$

$\Rightarrow$ de volgorde v/d creatie-operatoren is niet belangrijk.

(iv) $$\hat a_j|\dots,n_j,\dots\rangle = \hat a_j\prod_k \frac{(\hat a_k^\dagger)^{n_k}}{\sqrt{n_k!}}|\Psi^{(0)}\rangle \overset{(B)}{=} \Big(\prod_{k<j}\frac{(\hat a_k^\dagger)^{n_k}}{\sqrt{n_k!}}\Big)\underbrace{\hat a_j\frac{(\hat a_j^\dagger)^{n_j}}{\sqrt{n_j!}}}\Big(\prod_{l>j}\frac{(\hat a_l^\dagger)^{n_l}}{\sqrt{n_l!}}\Big)|\Psi^{(0)}\rangle$$

$$\overset{(B)}{=} \sqrt{n_j}\,|\dots,n_j-1,\dots\rangle + \cancel{\Big(\prod_k \frac{(\hat a_k^\dagger)^{n_k}}{\sqrt{n_k!}}\Big)\hat a_j|\Psi^{(0)}\rangle}, \qquad ①:\ \frac{(\hat a_j^\dagger)^{n_j}}{\sqrt{n_j!}}\hat a_j + n_j\frac{(\hat a_j^\dagger)^{n_j-1}}{\sqrt{n_j!}}$$

$$\hat a_j^\dagger|\dots,n_j,\dots\rangle = \hat a_j^\dagger\prod_k \frac{(\hat a_k^\dagger)^{n_k}}{\sqrt{n_k!}}|\Psi^{(0)}\rangle \overset{(B)}{=} \Big(\prod_{k<j}\frac{(\hat a_k^\dagger)^{n_k}}{\sqrt{n_k!}}\Big)\frac{(\hat a_j^\dagger)^{n_j+1}}{\sqrt{n_j!}}\Big(\prod_{l>j}\frac{(\hat a_l^\dagger)^{n_l}}{\sqrt{n_l!}}\Big)|\Psi^{(0)}\rangle$$

$$= \sqrt{n_j+1}\,|\dots,n_j+1,\dots\rangle .$$

(v) Inwendig product van twee toestanden: $\langle n_1',n_2',\dots|n_1,n_2,\dots\rangle$

$$= \langle\Psi^{(0)}|\cdots\frac{(\hat a_2)^{n_2'}}{\sqrt{n_2'!}}\frac{(\hat a_1)^{n_1'}}{\sqrt{n_1'!}}|n_1,n_2,\dots\rangle \overset{(iv)}{=} \prod_j \underbrace{\frac{\sqrt{n_j(n_j-1)\cdots(n_j-n_j'+1)}}{\sqrt{n_j'!}}}_{0 \text{ als } n_j'>n_j}\ \underbrace{\langle\Psi^{(0)}|n_1-n_1',n_2-n_2',\dots\rangle}_{0\text{ als } n_j'<n_j,\ \text{want } \langle\Psi^{(0)}|\hat a_j^\dagger=0}$$

$$= \prod_j\delta_{n_j',n_j}\frac{\sqrt{n_j!}}{\sqrt{n_j!}}\langle\Psi^{(0)}|\Psi^{(0)}\rangle = \prod_j\delta_{n_j',n_j}$$ $\Rightarrow$ de toest. vormen een orthonormale set.

De truc is dat er eventueel $\hat a_j$'s als $\hat a_j^\dagger$'s tussen $\langle\Psi^{(0)}|$ en $|\Psi^{(0)}\rangle$ moeten staan i.v.m. $\langle\Psi^{(0)}|\hat a_j^\dagger = 0$ en $\hat a_j|\Psi^{(0)}\rangle=0$: bijv. $\langle\Psi^{(0)}|\hat a_j^2\hat a_j^\dagger|\Psi^{(0)}\rangle \overset{(B)}{=} \langle\Psi^{(0)}|(\hat a_j\hat a_j^\dagger\cancel{\hat a_j}+\cancel{\hat a_j})|\Psi^{(0)}\rangle = 0$ en

$\langle\Psi^{(0)}|\hat a_j\hat a_j^\dagger\hat a_j\hat a_j^\dagger|\Psi^{(0)}\rangle \overset{(B)}{=} \langle\Psi^{(0)}|(\hat a_j^\dagger\hat a_j+\hat 1)(\hat a_j^\dagger\cancel{\hat a_j}+\hat 1)|\Psi^{(0)}\rangle = 1$.

Opg. 2) $|n_1,n_2,\dots\rangle = \frac{(\hat a_1^\dagger)^{n_1}}{\sqrt{n_1!}}\frac{(\hat a_2^\dagger)^{n_2}}{\sqrt{n_2!}}\cdots|\Psi^{(0)}\rangle \equiv \prod_j \frac{(\hat a_j^\dagger)^{n_j}}{\sqrt{n_j!}}|\Psi^{(0)}\rangle$

$|\Psi^{(0)}\rangle \equiv |0,0,\dots\rangle$, $\langle\Psi^{(0)}|\Psi^{(0)}\rangle = 1$ en $\hat a_j|\Psi^{(0)}\rangle = 0$

} basistoestanden die de Fock-ruimte opspannen ($n_j = 0,1$: uitsluitingsprincipe)

Fermionische anticommutatierelaties: $\{\hat a_j^\dagger,\hat a_k^\dagger\} = \{\hat a_j,\hat a_k\} = 0$ en $\{\hat a_j,\hat a_k^\dagger\} = \delta_{jk}\hat 1$ (F)

(i) Overgang naar andere 1-deeltjesrepr.: creatie/annihilatie-operatoren $\hat b_r^\dagger$ en $\hat b_r$.

Relaties: $\hat b_r^\dagger = \sum_j \hat a_j^\dagger\langle q_j|p_r\rangle$ en $\hat b_r = \sum_j \hat a_j\langle p_r|q_j\rangle$

$\Rightarrow \{\hat b_r^\dagger,\hat b_v^\dagger\} = \sum_{j,k}\langle q_j|p_r\rangle\langle q_k|p_v\rangle\{\hat a_j^\dagger,\hat a_k^\dagger\} \overset{(F)}{=} 0 \overset{\text{evenzo}}{=} \{\hat b_r,\hat b_v\}$,

$\{\hat b_r,\hat b_v^\dagger\} = \sum_{j,k}\langle p_r|q_j\rangle\langle q_k|p_v\rangle\{\hat a_j,\hat a_k^\dagger\} \overset{(F)}{=} \sum_j \langle p_r|q_j\rangle\langle q_j|p_v\rangle\hat 1 \overset{(13)}{=} \delta_{rv}\hat 1$.

De fermionische anticommutatierelaties zijn dus vorminvariant!

(ii) Alle creatie-operatoren $\hat a_1^\dagger, \hat a_2^\dagger,\dots$ anticommuteren $\Rightarrow$ een permutatie van twee creatie-operatoren in $|n_1,n_2,\dots\rangle$ bestaat uit een oneven aantal anticommuterende verwisselingen (2x verwisseling met tussenliggende operatoren + onderlinge verwisseling) en geeft dus een minteken $\dots x\dots\dots x\dots$

$\Rightarrow$ de volgorde v/d creatie-operatoren is wel belangrijk.

(iii) $\hat a_j|\dots,n_j,\dots\rangle = \hat a_j\prod_k \frac{(\hat a_k^\dagger)^{n_k}}{\sqrt{n_k!}}|\Psi^{(0)}\rangle \overset{(F)}{=} (-1)^{N_{<j}}\left(\prod_{k<j}\frac{(\hat a_k^\dagger)^{n_k}}{\sqrt{n_k!}}\right)\underbrace{\hat a_j\frac{(\hat a_j^\dagger)^{n_j}}{\sqrt{n_j!}}}_{\cancel{\delta_{n_j,0}\hat a_j} + \delta_{n_j,1}(\hat 1 - \cancel{\hat a_j^\dagger\hat a_j})}\left(\prod_{l>j}\frac{(\hat a_l^\dagger)^{n_l}}{\sqrt{n_l!}}\right)|\Psi^{(0)}\rangle$

$N_{<j} = \sum_{k=1}^{j-1} n_k$

$= \delta_{n_j,1}(-1)^{N_{<j}}|\dots,\overset{0}{\widehat{n_j-1}},\dots\rangle$,

$\hat a_j^\dagger|\dots,n_j,\dots\rangle = \hat a_j^\dagger\prod_k \frac{(\hat a_k^\dagger)^{n_k}}{\sqrt{n_k!}}|\Psi^{(0)}\rangle \overset{(F)}{=} (-1)^{N_{<j}}\left(\prod_{k<j}\frac{(\hat a_k^\dagger)^{n_k}}{\sqrt{n_k!}}\right)\underbrace{\frac{(\hat a_j^\dagger)^{n_j+1}}{\sqrt{n_j!}}}_{\substack{\delta_{n_j,0}\hat a_j^\dagger \\ \text{(Pauli)}}}\left(\prod_{l>j}\frac{(\hat a_l^\dagger)^{n_l}}{\sqrt{n_l!}}\right)|\Psi^{(0)}\rangle$

$= \delta_{n_j,0}(-1)^{N_{<j}}|\dots,\overset{1}{\widehat{n_j+1}},\dots\rangle$.

(iv) Inwendig product van twee toestanden: $\langle n_1',n_2',\dots|n_1,n_2,\dots\rangle$

$$= \langle\Psi^{(0)}|\cdots\frac{(\hat a_2)^{n_2'}}{\sqrt{n_2'!}}\frac{(\hat a_1)^{n_1'}}{\sqrt{n_1'!}}|n_1,n_2,\dots\rangle \underset{\langle\Psi^{(0)}|\hat a_j^\dagger = 0}{\overset{(iii)}{=}} \begin{cases}\delta_{n_1,1}\langle\Psi^{(0)}|\cdots\frac{(\hat a_2)^{n_2'}}{\sqrt{n_2'!}}|0,n_2,\dots\rangle & \text{als } n_1'=1\\ \delta_{n_1,0}\langle\Psi^{(0)}|\cdots\frac{(\hat a_2)^{n_2'}}{\sqrt{n_2'!}}|0,n_2,\dots\rangle & \text{als } n_1'=0\end{cases}$$

$$= \delta_{n_1,n_1'}\langle\Psi^{(0)}|\cdots\frac{(\hat a_2)^{n_2'}}{\sqrt{n_2'!}}|0,n_2,\dots\rangle = \dots = \prod_j\delta_{n_j,n_j'}\langle\Psi^{(0)}|0,0,\dots\rangle = \prod_j\delta_{n_j,n_j'}$$

$\Rightarrow$ de toestanden vormen een orthonormale set.