Quantum Field Theory 2: exercises for week 3

Exercise 5: propagator for a massive gauge boson

The Proca theory describes free, massive gauge bosons. It gives the following Lagrangian

$$\mathcal{L}_{\text{Proca}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu, \tag{1}$$

where $m \neq 0$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field tensor for the real gauge field $A_\mu(x)$. We ignore for the moment the gauge invariance problems that massive gauge fields have.

(a) Derive the following equation of motion for the lagrangian:

$$0 \;=\; (\Box + m^2)A^\nu - \partial^\nu(\partial \cdot A) \;\equiv\; D^{\nu\rho}_{\text{Proc}}\, A_\rho,$$

which contains a $\partial \cdot A$ term alongside a Klein-Gordon term.

(b) Contract the free index in this equation of motion with a derivative ∂, i.e. consider $\partial_\nu D^{\nu\rho}_{\text{Proc}}\, A_\rho \;=\; 0$, and show that this leads to the Lorenz condition $\partial^\mu A_\mu = 0$.

(c) What does the fact that the Lorenz condition holds imply for the equation of motion and its solutions $A_\mu(x)$?

(d) Consider the plane-wave expansion of the corresponding quantum-field solution:

$$\hat{A}_\mu(x) = \sum_\lambda \int \frac{\mathrm{d}\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left(\epsilon^\lambda_\mu(p)\hat{a}(\vec{p},\lambda)\mathrm{e}^{-ip\cdot x} + \epsilon^{\lambda\,*}_\mu(p)\hat{a}^\dagger(\vec{p},\lambda)\mathrm{e}^{+ip\cdot x}\right)|_{p^0=E_{\vec{p}}}\,, \tag{2}$$

with $\epsilon^\lambda_\mu(p)$ the polarization vector belonging to the particles in the Proca theory with on-shell momentum 4-vector p^μ and polarization quantum number λ. The operators $\hat{a}^\dagger(\vec{p},\lambda)$ and $\hat{a}(\vec{p},\lambda)$ are the bosonic creation and annihilation operators belonging to these free-particle modes. Use this plane-wave expansion to argue that $p^\mu\epsilon^\lambda_\mu(p) = 0$.

Now that we have a bit of a mathematical understanding of the theory, let us try to find the mathematical formulation of the propagator in this theory. This object should encode the creation of a particle/antiparticle at one spacetime point ("source") and its subsequent destruction at another ("sink"). The probability for this to happen can be expressed as

$$\langle 0|T(\hat{A}_\mu(x)\hat{A}^\dagger_\nu(y))|0\rangle\,, \tag{3}$$

where $|0\rangle$ represents the vacuum state and $T(\ldots)$ orders the arguments according to their time component (time ordering). If we were to substitute the plane-wave expansion (2) into this expression, we would end up with a sum over the polarizations, which we will call $R_{\mu\nu}$:

$$R_{\mu\nu} = \sum_\lambda \epsilon^\lambda_\mu(p)\epsilon^{\lambda\,*}_\nu(p) \tag{4}$$

Let us try to see what we can already deduce from this polarization sum alone.

(e) Argue that the generic result for the sum in equation (4) should be

$$R_{\mu\nu} \propto c_1 g_{\mu\nu} + c_2 p_\mu p_\nu, \tag{5}$$

where c_1 and c_2 are scalar factors. What is the mass dimension of c_1 and c_2? Explain why we can simplify this equation to $R_{\mu\nu} \propto c_1\left(g_{\mu\nu} - \frac{1}{m^2}p_\mu p_\nu\right)$.

In order to find the full expression for the propagator, we notice that equation (5) is given in momentum space. Since the propagator describes the motion of a particle, there is a conceptual link between the propagator and the equation of motion. The equation of motion you derived earlier is however given in position space.

(f) Transform the differential operator $D^{\mu\nu}_{\text{Proc}}$ of the Proca theory to its momentum-space counterpart $D^{\mu\nu}_{\text{Proc}}(p)$. Note that this momentum corresponds to a generic Fourier transform and is therefore in general not on-shell.

Since we know that the tensorial structure of the propagator is fully given by the polarization sum of equation (5), we know that the full propagator can be written as

$$P_{\mu\nu} = c_3(p^2)g_{\mu\nu} + c_4(p^2)p_\mu p_\nu, \tag{6}$$

(g) Determine the values for c_3 and c_4 if we would demand that

$$D^{\mu\nu}_{\text{Proc}}(p)P_{\nu\lambda}(p) = i\delta^\mu_\lambda, \tag{7}$$

which defines the propagator to be the inverse of the equation of motion. What does this imply for the expression for the full propagator?

(h) Inspect the momentum dependence of your massive gauge boson propagator and reason why its high-momentum behaviour might pose a problem in theories where the interactions are being mediated by such a massive gauge boson.

(i) In case you have some time left, you can substantiate on this by replacing the Maxwell part of the QED Lagrangian by the Proca Lagrangian and subsequently perform a standard naive power counting analysis (cf. pages 87 and 88 of the QFT lecture notes) to figure out whether this so-called "massive QED" is renormalizable or not.

In this course we will encounter different types of propagators. They all share that they can be calculated via the approach you took here: inverting the equation of motion. Any propagator can be calculated in this way! If you would do this rigorously, you'd find that the denominator will always be the same (for all stable particles), because it essentially originates from the Klein-Gordon theory (or, to speak in more physical terms, from the energy-momentum relation). The only thing that changes every time is the numerator, which describes the polarization sum.

In addition, you have seen a first glimpse of the problems that occur when using a gauge theory that is based on a massive gauge field!