Tensors and Applications

SET 4

(4.1)

$$g_{\mu\nu} \to g_{\mu\nu} + \Gamma^{\alpha}_{\mu\beta}\, g_{\alpha\nu}\, dx^{\beta} + \Gamma^{\alpha}_{\nu\beta}\, g_{\mu\alpha}\, dx^{\beta} =$$

$$= g_{\mu\nu} + \left(\Gamma_{\nu\mu\beta} + \Gamma_{\mu\nu\beta}\right) dx^{\beta} =$$

$$= g_{\mu\nu} + g_{\mu\nu,\beta}\, dx^{\beta} = g_{\mu\nu}(x+dx)$$

$$g^{\mu\nu} \to g^{\mu\nu} - \Gamma^{\mu}_{\alpha\beta}\, g^{\alpha\nu}\, dx^{\beta} - \Gamma^{\nu}_{\alpha\beta}\, g^{\mu\alpha}\, dx^{\beta}$$

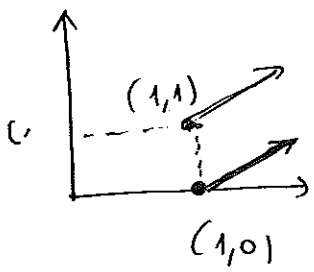
$$\rightsquigarrow g^{\mu\nu} + g^{\mu\nu}{}_{,\beta}\, dx^{\beta} = g^{\mu\nu}(x+dx)$$

$$\delta^{\mu}{}_{\nu} \to \delta^{\mu}{}_{\nu} - \Gamma^{\mu}{}_{\alpha\beta}\, \delta^{\alpha}{}_{\nu}\, dx^{\beta} +$$

$$+ \Gamma^{\alpha}{}_{\nu\beta}\, \delta^{\mu}{}_{\alpha}\, dx^{\beta} =$$

$$= \delta^{\mu}{}_{\nu} + dx^{\beta}\left(-\Gamma^{\mu}{}_{\nu\beta} + \Gamma^{\mu}{}_{\nu\beta}\right) = \delta^{\mu}{}_{\nu}$$

(4.3)



$$x^1 = \xi^1 \cos \xi^2$$

$$x^2 = \xi^1 \sin \xi^2$$

$$(1,0) \Rightarrow \xi^2 = 0$$
$$\xi^1 = 1$$

$$A^\mu(\xi) = \underbrace{A^\alpha(\xi(x))}_{(1,1)} \frac{\partial \xi^\mu}{\partial x^\alpha}$$

$$A^1(\xi) = \frac{\partial \xi^1}{\partial x^1} + \frac{\partial \xi^1}{\partial x^2} = \frac{x^1}{\xi^1} + \frac{x^2}{\xi^1} = \cos \xi^2 + \sin \xi^2$$

$$@ \ \xi^2 = 0 \Rightarrow A^1(\xi) = 1$$

$$A^2(\xi) = \frac{\partial \xi^2}{\partial x^1} + \frac{\partial \xi^2}{\partial x^2} = -\frac{x_2}{\xi_1^2} + \frac{x_1}{\xi_1^2} = 1$$

(Par. transp. no effect in Cartesian (all Γ's are $\emptyset$))

$(1,0)$: $\xi^1 = 1$, $\xi^2 = 0$

$(1,1)$: $\xi^1 = \sqrt{2}$, $\xi^2 = \frac{\pi}{4}$

ξ: $\quad g_{11} = 1$

$\quad g_{22} = (\xi_1)^2$

$\Gamma_{111} = 0$

$\Gamma_{112} = 0 = \Gamma_{121}$

$\Gamma_{122} = -\xi^1$

$\Gamma_{211} = 0 \;,\; \Gamma_{212} = \Gamma_{221} = \xi^1$

$\Gamma_{222} = 0$

$\Gamma^1_{22} = g^{11}\Gamma_{122} + g^{12}\ldots = 0$

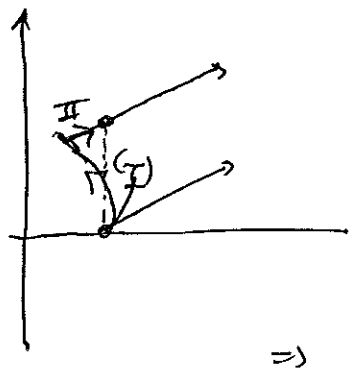
$\Gamma^2_{12} = g^{21}\Gamma_{112} + g^{22}\Gamma_{212} = \dfrac{1}{\xi^1}$

$A^\Gamma(\xi) = (1, 1)$

Now: $\quad dA^\Gamma = -\Gamma^\Gamma_{\alpha\beta} A^\alpha dx^\beta$

$$\Rightarrow \boxed{\begin{aligned} dA^1 &= \xi^1 A^2 d\xi^2 \\ dA^2 &= -\frac{1}{\xi^1}\left(A^1 d\xi^2 + A^2 d\xi^1\right) \end{aligned}} \quad (1)$$

Do it in 2 ways (paths)



(I) constant ξ^1

(II) constant ξ^2

$\Rightarrow$ (I) $\quad d\xi^1 = 0 \qquad \xi^2 : 0 \to \pi/4$

(II) $\quad \xi^1 : 1 \to \sqrt{2} \;,\; d\xi^2 = 0$

In these 2 regions the differential equations (1) become:

I) $$\frac{dA^1}{d\xi^2} = \xi^1 A^2$$

$$\frac{dA^2}{d\xi^2} = -\frac{1}{\xi^1} A_1$$

$$\Rightarrow \frac{d^2A^1}{(d\xi^2)^2} = -A^1 \Rightarrow A^1 = k\cos(\xi^2 + c)$$

$$A^2 = -\frac{k}{\xi^1}\sin(\xi^2 + c)$$

if $\xi^2 = 0$ $\quad \left.\begin{array}{l} 1 = k\cos c \\ \text{and } 1 = -k\sin c \end{array}\right\} \Rightarrow \begin{array}{l} k = \sqrt{2} \\ c = -\pi/4 \end{array}$

$$\Rightarrow A^r(\xi^1 = 1, \xi^2 = \tfrac{\pi}{4}) = (\sqrt{2}\cos 0, 0) = (\sqrt{2}, 0)$$

II) $$dA_1 = 0$$
$$dA_2 = -\frac{A^2}{\xi^1} d\xi^1$$

$$\Rightarrow \frac{dA^2}{A^2} = -\frac{d\xi^1}{\xi^1} \Rightarrow \ln A_2 = c - \ln \xi^1$$

$$\Rightarrow A^2 = \frac{c}{\xi^1}$$ with $\underline{c = 0}$ though

$$\Rightarrow \boxed{A^r(\xi^1 = \sqrt{2}, \xi^2 = \tfrac{\pi}{4}) = (\sqrt{2}, 0)}$$

Now back to the Cartesian

$$A^{\mu}(x) = A^{\alpha}(x(\xi)) \frac{\partial x^{\mu}}{\partial \xi^{\alpha}}$$

$$A^{1}(x) = A^{(1)}(\xi) \frac{\partial x^{1}}{\partial \xi^{1}} + A^{2}(\xi) \frac{\partial x^{1}}{\partial \xi^{2}} = \sqrt{2} \cos \xi^{2} =$$

$$= 1$$

$$(\xi^{2} = \pi/4)$$

and $$A^{2} = A^{2}(\xi) \frac{\partial x^{2}}{\partial \xi^{1}} + A^{2}(\xi) \frac{\partial x^{2}}{\partial \xi^{2}} =$$

$$= \sqrt{2} \sin \xi^{2} = 1$$

$$\Rightarrow \quad \underline{A^{\mu}(x) = (1, 1)}$$

<u>(4.4)</u>

$$x^{1} = R \sin\vartheta \sin\varphi$$
$$x^{2} = R \sin\vartheta \cos\varphi$$
$$x^{3} = R \cos\vartheta$$

$$(ds)^{2} = R^{2} \Big[(\cos\vartheta \, d\vartheta + \sin\vartheta \cos\varphi \, d\varphi)^{2} +$$

$$+ (\cos\vartheta \cos\varphi \, d\vartheta - \sin\vartheta \sin\varphi \, d\varphi)^{2} + (-\sin\vartheta \, d\vartheta)^{2} \Big]$$

$$= R^{2} \left[(d\vartheta)^{2} + \sin^{2}\vartheta (d\varphi)^{2} \right]$$

$$\Rightarrow \quad g_{\vartheta\vartheta} = R^{2}, \quad g_{\varphi\varphi} = \sin^{2}\vartheta \, R^{2}$$

Then $\Gamma_{\vartheta\varphi\varphi} = -\sin\vartheta\cos\vartheta\, R^2$

$$\Gamma_{\varphi\vartheta\varphi} = R^2 \sin\vartheta\cos\vartheta = \Gamma_{\varphi\varphi\vartheta}$$

$$\Gamma^{\vartheta}_{\varphi\varphi} = -\sin\vartheta\cos\vartheta$$

$$\Gamma^{\varphi}_{\vartheta\varphi} = \Gamma^{\varphi}_{\varphi\vartheta} = \frac{\cos\vartheta}{\sin\vartheta^2}$$

$$dA^{\mu} = -\Gamma^{\mu}_{\alpha\beta} A^{\alpha} dx^{\beta}$$

$$\begin{cases} dA^{\vartheta} = \sin\vartheta\cos\vartheta\, A^{\varphi} d\varphi \\ dA^{\varphi} = -\dfrac{\cos\vartheta}{\sin\vartheta}\left(A^{\vartheta} d\varphi + A^{\varphi} d\vartheta\right) \end{cases}$$

<u>1st case</u>

If $d\varphi = 0$: $dA^{\vartheta} = 0$

$$\frac{dA^{\varphi}}{d\vartheta} = -\frac{\cos\vartheta}{\sin\vartheta} A^{\varphi}$$

$\Rightarrow$ $A^{\vartheta} = \text{const}$ and $\dfrac{dA^{\varphi}}{A^{\varphi}} = -\dfrac{\cos\vartheta}{\sin\vartheta} d\vartheta$

$$d(\ln A^{\varphi}) = d(\ln \sin\vartheta)$$

$\Rightarrow$ $A^{\vartheta} = k_1$

$$A^{\varphi} = \frac{k_2}{\sin\vartheta}$$

Second case

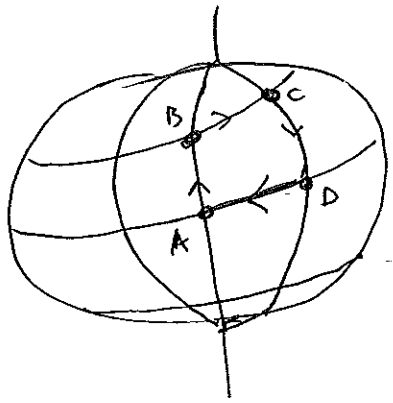
If $d\vartheta = 0$

$$\left.\begin{aligned} \frac{dA^{\vartheta}}{d\varphi} &= \sin\vartheta\cos\vartheta\, A^{\varphi} \\ \frac{dA^{\varphi}}{d\varphi} &= -\frac{\cos\vartheta}{\sin\vartheta} A^{\vartheta} \end{aligned}\right\} \quad \frac{d^2A^{\vartheta}}{(d\varphi)^2} = -(\cos\vartheta)^2 \cdot A^{\vartheta}$$

$$\Rightarrow \quad A^{\vartheta} = k \sin(\varphi\cos\vartheta + L)$$

$$A^{\varphi} = \frac{k}{\sin\vartheta} \cos(\varphi\cos\vartheta + L)$$

for our exercise



$A = (\vartheta_1, 0) \qquad B = (\vartheta_2, 0)$

$C = (\vartheta_3, \Delta\varphi)\ , \quad D = (\vartheta_1, \Delta\varphi)$

(a) A: $\quad V^{\theta} = a\ , \quad V^{\phi} = \dfrac{b}{\sin\vartheta}\ , \quad a^2 + b^2 = 1$

and $\vec{V}\cdot\vec{V} = V^{\vartheta 2} g_{\vartheta\vartheta} + V^{\varphi 2} g_{\varphi\varphi} = R^2$

$A \to B$: $\quad V^{\vartheta} = a\ , \quad V^{\phi} = \dfrac{k_2}{\sin\vartheta} \longrightarrow b$

(solutions from the differential equations for $d\varphi = 0$)

@ B : $V^\theta = a$, $V^\varphi = \dfrac{b}{\sin\vartheta_2}$

$B \to C$: $\left.\begin{aligned} A^\theta &= k \sin(L + \varphi\cos\vartheta) \\ A^\varphi &= \frac{k}{\sin\vartheta} \cos(L + \varphi\cos\vartheta) \end{aligned}\right\}$

$$\Rightarrow \quad k \sin(L + \varphi\cos\vartheta) = a$$

$$k \cos(L + \varphi\cos\vartheta) = b$$

from $a^2 + b^2 = 1 \quad \Rightarrow \quad k = 1$

and $\quad L + \varphi \underbrace{\cancel{\cos\vartheta}}_{\text{at this point } \varphi = 0.} = \arctan\left(\frac{a}{b}\right)$

$$\Rightarrow \quad V^\theta = \sin(L + \varphi\cos\vartheta) \ , \ V^\phi = \frac{1}{\sin\vartheta} \cos(L + \varphi\cos\vartheta)$$

$$\left(\text{with } \tan(L) = \frac{a}{b}\right)$$

@ C : $V^\theta = \sin(L_0 + \Delta\varphi\cos\vartheta_2)$

$$V^\varphi = \frac{\cos(L_0 + \cos\vartheta_2 \Delta\varphi)}{\sin\vartheta_2}$$

$$a\cos(\Delta\varphi\cos\vartheta_2) + b\sin(\Delta\varphi\cos\vartheta_2)$$

$$\frac{b\cos(\Delta\phi\cos\vartheta_2) - a\sin(\Delta\phi\cos\vartheta_2)}{\sin\vartheta_2}$$

$C \to D$ ... (similar)

@ D :

$$V^{\vartheta} = a\cos(\Delta\phi\cos\vartheta_2) + b\sin(\Delta\phi\cos\vartheta_2)$$

$$V^{\varphi} = \frac{[b\cos(\Delta\phi\cos\vartheta_2) - a\sin(\Delta\phi\cos\vartheta_2)]}{\sin\vartheta_1}$$

$$\sin(L_1 + \varphi\cos\vartheta_1) \qquad \cos(L_1 + \varphi\cos\vartheta_1)$$

different than the one before (L_0)

$D \to A$: k still $\underline{\underline{1}}$

But $B \to C$ and $D \to A$

I have that:

$$\sin(L_0 + \Delta\phi\cos\vartheta_2) = \sin(L_1 + \Delta\phi\cos\vartheta_1)$$

$$\Rightarrow \quad L_1 = L_0 + \Delta\phi\Delta C$$

with $\Delta C = \cos\vartheta_2 - \cos\vartheta_1$

$$V^{\vartheta} = \sin(L_0 + \Delta\phi\Delta C + \varphi\cos\vartheta_1)$$

$$V^{\varphi} = \frac{1}{\sin\vartheta_1}\{\cos(L_0 + \Delta\phi\Delta C + \varphi\cos\vartheta_1\}$$

$\Rightarrow$ @ A: $\left.\begin{aligned} V^{\theta} &= \sin(L_0 + \Delta\phi\Delta C) \\ V^{\varphi} &= \frac{1}{\sin\vartheta_1} \cos(L_0 + \Delta\phi\Delta C) \end{aligned}\right\} \rightarrow$

$\rightarrow$ components of V'

$$\vec{V_1} \cdot \vec{V_1}' = R^2 V^{\vartheta} V'^{\vartheta} + R^2 \sin\vartheta_1^2 \, V^{\varphi} V'^{\varphi} =$$

$$= R^2 \left(\sin L_0 \cdot \sin(L_0 + \Delta\phi\Delta C) + \cos L_0 \cdot \cos(L_0 + \Delta\phi\Delta C) \right) =$$

$$= R^2 \cos(\Delta\phi\Delta C)$$

$\Rightarrow$ angle between $\vec{V}, \vec{V}'$ is $\boxed{(\vec{V}, \vec{V}') = \Delta\phi\Delta C}$