Loop Quantum Cosmology

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QUANTUM GRAVITY IN CRACOW 2
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Open Issues

LQC provides the most successful physical application of loop gravity, and one of the most promising avenues towards a possible empirical test, but...

1. Which is the relationship between LQC and the full LQG?
   Brunnemann, Fleischhack, Engle, Bojowald, Kastrup, Koslowski...
2. Can we describe the full quantum geometry at the bounce?
3. Can we include inhomogeneities ? Bojowald, Hernandez, Kagan, Singh, Skirzewski, Martin-Benito, Garay, Mena Marugan...
   - Structure Formation
   - Inflation
   - Dark Energy

A strategy to address these questions

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The Cosmological Principle

- The dynamics of a homogeneous and isotropic space describes our real universe.
- The effect of the inhomogeneities on the dynamics of its largest scale, described by the scale factor, can be neglected in a first approximation.
- This is not a large scale approximation, because it is supposed to remain valid when the universe was small! It is an expansion in $n \sim \frac{a}{\lambda}$!
- The full theory may be expanded by adding degrees of freedom one by one, starting from the cosmological ones.
- We can define an approximated dynamics of the universe for a finite number of d.o.f. .

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Mode expansion $\longleftrightarrow$ sum over triangulations

- The large scale d.o.f. can be captured by averaging the metric over the simplices of a triangulation formed by n simplices.
- The full theory can be regarded as an expansion for growing n. Cosmology corresponds to the lower order where there is only a tetrahedrum: the only d.o.f. is given by the volume.
- Restrict the dynamics to a finite n. Define an approximated dynamics of the universe, inhomogeneous but truncated at a finite number of tertrahedra.
- At fixed n, approximate the dynamics by the non-graph changing Hamiltonian constraint. This gives a consistent classical and quantum model for each n.

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Definition of the classical theory

Oriented triangulation Δ_n of a 3-sphere (n tetrahedra t and $2n$ triangles f)

Variables $\begin{cases} U_f & \in SU(2), \\ E_f = E_f^i \tau_i & \in su(2)\,. \end{cases}$

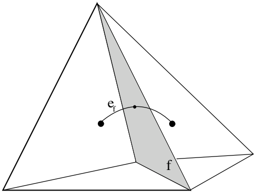
Poisson brackets $\begin{cases} \{U_f, U_{f'}\} & = \quad 0\,, \\ \{E_f^i, U_{f'}\} & = \quad \delta_{ff'}\ \tau^i U_f\,, \\ \{E_f^i, E_{f'}^j\} & = \; -\,\delta_{ff'}\ \epsilon^{ijk} E_f^k\,. \end{cases}$

Dynamics $\begin{cases} \text{Gauge} & G_t \equiv \sum_{f \in t} E_f \sim 0, \\ \text{Hamiltonian} & C_t \equiv \sum_{ff' \in t} Tr[(U_{ff'}\ E_{f'} E_f] \sim 0 \end{cases}$

Interpretation

- Cosmological approximation to the dynamics of the geometry of a closed universe.
- (U_f, E_f) average gravitational d.o.f. over a triangulation Δ_n of space:

- U_f: parallel transport of the Ashtekar connection A_a along the link e_f of Δ_n^* dual to the f;
- E_f: flux Φ_f of the Ashtekar's electric field E^a across the triangle f, parallel transported to the center of the tetrahedron: $E_f = U_{e_1}^{-1} \Phi_f U_{e_1}$.



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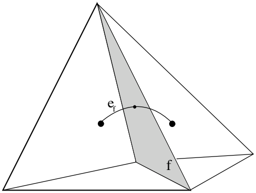
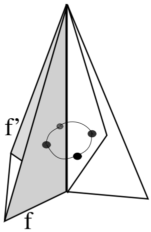


$$U_{f-1} = U_f^{-1} \quad \text{and} \quad E_{f-1} = -U_f^{-1} E_f U_f$$

- The constraints approximate the Ashtekar's gauge and Hamiltonian constraint $\mathrm{Tr}[F_{ab}E^aE^b] \sim 0$.
- C_t: Non-graph-changing hamiltonian constraint.
- $U_{ff'} \sim \mathbb{1} + |\alpha|^2 F_{ab} + o(|\alpha|^2 F)$

The expansion can be performed:

- for small loops whatever were F_{ab}
- but also for large loops if F_{ab} is small.



$U_{ff'} = U_f U_n \dots U_1 U_{f'}$

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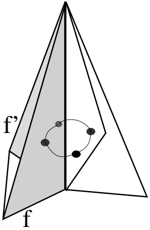

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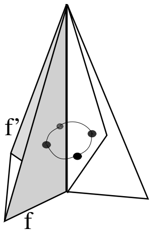


$U_{ff'} = U_f U_n \ldots U_1 U_{f'}$

Adding a scalar field

- Add a variable (ϕ_t, p_{ϕ_t}). Represents matter, defines an *n*-fingered time.
- Hamiltonian constraint

$$S_t = \frac{1}{V_t} C_t + \frac{\kappa}{2V_t}\, p^2_{\phi_t} \sim 0.$$

where

$$V_t = \sum_{ff'f'' \in t} \sqrt{Tr[E_f E_{f'} E_{f''}]}.$$

- Ultralocal.
- Easy to add spatial derivative terms, or extend to fermions and gauge fields.

Quantum theory

1. Hilbert space: $H_{aux} = L_2[SU(2)^{2n}, dU_f]$. States $\psi(U_f)$.

2. Operators:
   U_f are diagonal and E_f are the left invariant vector fields on each $SU(2)$.
   The operators $E_{f^{-1}}$ turn then out to be the right invariant vector fields!

3. States that solve gauge constraint: $SU(2)$ spin networks on graph Δ_n^*

$$\psi_{j_f \iota_t}(U_f) \equiv \langle U_f | j_f, \iota_t \rangle \equiv \otimes_f \Pi^{(j_f)}(U_f) \cdot \otimes_t \iota_t. \quad (1)$$

4. With a scalar field: $H_{aux} = L_2[SU(2)^{2n}, dU_f] \otimes L_2[\mathbb{R}^n]$, with

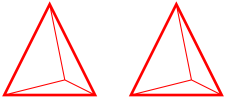
$$\psi(j_f, \iota_t, \phi_t) \equiv \langle j_f, \iota_t, \phi_t | \psi \rangle. \quad (2)$$

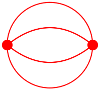
5. Quantum Hamiltonian constraint: rewriting it in the Thiemann's form

$$C_t = \sum_{ff'f'' \in t} \epsilon^{ff'f''} \mathrm{Tr}\left[U_{ff'} U_{f''}^{-1} [U_{f''}, V_t] \right] \sim 0. \quad (3)$$

Dipole cosmology

Take $n = 2$ so that Δ_2 is formed by two tetrahedra glued along all their faces.

$\Delta_2 =$ 

$\Delta_2^* =$ 

$\mathcal{H}_{aux} = L_2[\,SU(2)^4] \otimes L_2[\mathbb{R}^2]$. Gauge invariant states $\psi(j_f, \iota_t, \phi_t)$.

Spin networks basis $|\, j_f, \iota_t, \phi_t\rangle = |\, j_1, j_2, j_3, j_4, \iota_1, \iota_2, \phi_1, \phi_2\rangle$.

Dynamics:

$$\begin{cases} \dfrac{\partial^2}{\partial\phi_1^2}\,\psi(j_f, \iota_t, \phi_t) = \dfrac{2}{\kappa} \displaystyle\sum_{\epsilon_f = 0, \pm 1} C_1{}^{\epsilon_f \iota'_t}_{j_f \iota_t}\; \psi\left(j_f + \frac{\epsilon_j}{2}, \iota'_t, \phi_t\right), \\[2ex] \dfrac{\partial^2}{\partial\phi_2^2}\,\psi(j_f, \iota_t, \phi_t) = \dfrac{2}{\kappa} \displaystyle\sum_{\epsilon_f = 0, \pm 1} C_2{}^{\epsilon_f \iota'_t}_{j_f \iota_t}\; \psi\left(j_f + \frac{\epsilon_j}{2}, \iota'_t, \phi_t\right). \end{cases}$$

Born-Oppenheimer approximation

1. d.o.f.: “heavy" (R, p_R) (nuclei) and “light" (r, p_r) (electrons).
2. B-O Ansatz: $\psi(R, r) = \Psi(R)\phi(R; r)$, where $\partial_R \Phi(R; r)$ is small.
3. Hamiltonian splits as $H(R, r, p_R, p_r) = H_R(R, p_R) + H_r(R; r, p_r)$

Time independent Schrödinger equation $H\psi = E\psi$ becomes

$H\psi = (H_R + H_r)\Psi\Phi = \Phi H_R \Psi + \Psi H_r \Phi = E\Psi\Phi \;\Rightarrow\; \frac{H_R\Psi}{\Psi} - E = -\frac{H_r\Phi}{\Phi}$.

Since the lhs does not depend on r, each side is equal to a function $\rho(R)$.
Therefore we can write two equations

$$\begin{cases} H_R\Psi(R) + \rho(R)\Psi(R) = E\Psi(R) & (1) \quad \text{Schr. eq. for nuclei, with additional term.} \\ H_r\Phi(R, r) = \rho(R)\Phi(R, r) & (2) \quad \text{Schr. eq. for electrons, in the background } R. \end{cases}$$

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Homogeneous and inhomogeneous d.o.f.

Let us apply this idea to dipole cosmology: $R \to$ hom d.o.f. $r \to$ inhom d.o.f.

$U_f = \exp A_f$. ω_f: fiducial connection ($|\omega_f| = 1$).

1. d.o.f.

$$\begin{cases} A_f = c\,\omega_f + a_f, \\ E_f = p\,\omega_f + h_f. \end{cases} \qquad \begin{cases} V = p^{\frac{3}{2}}, \\ \{c, p\} = \frac{8\pi G}{3} = 1. \end{cases}$$

Also $\phi_{1,2} = \frac{1}{2}(\phi \pm \Delta\phi)$, and $V_{1,2} = \frac{1}{2}(V \pm \Delta V)$.

2. B-O Ansatz: $\psi(c, a, \phi, \Delta\phi) = \Psi(c, \phi)\phi(c, \phi; a, \Delta\phi)$.
$c \in [0, 4\pi]$ is a periodic variable. We can therefore expand $\Psi(c, \phi)$

$$\Psi(c, \phi) = \sum_{\text{integer } \mu} \psi(\mu, \phi)\, e^{i\mu c/2}.$$

The basis of states $\langle c\,|\mu\rangle = e^{i\mu c/2}$ satisfies

$$p|\mu\rangle = \frac{1}{2}|\mu\rangle \qquad e^{ic}|\mu\rangle = |\mu + 2\rangle$$

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3. Hamiltonian constraint: $C_t = C_t^{hom} + C_t^{in}$.

- With some technicalities:

$$\begin{cases} \frac{\partial^2}{\partial\phi^2}\Psi(c,\phi) - C^{hom}\Psi(c,\phi) - \rho(c,\phi)\Psi(c,\phi) = 0, & (1) \\ \frac{\partial^2}{\partial\phi^2}\phi(c,\phi;a,\Delta\phi) + C^{inh}\phi(c,\phi;a,\Delta\phi) = \rho(c,\phi)\phi(c,\phi;a,\Delta\phi). & (2) \end{cases}$$

(1) Quantum Friedmann equation for the homogeneous d.o.f. (c,ϕ), corrected by the energy density $\rho(c,\phi)$ of the inhomogeneous modes.

(2) The Schrödinger equation for the inhomogeneous modes in the background homogeneous cosmology (c,ϕ). $\rho(c,\phi)$ energy eigenvalue.

- At the order zero of the approximation, where we disregard entirely the effect of the inhomogeneous modes on the homogeneous modes, we obtain

$$\frac{\partial^2}{\partial\phi^2}\Psi(c,\phi) = C^{hom}\Psi(c,\phi). \quad (3)$$

Quantum Friedmann equation

The Hamiltonian constraint

$$C_t(c,a,p,h) = \sum_{ff'f''\in t} \mathrm{Tr}\big[e^{c\omega_f + a_f}\, e^{-c\omega_{f'} - a_{f'}}\, e^{-c\omega_{f''} - a_{f''}}\,[e^{c\omega_{f''}+a_{f''}}, V \pm \Delta V]\big] .$$

becames in the B-O approximation disregarding the inhomogeneus variables

$$C_t^{hom}(c,p) = \frac{1}{12} \sum_{ff'f''\in t} \mathrm{Tr}\Big[e^{c\omega_f}\, e^{-c\omega_{f'}}\ \ e^{-c\omega_{f''}}\,[e^{c\omega_{f''}}, p^{\frac{3}{2}}]\Big]$$

and writing the holonomies using the Eulero's formulas

$$C_t^{hom} = \sum_{ff'f''} Tr\left[\left(\cos\left(\frac{c}{2}\right)\mathbb{I} + 2\sin\left(\frac{c}{2}\right)\omega_f\right)\left(\cos\left(\frac{c}{2}\right)\mathbb{I} - 2\sin\left(\frac{c}{2}\right)\omega_{f'}\right) e^{-c\omega_{f''}}\,[e^{c\omega_{f''}}, p^{\frac{3}{2}}]\right]$$

$$= \sum_{ff'f''} \mathrm{Tr}\left[\left(\cos^2\left(\frac{c}{2}\right)\mathbb{I} + 2\sin\left(\frac{c}{2}\right)\cos\left(\frac{c}{2}\right)(\omega_f - \omega_{f'}) - 4\sin^2\left(\frac{c}{2}\right)[\omega_f, \omega_{f'}]\right) e^{-c\omega_{f''}}\,[e^{c\omega_{f''}}, p^{\frac{3}{2}}]\right] .$$

- Consider the action of the last factor on the state $|\mu\rangle$

$$e^{-c\omega_{f''}}[e^{c\omega_{f''}}, p^{\frac{3}{2}}]e^{i\mu c/2} = \left(-ik\frac{\partial}{\partial c}\right)^{\frac{3}{2}} e^{i\mu c/2} - e^{-c\omega_{f''}}\left(-ik\frac{\partial}{\partial c}\right)^{\frac{3}{2}} e^{ic(\mu/2 - i\omega_{f''})}$$
$$= k\left(\mu^{\frac{3}{2}}\mathbb{I} - (\mu\mathbb{I} - i2\omega_{f''})^{\frac{3}{2}}\right) e^{i\mu c/2}. \quad (1)$$

- We can write $(\mu\mathbb{I} - i2\omega_{f''})^{\frac{3}{2}} = \alpha(\mu)\mathbb{I} + \beta(\mu)\omega_{f''}$, compute the coefficients $\alpha(\mu)$ and $\beta(\mu)$ squaring this equation and take $\tilde{\alpha}(\mu) = (\mu/2)^{\frac{3}{2}} - \alpha(\mu)$.

- $$C_t^{hom} e^{i\mu c/2} = \left[\tilde{\alpha}(\mu) + \left(\frac{7}{4}\tilde{\alpha}(\mu) + 3^{\frac{1}{4}}\, 2^{\frac{5}{2}}\, \beta(\mu)\right)\sin^2\frac{c}{2}\right] e^{i\mu c/2}$$

- Bringing everything together, the quantum Friedmann equation reads

$$\boxed{\frac{\partial^2}{\partial\phi^2}\Psi(\mu,\phi) = C^+(\mu)\ \Psi(\mu+2,\phi) + C^0(\mu)\ \Psi(\mu,\phi) + C^-(\mu)\ \Psi(\mu-2,\phi)}$$

where $C^+(\mu) = C^-(\mu) = \frac{\mu^{3/2}}{\kappa\sqrt{2}}\left[-\frac{7}{16}\,\tilde{\alpha}(\mu) - \sqrt{2\sqrt{3}}\,\beta(\mu)\right]$

and $C^0(\mu) = \frac{\mu^{3/2}}{\kappa\sqrt{2}}\left[\frac{11}{8}\tilde{\alpha}(\mu) + 2\sqrt{2\sqrt{3}}\,\beta(\mu)\right]$.

- This eq. has precisely the structure of the LQC dynamical equation.
- μ is discrete without ad hoc hypotheses, or area-gap argument.

Summary

1. Family of models opening a systematic way
   for describing the inhomogeneous d.o.f. in quantum cosmology.
2. Derivation of the structure of LQC as a B-O approximation:
   light on LQC/LQG relation.

Comments

1. Does bounce scenario survives?
   How do quantum inhomogeneous fluctuations affect structure formation?
   and for inflation?
2. $\rho(c, \phi)$ term in quantum Friedmann eq.
   Physics? Affects cosmological constant?
3. Intuition that near-flat-space dynamics can *only* be described by many nodes is misleading. Relevant for the *n*-point functions calculations.

To be done

1. Immirzi parameter γ.
2. Realistic matter fields.
3. Relation between the $\psi(\mu, \phi)$ homogeneous states and the full $\psi(j_f, \iota_t, \phi_t)$ states in the spinnetwork basis.
4. Relation to $\bar{\mu}$ quantization scheme.
5. Spinfoam version. Cosmological Regge calculus (Barrett, Williams et al). $1 \to 4, 4 \to 1$ Pachner moves.