a new twist on spin connections

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PLAN OF THE TALK

- **Twisted Geometry**
- **A connection for twisted geometry**
  - Interpolation
  - Which interpolation?
- **Geometrical interpretation**
- Is twisting related to torsion?

CLASSICAL VS CONTINUOUS

$$\begin{array}{ccc}
\begin{array}{c}\text{Exact quantum gravity}\\ \text{transition amplitudes}\\ W(h_l)\end{array} & \xrightarrow{\hbar\to 0} & \begin{array}{c}\text{General relativity}\\ \text{Hamilton function}\\ S(q)\end{array} \\
\Big\uparrow{\scriptstyle \mathcal{C}\to\infty} & & \Big\uparrow{\scriptstyle \Delta\to\infty} \\
\begin{array}{c}\text{LQG}\\ \text{transition amplitudes}\\ W_{\mathcal{C}}(h_l)\end{array} & \xrightarrow{\hbar\to 0} & \begin{array}{c}\text{Regge}\\ \text{Hamilton function}\\ S_{\Delta}(l_{i_b})\end{array}
\end{array}$$

Continuous limit $\uparrow$

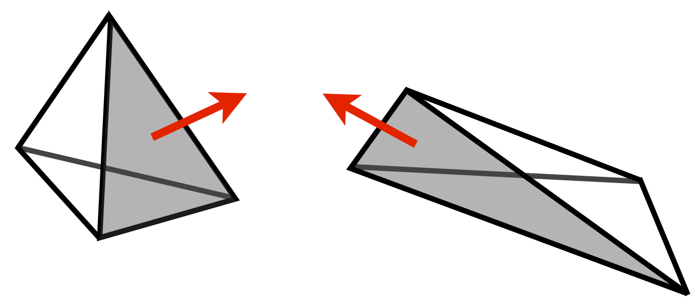
Classical limit $\longrightarrow$

- No critical point
- No infinite renormalization
- Physical scale: Planck length

Regime of validity of the expansion:

$$L_{Planck} \ll L \ll \sqrt{\frac{1}{R}}$$

twisted geometry

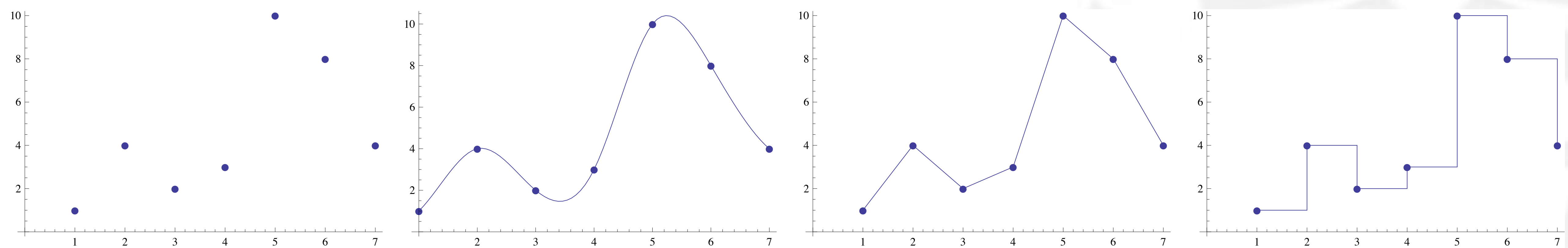


- By a twisted geometry we mean: an oriented 3d simplicial complex (a triangulation) $\mathcal{T}$, equipped with a flat metric on each 3-simplex (which makes it a flat tetrahedron)
- For any two tetrahedra sharing a face **the area of the face is the same** whether it is computed from the metric on one side or the other.
- Regge geometry: impose to the length of the edges to be the same.
- In a twisted geometry two adjacent triangles have the same area and the same normal, but the angles and the edge lengths can differ.
- The two triangles can be identified => discontinuity of the metric across the triangle

ASSIGNING GEOMETRIES TO STATES

Freidel Speziale 2010
Rovelli, Speziale 2010

- Loop gravity on a fixed graph describes a truncation of general relativity. The variables capture only a finite number of the degrees of freedom of the metric. There is no unique geometric interpretation associated to a single graph.
- "Interpolating" geometries are not strictly needed for the physical interpretation of the theory, but provide useful approximations of a continuous geometry.



- data set
- polynomial (mode expansion)
- piecewise linear (Regge geometries)
- piecewise flat (twisted geometries for generic holonomy-fluxes) discontinuous

- A twisted geometry is a specific choice of "interpolating geometry", chosen among *discontinuous* metrics. To any graph and any holonomy-flux configuration, we can associate a twisted geometry: a discrete discontinuous geometry on a cellular decomposition space into polyhedra.
The phase space of LQG on a graph can be visualized not only in terms of holonomies and fluxes, but also in terms of a simple geometrical picture of adjacent flat polyhedra.

REALATIONS

$$
\begin{tikzcd}
\textit{twistor space} \arrow[d, "\text{area matching}"] & \\
\textit{``open'' twisted geometry} \arrow[r, leftrightarrow] \arrow[d, "\text{closure}"] & \textit{SU(2) LQG} \arrow[d, "\text{Gauss law}"] \\
\textit{``closed'' twisted geometry} \arrow[r, leftrightarrow] \arrow[d, "\text{shape matching}"] & \textit{gauge-inv LQG} \\
\textit{Regge phase space} &
\end{tikzcd}
$$

the connection

"open" twisted geometry	$\longleftrightarrow$	*SU(2) LQG*
$\downarrow$ closure		$\downarrow$ Gauss law
"closed" twisted geometry	$\longleftrightarrow$	*gauge-inv LQG*

E => **flux operators** *define the* 3d geometry

A => **holonomy operators** *define the* SU(2) connection

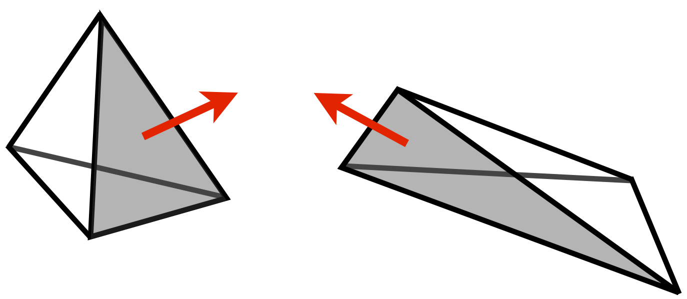
- independent: A not chosen to be compatible with e => in general there is torsion
- Ashtekar-Barbero connection: $A = \Gamma + \gamma K$

spin connection *extrinsic curvature*

CARTAN STRUCTURE EQUATION

- The spin connection is determined by $de^i + \epsilon^i{}_{jk}\, \omega^j \wedge e^k = \mathrm{T}$ *torsion*
- What can we carry in a discrete setting, i.e. *twisted geometry*?

  2 tets that meet along one face: opposite normals, same face area, different face shape

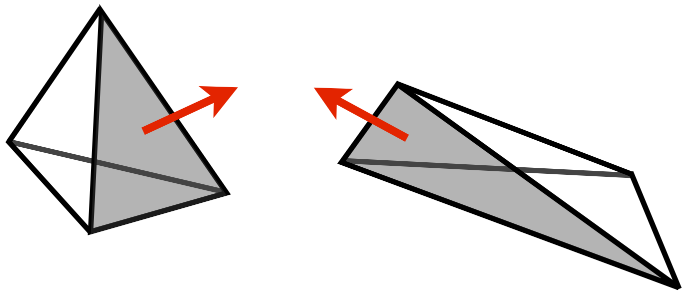


- The metric is discontinuous across the triangle!
  - no Cartan equation
  - Γ cannot be defined let alone the Ashtekar Barbero decomposition

CARTAN STRUCTURE EQUATION

- The spin connection is determined by $de^i + \epsilon^i{}_{jk}\, \omega^j \wedge e^k = 0$
- What can we carry in a discrete setting, i.e. *twisted geometry*?

  2 tets that meet along one face: opposite normals, same face area, different face shape



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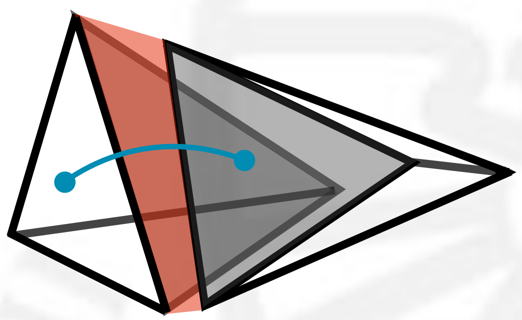
interpolated connection

AN EXPLICIT CONSTRUCTION OF Γ

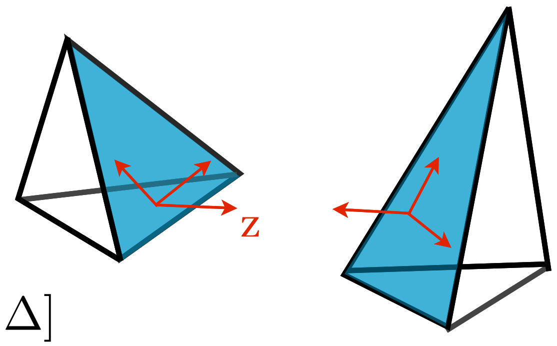
- Idea:
  - introduce interpolating geometry
  - calculate connection and holonomy
  - removing the interpolation by taking $\Delta \to 0$



$z=[0,\Delta]$

- **left**

  dx^i



$z \in [0, \Delta]$

- **right**

$$e^1 = e^1_x dx + e^1_y dy$$
$$e^2 = e^2_x dx + e^2_y dy$$
$$e^3 = dz$$

- area matching: $\det e = 1$

- linear transformation that matches shapes: $e = \{e^i{}_a\} = \begin{pmatrix} e^1_x & e^1_y & 0 \\ e^2_x & e^2_y & 0 \\ 0 & 0 & 1 \end{pmatrix}$

  $SL(2,\mathbb{R})$ upper block diagonal subgroup of $SL(3,\mathbb{R})$

- extend the definition of e to e(z): $e(0) = \mathbb{1}$ on the left

  $e(\Delta) = e$ on the right

uniqueness

INTERPOLATION

- Guiding principle: **the resulting holonomy must transform as an holonomy under a change of frame on either tetrahedron**

$$U(\Lambda_s e \Lambda_t^{-1}) = \Lambda_s U(e) \Lambda_t^{-1} \qquad \text{for every} \qquad \Lambda_s, \Lambda_t \in SO(3).$$

- Idea: interpolate using the Lie algebra of $sl(3,\mathbb{R})$

- Notice that a triad that is consistent with this condition is a matrix that can be written in a **polar decomposition**: $e = e^A e^S$

  a matrix $\mathbf{M}$ can be always written as $\mathrm{M{=}PU{=}U\widetilde{P}}$

  positive *unitary*

  and $\mathrm{M{=}e^A e^S{=}UP}$

  symmetric *antisymmetric*

- Interpolation: $e(z) = e^{zA} e^{zS}$ $\qquad e(z) \in SL(2,\mathbb{R}) \subset SL(3,\mathbb{R})$

- Cartan equation: $de^i = (A + e^{zA} S e^{-zA})^i{}_j \; dz \wedge e^j$

THE INTERPOLATED CONNECTION

- It is convenient to lower an index and antisymmetrize: $\omega_{ij} = \varepsilon_{ijk}\,\omega^k$
- The solution of the Cartan equation is: $\omega^i = B^i{}_j\, e^j$

  where $B^i{}_j = -\epsilon^{ikl}(A + e^{zA} S e^{-zA})_{jk} e^z_l + \frac{1}{2}\epsilon^{klm} A_{kl} e^z_m\ \delta^i_j$

- The holonomy across Δ is given by $U \;=\; \mathcal{P}\, e^{-\int_\gamma \omega} = \mathcal{P}\, e^{-\int_0^\Delta \omega(\partial_z) dz}$
- Notice that B is upper block diagonal: $\omega^k(\partial_z) = \frac{1}{2}\epsilon^{kij} A_{ij}$
- **The holonomy is just the rotation part of the polar decomposition:** $U = \exp A$
- **Distributional torsionless spin connection:** $\Gamma = -A\, d\tau$

  where $\tau : (\sigma^1, \sigma^2) \mapsto x^a(\sigma)$

$$d\tau_a(x) \equiv \int_\tau d^2\sigma\, \frac{\partial x^b}{\partial\sigma^1} \frac{\partial x^c}{\partial\sigma^2}\, \epsilon_{abc}\, \delta(x - x(\sigma))$$

a function of the normals

CONNECTION AS A FUNCTION OF THE NORMALS

- $U = e^{A}$ explicitly written in terms of the tetrads is $U(e) = e(e^T e)^{-1/2}$
- Explicit characterization: solve U given the normals to the tetrahedra! $\mathbf{n}_1 = \frac{1}{2}\,\mathbf{v}_2 \times \mathbf{v}_3$
- A tetrahedron is completely define by a triplet of vectors: $\mathbf{v}_a \in R^3, a = 1, 2, 3.$
- linear coordinates:

$$\begin{aligned} \mathbf{v}_1 &= (a, 0, 0) \qquad a = |\mathbf{v}_1|, \quad b = \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{|\mathbf{v}_1|}, \\ \mathbf{v}_2 &= (b, c, 0) \\ \hat{\mathbf{n}}_3 &= (0, 0, 1) \qquad c = \frac{\sqrt{|\mathbf{v}_1|^2 |\mathbf{v}_2|^2 - (\mathbf{v}_1 \cdot \mathbf{v}_2)^2}}{|\mathbf{v}_1|}. \end{aligned}$$

- Metric:

$$g = \begin{pmatrix} |\mathbf{v}_1|^2 & \mathbf{v}_1 \cdot \mathbf{v}_2 & 0 \\ \mathbf{v}_1 \cdot \mathbf{v}_2 & |\mathbf{v}_2|^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- The triad in this metric is $e^i = v_1^i dx + v_2^i dy + \hat{n}_3^i dz$
- $SL(3, \mathbb{R})$ matrix that transforms τ left in τ right: $s = \{e^i{}_a (\tilde{e}^{-1})^a{}_j\} = \begin{pmatrix} a/\tilde{a} & 0 & 0 \\ (b\tilde{c} - c\tilde{b})/\tilde{a}\tilde{c} & c/\tilde{c} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$e = \{e^i{}_a\} = \begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\cos(\theta) = (c\tilde{a} + a\tilde{c})/\sqrt{D}, \quad \sin(\theta) = (b\tilde{c} - c\tilde{b})/\sqrt{D},$$
$$D = \tilde{c}^2(a^2 + b^2) + c^2(\tilde{a}^2 + \tilde{b}^2) + 2c\tilde{c}(a\tilde{a} - b\tilde{b}).$$

- **Spin connection:**

$$\Gamma = \theta\, e(\hat{\mathbf{n}}_3)\, d\tau$$

curvature

- Proposition: If the twisted geometry is Regge, then the holonomy of Γ on a path around a bone, is a rotation around the the bone by an angle equal to the Regge deficit angle.
$$\delta_l = 2\pi - \textstyle\sum_i \theta_i$$

The spin connection reduces nicely to the Regge one, if shape matching is imposed.

- $U_l = U_{\sigma_n} U_{\tau_{n-1}} \cdots U_{\sigma_1} U_{\tau_1}$

The changes of frame within each tetrahedron bring the initial triangle's inward normal into the final triangle's outward normal and this is just a rotation about the bone by the dihedral angle.

- $$F^{ij} = d\omega^{ij} + \omega^i{}_k \wedge \omega^{kj} \xrightarrow{g_{ab} = e_{ai} e^i_b} F^{ij}[\omega(e)] = \frac{1}{2} e^i_c \, e^{jd} \, R^c{}_{dab}[g(e)] \; dx^a \wedge dx^b$$

- In general, the curvature in twisted geometry is not of the Regge form $R_{abcd} \sim e^{\delta \epsilon_{abe} l^e} \, \epsilon_{cdf} l^f$

- It may be possible to characterize what *Petrov classes* are possible in a twisted geometry and to see if they are more general than the single class that Regge geometry captures.

conclusions

CONCLUSIONS

- Two confusions:

  - Twisting does not encode torsion.

    We can define a torsionless connection in a twisted geometry.

    The twisting is purely metrical!

  - No need to suppress twisting :-)

thank you!