Extra Exercise 1: Minimal substitution

Classical electromagnetic fields and minimal substitution

Consider a spin-0 particle with mass m and charge q in a potential $V(\vec{r})$. The classical lagrangian of this system is

$$L_0(\vec{r},\dot{\vec{r}}) = \frac{1}{2}m\dot{\vec{r}}^2 - V(\vec{r}), \tag{1}$$

with corresponding lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_0}{\partial \dot{\vec{r}}} - \frac{\partial L_0}{\partial \vec{r}} = 0 \quad \Rightarrow \quad m\ddot{\vec{r}} = -\vec{\nabla}V(\vec{r}). \tag{2}$$

We can transform this into a hamiltonian H_0 with the help of the definition of $\vec{p}$

$$\vec{p} \equiv \frac{\partial L_0}{\partial \dot{\vec{r}}} = m\dot{\vec{r}}, \tag{3}$$

such that

$$H_0(\vec{r},\vec{p}) \equiv \vec{p}\cdot\dot{\vec{r}} - L_0(\vec{r},\dot{\vec{r}}) = \frac{\vec{p}^2}{2m} + V(\vec{r}). \tag{4}$$

We subject this system to a (classical) electromagnetic field, characterized by the real scalar potential $\phi(\vec{r},t)$ and the real vector potential $\vec{A}(\vec{r},t)$. This changes the lagrangian into

$$L(\vec{r},\dot{\vec{r}},t) = L_0(\vec{r},\dot{\vec{r}}) + q\vec{A}(\vec{r},t)\cdot\dot{\vec{r}} - q\phi(\vec{r},t). \tag{5}$$

a) How does the (canonical) momentum $\vec{p} \equiv \partial L/\partial\dot{\vec{r}}$ change by applying this electromagnetic field?

b) Show that the lagrange equation now has the following form:

$$m\ddot{\vec{r}} = q\left(\vec{\mathcal{E}}(\vec{r},t) + \dot{\vec{r}}\times\vec{\mathcal{B}}(\vec{r},t)\right) - \vec{\nabla}V(\vec{r}), \tag{6}$$

where the term linear in q is the Lorentz force.
Hint: use that $\vec{\nabla}\left(\vec{A}\cdot\dot{\vec{r}}\right) - \left(\dot{\vec{r}}\cdot\vec{\nabla}\right)\vec{A} = \dot{\vec{r}}\times\left(\vec{\nabla}\times\vec{A}\right)$, together with the following definitions for the fields

$$\vec{\mathcal{E}}(\vec{r},t) \equiv -\vec{\nabla}\phi(\vec{r},t) - \frac{\partial}{\partial t}\vec{A}(\vec{r},t), \qquad \vec{\mathcal{B}}(\vec{r},t) \equiv \vec{\nabla}\times\vec{A}(\vec{r},t).$$

c) Show that the Hamiltonian is obtained through minimal substitution:

$$H(\vec{r},\vec{p},t) = \frac{\left[\vec{p} - q\vec{A}(\vec{r},t)\right]^2}{2m} + q\phi(\vec{r},t) + V(\vec{r}). \tag{7}$$

As we have seen now, minimal substitution adds in a direct way the interaction between classical particles and classical electromagnetic fields within the hamiltonian formalism. It also works in the relativistic lagrangian formalism, however, where we can describe interactions between matter particles and gauge fields (e.g. A_μ) directly via minimal substitution in the momentum operator (e.g. by converting the ∂_μ to the covariant derivative $D_\mu = \partial_\mu + igA_\mu$).