Ex. 4 Consider the potential of the 2HDM:

$$V_H = \mu_1^2 H_1^\dagger H_1 + \mu_2^2 H_2^\dagger H_2 - \mu_3^2 (\underbrace{H_1 \times H_2}_{H_{1\ell}\epsilon_{\ell m} H_{2m}} + h.c.) + \frac{m_Z^2}{2v^2}(H_1^\dagger H_1 - H_2^\dagger H_2)^2 + \frac{2m_W^2}{v^2}|H_1^\dagger H_2|^2$$

$$H_{1\ell}\epsilon_{\ell m} H_{2m}\,, \quad (\epsilon_{\ell m}) = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}$$

with $\mu_{1,2,3}^2$ and m_Z^2, m_W^2, v^2 real and positive.

Ingredients: $H_1 = \begin{pmatrix} \phi_1^{0*} \\ -\phi_1^- \end{pmatrix} \begin{matrix} \leftarrow \in \mathbb{C}, \text{neutral} \\ \leftarrow \in \mathbb{C}, \text{charged} \end{matrix}$, $H_2 = \begin{pmatrix} \phi_2^+ \\ \phi_2^0 \end{pmatrix} \begin{matrix} \leftarrow \in \mathbb{C}, \text{charged} \\ \leftarrow \in \mathbb{C}, \text{neutral} \end{matrix}$

(i) Show that V_H is always bounded from below if $\underline{\mu_1^2 + \mu_2^2 \geq 2\mu_3^2}$.
Hint: you may use that $H_1 \times H_2 + h.c. = 2\cos(\varphi)\,|H_1 \times H_2|$
$= 2\cos(\varphi)\sqrt{|H_1|^2 |H_2|^2 - |H_1^\dagger H_2|^2}$, with φ the complex phase of $H_1 \times H_2$ and $|H_i| \equiv \sqrt{H_i^\dagger H_i}$ $(i = 1,2)$.

(ii) Minimalization, step 1: take fixed values for $|H_1|$ and $|H_2|$, and subsequently minimize V_H given those fixed values.

(iii) Minimalization, step 2: consider the minimized expression obtained in part (ii).

- Show that a minimum away from $|H_1| = |H_2| = 0$ can be obtained for $\underline{\mu_3^4 > \mu_1^2 \mu_2^2}$.
- Next the minimized expression of part (ii) should be minimized as a function of $|H_1|$ and $|H_2|$, resulting in $|H_1^{min}|$ and $|H_2^{min}|$: argue that no electroweak symmetry breaking can occur if $|H_1^{min}| = |H_2^{min}|$ and derive the minimalization conditions

$$\boxed{\begin{aligned} &2\mu_1^2\,|H_1^{min}| + \frac{2m_Z^2}{v^2}|H_1^{min}|\left(|H_1^{min}|^2 - |H_2^{min}|^2\right) - 2\mu_3^2\,|H_2^{min}| = 0 \\ &2\mu_2^2\,|H_2^{min}| - \frac{2m_Z^2}{v^2}|H_2^{min}|\left(|H_1^{min}|^2 - |H_2^{min}|^2\right) - 2\mu_3^2\,|H_1^{min}| = 0 \end{aligned}}$$