Ex. 5 Consider the inverse running gauge couplings

$$\alpha_1^{-1}(Q) \equiv \frac{4\pi}{\frac{5}{3} g_B^2(Q)}, \quad \alpha_2^{-1}(Q) \equiv \frac{4\pi}{g_W^2(Q)} \quad \text{and} \quad \alpha_3^{-1}(Q) \equiv \frac{4\pi}{g_S^2(Q)},$$

which satisfy the renormalization group equations

$$\frac{\partial}{\partial \log(Q)} \begin{pmatrix} \alpha_1^{-1}(Q) \\ \alpha_2^{-1}(Q) \\ \alpha_3^{-1}(Q) \end{pmatrix} = -\frac{1}{2\pi} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \quad \text{at 1-loop level.}$$

In the SM:

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ -22/3 \\ -11 \end{pmatrix}}_{\text{gauge bosons}} + \underbrace{N_g \begin{pmatrix} 4/3 \\ 4/3 \\ 4/3 \end{pmatrix}}_{N_g \text{ generations of fermions}} + \underbrace{N_H \begin{pmatrix} 1/10 \\ 1/6 \\ 0 \end{pmatrix}}_{N_H \text{ Higgs doublets}} = \begin{pmatrix} 41/10 \\ -19/6 \\ -7 \end{pmatrix}.$$

(with $N_g = 3$, $N_H = 1$)

In the MSSM:

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -6 \\ -9 \end{pmatrix} + N_g \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} + N_H \begin{pmatrix} 3/10 \\ 1/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 33/5 \\ 1 \\ -3 \end{pmatrix},$$

(with $N_g = 3$, $N_H = 2$)

including the corresponding superpartners in all three components.

Introduce the related quantities $\sin^2\theta_W(Q) = \dfrac{g_B^2(Q)}{g_W^2(Q) + g_B^2(Q)}$ and $\alpha^{-1}(Q) = \dfrac{\alpha_2^{-1}(Q)}{\sin^2\theta_W(Q)}$.

Assume that the gauge couplings unify at $Q = M_X$, i.e. $\alpha_i^{-1}(M_X) \equiv \alpha_G^{-1}$ for $i = 1,2,3$ and prove the following relations

(i) $\sin^2\theta_W(M_X) = 3/8$

(ii) $I_\alpha = (b_2 - b_3)\,\alpha_1^{-1}(Q) + (b_3 - b_1)\,\alpha_2^{-1}(Q) + (b_1 - b_2)\,\alpha_3^{-1}(Q) = 0$

Hint: first figure out what is special about I_α.

(iii) $\sin^2\theta_W(Q) = \dfrac{1}{5b_1 + 3b_2 - 8b_3}\left[3(b_2 - b_3) + 5(b_1 - b_2)\,\alpha_3^{-1}(Q)/\alpha^{-1}(Q)\right]$.

(iv) Finally, plug in the actual experimental values $\alpha^{-1}(m_Z) = 127.916 \pm 0.015$ and $\alpha_3^{-1}(m_Z) = 8.45 \pm 0.05$. Compare your SM/MSSM prediction for $\sin^2\theta_W(m_Z)$ with the actual experimental value 0.23131 ± 0.00007.