Problem set for the course "Beyond the Standard Model" (theoretical aspects)

Ex. 1 Consider the Lie group $SU(N)$ with elements $g = e^{i\vartheta^a T^a}$, where ϑ^a are real parameters and T^a are the generators of the associated Lie algebra with fundamental commutation relations $[T^a, T^b] = i f^{abc} T^c$.
In the defining (fundamental) representation the group elements are given by $N \times N$ matrices U, with $UU^\dagger = 1$ and $\det U = 1$.

(i) Show that T^a is hermitian and traceless.

(ii) Argue that $SU(N)$ has $N^2 - 1$ independent generators.

(iii) Prove that the structure constants f^{abc} are real.

(iv) Consider the subgroup $SO(N) \subset SU(N)$, with group elements that are given by $N \times N$ matrices O in the fundamental representation. These matrices satisfy the additional constraint $OO^T = 1$. Deduce that T^a is purely imaginary and that $SO(N)$ has $\frac{1}{2}N(N-1)$ independent generators.

Ex. 2 Consider the local $SU(N)$ gauge theory for $N > 1$, described in the lecture notes on p. 8-11.

(i) Why is a QED-like charge scaling $\vartheta^a(x) \to Q\vartheta^a(x)$, $g \to Qg$ not an effective means of changing the interaction strength?

(ii) Consider the subset of global $SU(N)$ gauge transformations, for which $\mathcal{L}_{SU(N)}$ is invariant. Derive the conserved Noether currents $J^{a,\mu}(x)$ for each of the independent global transformations labeled by $\vartheta^a \in \mathbb{R}$ and link the combination $g W_\mu^a(x) J^{a,\mu}(x)$ to the three types of interactions contained in the $SU(N)$ gauge theory.

Hint: in part (i) you have to demand that after charge scaling W_μ^a transforms in the same way as before charge scaling. Only then one and the same gauge field could mediate the $SU(N)$ interaction between differently charged particles.

Ex. 3 Consider $U(x) = e^{\frac{i}{2}\vec{\tau}\cdot\vec{\varphi}(x)} \in SU(2)$, with $\varphi^1(x), \varphi^2(x), \varphi^3(x) \in \mathbb{R}$ and τ^1, τ^2, τ^3 the usual Pauli spin matrices. Introduce a doublet field $\Phi(x)$ that transforms under $SU(2)$ according to

$$\Phi(x) \to \Phi'(x) = U(x)\,\Phi(x)$$ (as expected for a fundamental doublet).

Prove that $\Phi^c(x) \equiv i\tau^2\,\Phi^*(x)$ (* = complex conjugation) has the same $SU(2)$ transformation property as $\Phi(x)$.

Hint: first figure out what happens if you bring τ^2 to the other side of $(\tau^j)^*$ for $j = 1, 2, 3$.