Part 4: Supersymmetry (chapter 4+5 of the textbook)

We have already encountered a few appealing features of supersymmetry (SUSY), in the context of gauge-coupling unification and as solution to the hierarchy problem. Some more theoretical motivations for SUSY are

- SUSY would complete the set of possible 4-dimensional spacetime symmetries
- local SUSY theories require the introduction of gravity
- the lightest SUSY particle might be a candidate for cold dark matter

⋮

Before going into the details of SUSY models we first consider a toy model that can exemplify some of the properties of SUSY, which is a transformation that turns fermions into bosons and vice versa.

Non-relativistic linear harmonic oscillator model for SUSY:

consider a system with Hamilton operator $\hat{H} = \omega_F(\underbrace{\hat{a}^\dagger\hat{a}}_{\hat{N}_F} - 1/2) + \omega_B(\underbrace{\hat{b}^\dagger\hat{b}}_{\hat{N}_B} + 1/2)$ $(\omega_{F,B} > 0)$, describing fermionic oscillator modes created by $\hat{a}^\dagger$ and annihilated by $\hat{a}$ as well as bosonic oscillator modes created by $\hat{b}^\dagger$ and annihilated by $\hat{b}$.

Algebra: $\{\hat{a},\hat{a}\} = \{\hat{a}^\dagger,\hat{a}^\dagger\} = 0$, $\{\hat{a},\hat{a}^\dagger\} = \hat{1}$; $[\hat{a},\hat{b}] = [\hat{a}^\dagger,\hat{b}^\dagger] = [\hat{a},\hat{b}^\dagger] = [\hat{a}^\dagger,\hat{b}] = 0$;
$[\hat{b},\hat{b}] = [\hat{b}^\dagger,\hat{b}^\dagger] = 0$, $[\hat{b},\hat{b}^\dagger] = \hat{1}$.

Introduce the SUSY operators $\hat{Q} = \hat{b}^\dagger\hat{a}$ and $\hat{Q}^\dagger = \hat{a}^\dagger\hat{b}$ that remove a fermion (boson) and replace it by a boson (fermion):

$[\hat{Q},\hat{H}] = (\omega_F - \omega_B)\hat{Q}$, $[\hat{Q}^\dagger,\hat{H}] = -(\omega_F - \omega_B)\hat{Q}^\dagger$

$\Rightarrow$ if $\omega_F = \omega_B = \omega$, then $\hat{H} = \omega(\hat{N}_F + \hat{N}_B)$ commutes with the SUSY operators $\hat{Q}$ and $\hat{Q}^\dagger$.

The system is supersymmetric and $\hat{Q}, \hat{Q}^\dagger$ are conserved (constants of motion).

Moreover: $[\hat{N}_B,\hat{Q}] = \hat{Q}$, $[\hat{N}_B,\hat{Q}^\dagger] = -\hat{Q}^\dagger$ and $[\hat{N}_F,\hat{Q}] = -\hat{Q}$, $[\hat{N}_F,\hat{Q}^\dagger] = \hat{Q}^\dagger$.

So, $\hat{Q}$ and $\hat{Q}^\dagger$ have the advertised actions on fermionic and bosonic modes:

$$\hat{N}_B\hat{Q}|\psi\rangle = \hat{Q}\overbrace{\hat{N}_B|\psi\rangle}^{\equiv n_B|\psi\rangle} + \hat{Q}|\psi\rangle = (n_B+1)\hat{Q}|\psi\rangle$$
$$\hat{N}_F\hat{Q}|\psi\rangle = \hat{Q}\underbrace{\hat{N}_F|\psi\rangle}_{\equiv n_F|\psi\rangle} - \hat{Q}|\psi\rangle = (n_F-1)\hat{Q}|\psi\rangle$$

and the opposite for $\hat{Q}^\dagger$!

Next we consider the algebra among the SUSY operators:

$\boxed{\{\hat{Q},\hat{Q}\} = \{\hat{Q}^\dagger,\hat{Q}^\dagger\} = 0}$ (i.e. $\hat{Q}$ and $\hat{Q}^\dagger$ are "fermionic" operators),

$\{\hat{Q},\hat{Q}^\dagger\} = \hat{b}^\dagger\hat{a}\hat{a}^\dagger\hat{b} + \hat{a}^\dagger\hat{b}\hat{b}^\dagger\hat{a} = \hat{a}\hat{a}^\dagger\hat{b}^\dagger\hat{b} + \hat{a}^\dagger\hat{a}\hat{b}\hat{b}^\dagger = -\hat{N}_F\hat{N}_B + \hat{N}_B + \hat{N}_F\hat{N}_B + \hat{N}_F = \hat{N}_F + \hat{N}_B = \boxed{\hat{H}/\omega = \{\hat{Q},\hat{Q}^\dagger\}}$

$\Rightarrow$ $\hat{Q}, \hat{Q}^\dagger, \hat{H}$ constitute a closed algebra!

Properties: *) $\hat{Q}$ and $\hat{Q}^\dagger$ exchange bosons and fermions, without changing the energy;

*) the SUSY operators can in some sense be interpreted as square root of the Hamilton operator (like the Dirac eqn. can be interpreted as the square root of the Klein-Gordon eqn.);

*) $\hat{H}$ = generator of temporal translations, $\hat{Q}, \hat{Q}^\dagger$ change spin and switch between odd states under 2π rotations (fermions) and even states (bosons)

↳ all three operators correspond to external (spacetime) transformations + $\hat{Q}, \hat{Q}^\dagger$ will not commute with the generators of Lorentz transformations;

*) $\hat{Q}|0\rangle = \hat{Q}^\dagger|0\rangle = 0$ for the ground state

$\Rightarrow$ $\langle 0|\hat{H}|0\rangle = \omega\langle 0|\{\hat{Q},\hat{Q}^\dagger\}|0\rangle = 0$ for the ground state energy, i.e. the harmonic-oscillator zero-point energies cancel between bosonic and fermionic degrees of freedom.

Let's now turn to the relativistic 4-dimensional case:

- since the spins of the modes linked by SUSY transformations differ by a half-integer amount, the SUSY generator $\hat{Q}$ must be spinorial

$\Rightarrow$ non-hermitean, $\hat{\bar{Q}} = \hat{Q}^\dagger\gamma^0$ for the adjoint, $\hat{Q}$ and $\hat{\bar{Q}}$ carry spin ½ and satisfy an anticommutator algebra, $[\hat{Q}] = [\hat{Q}^\dagger] = 1/2$;

- SUSY algebra for $\hat{Q}_\alpha$ and $\hat{\bar{Q}}_\beta$ (α,β = spinor index):

① $[\hat{Q}_\alpha, \hat{T}^a] = 0$ for any generator $\hat{T}^a$ of an internal (gauge) transformation,

② $[\hat{Q}_\alpha, \hat{P}_\mu] = 0$, $[\hat{Q}_\alpha, \hat{M}_{\mu\nu}] = \frac{i}{4}\underbrace{[\gamma_\mu,\gamma_\nu]_{\alpha\beta}}_{\frac{1}{2}(\sigma_{\mu\nu})_{\alpha\beta}}\hat{Q}_\beta$, ← ($\hat{Q}$ transforms as a spin-½ field) ($\hat{M}_{\mu\nu}$: gen. of homogeneous Lor. transf.)

③ $\{\hat{Q}_\alpha,\hat{Q}_\beta\} = \{\hat{\bar{Q}}_\alpha,\hat{\bar{Q}}_\beta\} = 0$, $\{\hat{Q}_\alpha,\hat{\bar{Q}}_\beta\} = 2(\gamma^\mu)_{\alpha\beta}\hat{P}_\mu$. ($\hat{P}_\mu$: gen. of spacetime translations)

The last operator identity implies that invariance under local (i.e. x-dependent) SUSY transformations would require local coordinate transformations and therefore the introduction of gravity!

Consequences of the above-given algebra for SUSY theories:

① → $\hat{Q}, \hat{\bar{Q}}$ connect states in the same representation of the gauge group ⇒ the same charges/couplings and weak/colour d.o.f.

② → ∃ simultaneous set of eigenstates of $\hat{Q}$ and $\hat{P}_\mu$, and $[\hat{Q}_\alpha, \hat{P}^2] = 0$ (and similarly for $\hat{\bar{Q}}$ and $\hat{P}_\mu$) ⇒ $\hat{Q}, \hat{\bar{Q}}$ connect states with the same Lorentz-invariant eigenvalue m^2 of $\hat{P}^2$ ← (not observed in nature ⇒ SUSY must be broken!)

Hence, single-particle states of a SUSY theory fall into irreducible representations of the SUSY algebra, called supermultiplets. These contain both fermionic and bosonic states, which are called each others superpartners. The corresponding particles have the same mass and gauge quantum numbers, but they differ by half a unit of spin.

③ → the Hamilton operator $\hat{H} = \hat{P}_0$ satisfies $\langle\psi|\hat{H}|\psi\rangle \geq 0$ for all states $|\psi\rangle$. Consider the SUSY ground state $|\Omega\rangle$ with $\hat{H}|\Omega\rangle = E_0|\Omega\rangle$, then it follows that $E_0 = 0 \Leftrightarrow \hat{Q}|\Omega\rangle = \hat{\bar{Q}}|\Omega\rangle = 0$. This implies that a spontaneously broken SUSY theory has a positive-energy ground state!

Consider the subspace of orthonormal states $\{|i\rangle\}$ (labeled by spin, polarization) in a supermultiplet with the same eigenvalue p_μ of $\hat{P}_\mu$ and define the operator $(-1)^{2s}$ with $(-1)^{2s}|B\rangle = |B\rangle$ and $(-1)^{2s}|F\rangle = -|F\rangle$. Then obviously $\{\hat{Q}_\alpha, (-1)^{2s}\} = \{\hat{\bar{Q}}_\beta, (-1)^{2s}\} = 0$ and

$$\sum_i \langle i|(-1)^{2s}\hat{Q}_\alpha\hat{\bar{Q}}_\beta|i\rangle + \underbrace{\sum_i \langle i|(-1)^{2s}\hat{\bar{Q}}_\beta\hat{Q}_\alpha|i\rangle}_{} = 0$$

$$\sum_i \langle i|(-1)^{2s}\hat{\bar{Q}}_\beta\hat{Q}_\alpha|i\rangle = \sum_{i,j} \langle i|(-1)^{2s}\hat{\bar{Q}}_\beta|j\rangle\langle j|\hat{Q}_\alpha|i\rangle = \sum_j \langle j|\hat{Q}_\alpha(-1)^{2s}\hat{\bar{Q}}_\beta|j\rangle$$

$$= -\sum_j \langle j|(-1)^{2s}\hat{Q}_\alpha\hat{\bar{Q}}_\beta|j\rangle$$

$$\overset{③}{\Rightarrow} \sum_i \langle i|(-1)^{2s}\,2(\gamma^\mu)_{\alpha\beta}\hat{P}_\mu|i\rangle = 2(\gamma^\mu)_{\alpha\beta}\,p_\mu \underbrace{\sum_i \langle i|(-1)^{2s}|i\rangle}_{n_B - n_F}.$$

(# bosonic d.o.f.) → $n_B - n_F$ ← (# fermionic d.o.f.)

Witten index

Hence, $n_B = n_F$ within a multiplet for non-zero momentum states and if $p_\mu = 0$ then $n_F \neq n_B$ is allowed !

Consequence : – SUSY ground state $\Rightarrow p_\mu = 0 \Rightarrow n_B - n_F$ can be anything,
– broken SUSY ground state $\Rightarrow p_0 \neq 0 \Rightarrow n_B = n_F$ mandatory !

§4.1 Simplest supermultiplets needed for a chiral theory :

1) Chiral (matter) supermultiplet : contains ψ_L, $\tilde{\psi}_L$, F_L.
ψ_L : spin-1/2, massless left-handed Dirac field (can be represented in terms of a 2-component Weyl spinor), $[\psi] = 3/2$,
d.o.f. = 2 (on-shell), 4 (off-shell) ;
$\tilde{\psi}_L$: spin-0, complex scalar field, generic name : sfermion, $[\tilde{\psi}] = 1$, # d.o.f. = 2 ;
F_L : spin-0, non-propagating complex scalar field, $[F] = 2$,
d.o.f. = 0 (on-shell), 2 (off-shell). (no kinetic term)
(auxiliary field needed for closing the SUSY algebra off-shell)
All fields are in the same (fundamental) representation of the gauge group.
Right-handed Dirac fields are first conjugated in order to bring them in left-handed form: $\psi_R \rightarrow (\psi_R)^c = i\tau^2\psi_R^*$ in 2-component notation. (because of $SU(2)_L$ transf. property)
SM : leptons, quarks and Higgs bosons live in chiral supermultiplets. (chiral)

2) Vector (gauge) supermultiplet : contains W_μ^a, $\tilde{w}^a$, D^a.
W_μ^a : spin-1, massless real gauge field ($W_\mu = W_\mu^a T^a = -W_\mu^c$), $[W] = 1$,
d.o.f. = 2 (on-shell), 3 (off-shell) ;
$\tilde{w}^a$: spin-1/2, massless Majorana fermion ($\tilde{w} = \tilde{w}^a T^a = \tilde{w}^c$), generic name : gaugino, $[\tilde{w}] = 3/2$, # d.o.f. = 2 (on-shell), 4 (off-shell) ;
D^a : spin-0, non-propagating real scalar field, $[D] = 2$,
d.o.f. = 0 (on-shell), 1 (off-shell). (no kinetic term)
(auxiliary field needed for closing the SUSY algebra off-shell)
All fields are in the same (adjoint) representation of the gauge group.
($\tilde{w} = \tilde{w}^c$, linking $\tilde{w}_L$ and $\tilde{w}_R$)
Since the gaugino fields are in the adjoint representation their left- and right-handed components transform in the same way under gauge transf.
$\Rightarrow$ no chiral fermions can live in vector multiplets.

Extended SUSY: if there would be more than one distinct copy of SUSY generators $\hat{Q}, \hat{\bar{Q}}$, then there actually would be no chiral fermions in 4 dimensions (in view of the connection between $s=0$, $s=1/2$ and $s=1$ fields)

$\Rightarrow$ the SUSY extension of the SM involves $N=1$ SUSY, i.e. SUSY based on one copy of SUSY generators!

§4.2 Interactions in SUSY models

We start with interactions among fields belonging to a chiral (matter) supermultiplet (called Wess-Zumino model), which can be trivially extended to multiple chiral supermultiplets:

$$\psi_L^T i\tau^2 \psi_L = \psi_{L_j} \epsilon_{jk} \psi_{L_k}$$

$$\mathcal{L}_{WZ} = (\partial_\mu \tilde{\psi}_L^*)(\partial^\mu \tilde{\psi}_L) + i\bar{\psi}_L \gamma^\mu \partial_\mu \psi_L + F_L^* F_L + \left[-i W'(\tilde{\psi}_L) F_L - \tfrac{1}{2} W''(\tilde{\psi}_L) (\bar{\psi}_L)^c \psi_L + \text{h.c.} \right].$$

(W' and W'' are derivatives)

Here $W(\tilde{\psi}_L)$ is an analytic polynomial function, called superpotential, containing up to three powers in the scalar field $\tilde{\psi}_L$. This implies that $W(\tilde{\psi}_L)$ depends exclusively on $\tilde{\psi}_L$ and not on $\tilde{\psi}_L$ AND $\tilde{\psi}_L^*$ at the same time!

This model is supersymmetric, since $\mathcal{L}_{WZ}$ changes by a 4-divergence (leaving the equations of motion invariant) under the infinitesimal global SUSY transformation originating from $e^{i(\bar{\epsilon}\hat{Q} + \hat{\bar{Q}}\epsilon)}$:

$$\delta\tilde{\psi}_L = i\bar{\epsilon}\psi_L = i\bar{\epsilon}_R\psi_L \quad , \quad \delta\tilde{\psi}_L^* = i\bar{\psi}_L\epsilon = i\bar{\psi}_L\epsilon_R$$

$$\delta\psi_L = P_L(\not{\partial}\tilde{\psi}_L\epsilon + F_L\epsilon) = \not{\partial}\tilde{\psi}_L\epsilon_R + F_L\epsilon_L \quad , \quad \delta\bar{\psi}_L = (-\bar{\epsilon}\not{\partial}\tilde{\psi}_L^* + \bar{\epsilon}F_L^*)P_R = -\bar{\epsilon}_R\not{\partial}\tilde{\psi}_L^* + \bar{\epsilon}_L F_L^*$$

$$\delta(\psi_L)^c = P_R(-\not{\partial}\tilde{\psi}_L^*\epsilon + F_L^*\epsilon) = -\not{\partial}\tilde{\psi}_L^*\epsilon_L + F_L^*\epsilon_R \quad , \quad \delta(\bar{\psi}_L)^c = (\bar{\epsilon}\not{\partial}\tilde{\psi}_L + \bar{\epsilon}F_L)P_L = \bar{\epsilon}_L\not{\partial}\tilde{\psi}_L + \bar{\epsilon}_R F_L$$

$$\delta F_L = -i\bar{\epsilon}\not{\partial}\psi_L = -i\bar{\epsilon}_L\not{\partial}\psi_L \quad , \quad \delta F_L^* = i(\partial_\mu\bar{\psi}_L)\gamma^\mu\epsilon = i(\partial_\mu\bar{\psi}_L)\gamma^\mu\epsilon_L$$

Properties of the spinorial transformation parameter ϵ:

- a Majorana spinor ("real") with mass dimension $[\epsilon] = -1/2$ $\Rightarrow$ it raises the dimensions of fields and derivatives by $1/2$ in SUSY transformations;
- independent of the spacetime coordinates (global transf.);
- its components commute with the bosonic fields and anticommute among themselves and with fermionic fields.

The interaction terms between the square brackets have been obtained by requiring

*) renormalizability ⇒ no couplings with negative mass dimension
⇒ maximally 2 fermionic fields, 2 auxiliary fields, 4 scalar fields;
*) supersymmetry ⇒ many terms are eliminated.

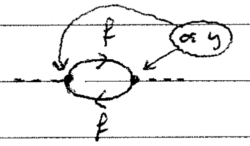
Using the equations of motion for the non-propagating fields F_L and F_L^*:

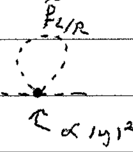
$$\frac{\partial \mathcal{L}_{WZ}}{\partial F_L} - \partial_\mu \frac{\partial \mathcal{L}_{WZ}}{\partial(\partial_\mu F_L)} = 0 \Rightarrow F_L^* = i W'(\tilde{\varphi}_L) \quad \text{and similarly} \quad F_L = -i\,[W'(\tilde{\varphi}_L)]^*$$

⇒ <u>F-terms</u>: $F_L^* F_L + (-i W'(\tilde{\varphi}_L) F_L + h.c.) \to -|W'(\tilde{\varphi}_L)|^2 \equiv \mathcal{L}_F$.

$[L]=2$, $[M]=1$, $[y]=0$

Most general form of the superpotential: $W(\tilde{\varphi}_L) = L\tilde{\varphi}_L + \frac{1}{2} M \tilde{\varphi}_L^2 + \frac{1}{6} y \tilde{\varphi}_L^3$, where
- the M-term gives the fermions and sfermions the same mass,
- the y-term links Yukawa interactions and quartic scalar interactions:

cancellation between 

and 

that solves the hierarchy problem,

- all three terms feature in the scalar potential: $V_F = |W'(\tilde{\varphi}_L)|^2 = |L + M\tilde{\varphi}_L + \frac{1}{2} y \tilde{\varphi}_L^2|^2 \geq 0$
⇒ • $L \neq 0 \to \langle\tilde{\varphi}_L\rangle_0 \neq 0$ possible, breaking SUSY in view of the absence of $\langle\varphi_L\rangle_0 \neq 0$
  • $L = 0 \to$ minimal for $\langle\tilde{\varphi}_L\rangle_0 = 0$ (EWSB)
  • $y \neq 0 \to$ triple and quartic scalar interactions!

(SU(N)-style)
Next we add <u>gauge interactions</u> and the associated vector (gauge) supermultiplet:
$\partial_\mu \to D_\mu = \partial_\mu + i g W_\mu^a T^a$ in the matter Lagrangian + add "kinetic" terms for the gauge supermultiplet

$$\mathcal{L}_V = -\frac{1}{4} W_{\mu\nu}^a W^{a,\mu\nu} + \frac{i}{2} \bar{\tilde{W}} \not{D} \tilde{W} + \frac{1}{2} D^a D^a, \quad \text{with } W_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a - g f^{abc} W_\mu^b W_\nu^c$$
$$\text{and } (D_\mu \tilde{W})^a = \partial_\mu \tilde{W}^a - g f^{abc} W_\mu^b \tilde{W}^c.$$
($-g f^{abc} \to + i g (T^b)^{ac}$)

In addition we need extra gauge interactions to make everything supersymmetric:

$$\mathcal{L}_{int} = -\sqrt{2} g (\bar{\tilde{W}}^a \tilde{\varphi}_L^* T^a \psi_L + h.c.) + g \tilde{\varphi}_L^* T^a \tilde{\varphi}_L D^a.$$

Extra requirement: <u>the superpotential should be a gauge singlet (i.e. gauge inv.)</u>
⇒ $L\tilde{\varphi}_L$ only allowed if $\tilde{\varphi}_L$ is a singlet!

Using the equations of motion for the non-propagating fields D^a:

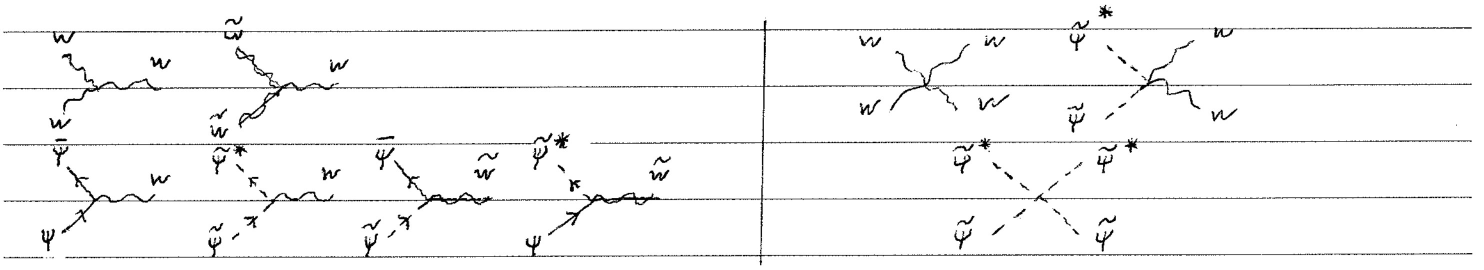
$$\frac{\partial}{\partial D^a}(\mathcal{L}_V + \mathcal{L}_{int}) = 0 \Rightarrow D^a = -g\,\tilde{\psi}_L^* T^a \tilde{\psi}_L$$ ← (this becomes $-g[\tilde{\psi}_L^* T^a \tilde{\psi}_L - \tilde{\psi}_R^* T^a \tilde{\psi}_R]$ for a non-chiral SU(N) theory)

$\Rightarrow$ D-terms: $-\frac{g^2}{2}(\tilde{\psi}_L^* T^a \tilde{\psi}_L)^2 \equiv \mathcal{L}_D$, giving rise to quartic couplings of scalar fields that are fixed by gauge couplings! This will have important phenomenological consequences in the Higgs sector: Higgs masses cannot all be arbitrarily large! Contribution to the scalar potential: $V_D = -\mathcal{L}_D = \frac{g^2}{2}(\tilde{\psi}_L^* T^a \tilde{\psi}_L)^2 \geq 0 \Rightarrow V = V_F + V_D$ is bounded from below, with minimum at $\tilde{\psi}_L = 0$ if $L = 0$!

↑ (quartic int. $\propto g^2, |y|^2$)

↑ (bad for EWSB)

- the gauginos are massless, since the gauge bosons have to be massless,
- no scalar - gaugino - gaugino interactions ⇒ no EWSB source for gaugino masses!
- all new interactions are gauge interactions, i.e. $\propto g$ or g^2



$\propto g$ $\qquad\qquad\qquad\qquad$ $\propto g^2$

§4.3 The ingredients for a Minimal Supersymmetric Standard Model (MSSM)

As mentioned before, all leptons, quarks and Higgs bosons live in chiral supermultiplets. They will have superpartners with the same gauge quantum numbers. The particles can only acquire mass through the Higgs mechanism, otherwise gauge invariance is broken ⇒ before electroweak symmetry breaking (EWSB) all particles, including superpartners, will be massless. In the SM one Higgs doublet Φ was able to do the job by means of the conjugate doublet $\Phi^c = i\tau^2 \Phi^*$. This is not allowed in the SUSY version of the SM, since in an analytic superpotential of scalar fields not both Φ and Φ^* can feature

⇒ we need at least two Higgs doublets $H_1 = \begin{pmatrix} h_1^0 \\ h_1^- \end{pmatrix}$ with $Y(H_1) = -1$ and $H_2 = \begin{pmatrix} h_2^+ \\ h_2^0 \end{pmatrix}$ with $Y(H_2) = +1$ in order to give masses to down- and up-type fermions [By the way: this is also needed for the cancellation of the chiral anomalies].