Quantum Field Theory 2: exercises for week 6

Exercise 8: Gauge-boson masses in the Standard Model

Consider the gauged kinetic Higgs Lagrangian term $(D_\mu\Phi)^\dagger(D^\mu\Phi)$, with

$$D_\mu\Phi = \Big(\partial_\mu + \frac{i}{2}\,g\,\vec{\sigma}\cdot\vec{W}_\mu + \frac{i}{2}\,g' B_\mu\Big)\Phi\ , \tag{1}$$

where g is the $SU(2)_L$ gauge coupling, g' the $U(1)_Y$ gauge coupling, $W_\mu^{1,2,3}$ the $SU(2)_L$ gauge bosons, $\sigma^{1,2,3}$ the Pauli spin matrices and B_μ the $U(1)_Y$ gauge boson.

(a) Insert the vacuum expectation value $\langle\Phi\rangle_0 = \begin{pmatrix} 0 \\ v/\sqrt{2}\end{pmatrix}$ in this lagrangian and use that $(\vec{\sigma}\cdot\vec{f}\,)^2 = \vec{f}^{\,2} I_2$ for arbitrary functions f^1, f^2 and f^3, with I_2 being the 2×2 identity matrix. Derive in this way the following expression for the gauge-boson mass terms:

$$\begin{aligned}\mathcal{L}^{\rm GB}_{\rm mass} = \frac{1}{2}\bigg\{&\frac{1}{4}v^2 g^2\left(W^1_\mu W^{1,\mu} + W^2_\mu W^{2,\mu}\right)\\ &+\frac{1}{4}v^2\Big(g^2 W^3_\mu W^{3,\mu} + g'^2 B_\mu B^\mu - g g'\left[W^3_\mu B^\mu + B_\mu W^{3,\mu}\right]\Big)\bigg\}\end{aligned} \tag{2}$$

(b) The diagonalized gauge-boson mass terms read $+\frac{1}{2}\sum_{a=1}^4 M_a^2\, V_\mu^a V^{a,\mu}$. Find the four eigenvalues M_a^2 and express the corresponding eigenvectors V_μ^a in terms of the original fields $W_\mu^{1,2,3}$ and B_μ.

Hint: use matrix notation for $\mathcal{L}^{\rm GB}_{\rm mass}$.

(c) What do the eigenvalues and eigenvectors you found in (b) represent?

(d) What would change if we would have used instead of Φ a scalar doublet $\tilde{\Phi}$ with opposite hypercharge, such that $\langle\tilde{\Phi}\rangle_0 = \begin{pmatrix} v/\sqrt{2} \\ 0\end{pmatrix}$ and $D_\mu\tilde{\Phi} = \Big(\partial_\mu + \frac{i}{2}g\,\vec{\sigma}\cdot\vec{W}_\mu - \frac{i}{2}g' B_\mu\Big)\tilde{\Phi}$?

In (c) you found two mixed states of W^3_μ and B_μ. If you write these fields in a vector (W^3_μ, B_μ), you can directly find the mixed states by applying a rotation matrix

$$\begin{bmatrix}\cos\theta_w & \sin\theta_w\\ -\sin\theta_w & \cos\theta_w\end{bmatrix} \equiv \begin{bmatrix} c_w & s_w \\ -s_w & c_w\end{bmatrix} \tag{3}$$

to this row vector. This leaves θ_w as a free parameter. It is called the electroweak mixing angle or Weinberg angle.

(e) Give an expression for $\tan\theta_w$ in terms of g and g'.

It is striking that the seemingly random sizes of the particle masses (0 GeV, 80.3 GeV and 91.2 GeV for the photon, W boson and Z boson respectively) have a common origin. The theoretically beautiful notion of unification of forces has inspired many to look for a theory that also includes the strong force, creating a so called unified gauge theory, or grand unified theory (GUT). Although the combination of the EM and weak force in the exercise above is a nice concept, it in itself is not a unified theory, because the coupling strengths g and g' are independent of each other. The theoretical power of the theory stops at exactly the point you finished exercise (e): the rotation angle you found there is one of those parameters that you have to measure to know. There is – as far as we know – no way to derive its value from the Standard Model theory.

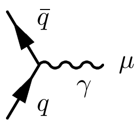
Exercise 9: The electroweak gauge interactions for quarks

Start from the electroweak interactions in terms of gauge eigenstates:

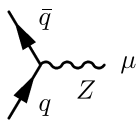
$$
\begin{aligned}
\mathcal{L}_{\text{quark}}^{\text{electroweak int.}} &= \sum_A \bar{Q}'_{A_L} i\gamma^\mu \Big(\frac{i}{2}\, g\, \vec{\sigma}\cdot\vec{W}_\mu + \frac{i}{2}\, g'\, Y(Q_L) B_\mu \Big) Q'_{A_L} \\
&\quad + \sum_A \bar{u}'_{A_R} i\gamma^\mu \Big(\frac{i}{2}\, g'\, Y(u_R) B_\mu \Big) u'_{A_R} + \sum_A \bar{d}'_{A_R} i\gamma^\mu \Big(\frac{i}{2}\, g'\, Y(d_R) B_\mu \Big) d'_{A_R} \ ,
\end{aligned}
$$

with $Y(f)$ indicating the hypercharge of multiplet f and A labeling the various generations of quarks. Subsequently you are requested to translate this in terms of mass eigenstates for both quarks and gauge bosons.

(a) Write down the $SU(2)_L \times U(1)_Y$ gauge interaction term for neutral current (NC) interactions (i.e. interactions that involve a neutral gauge boson: γ or Z) and derive the following Feynman rules for the indicated vertices (with all particles defined to be incoming):



$$= -i\,|e|\,Q_q\,\gamma^\mu$$

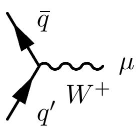


$$= -\frac{ig}{c_w}\,\gamma^\mu (C_V^q - C_A^q \gamma^5)$$

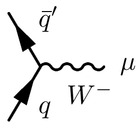
where $C_V^q = \frac{1}{2} I_3(q) - s_w^2 Q_q$ is the vector coupling of the Z boson, $C_A^q = \frac{1}{2} I_3(q)$ its axial vector coupling and s_w the sine of the Weinberg angle.

Hint: first figure out which components of the various generators contribute.

(b) Do the same for the charged current (CC) interactions (i.e. interactions that involve a charged gauge boson W^+ or W^-), where q denotes an up-type quark mass eigenstate and q' a down-type quark mass eigenstate:



$$= -\frac{ig}{\sqrt{2}}\,(V_{\text{CKM}})_{qq'}\,\gamma^\mu P_L$$



$$= -\frac{ig}{\sqrt{2}}\,(V_{\text{CKM}})^*_{qq'}\,\gamma^\mu P_L$$

with V_{CKM} the quark-mixing matrix and P_L the left-handed projection operator.

You might have noticed that neutral current interactions leave the particle type/flavour invariant, while charged current interactions can change it. This is experimentally an important notion: finding a flavour changing neutral current (FCNC) would be a huge sign of physics beyond the Standard Model and is therefore actively sought for in particle physics experiments.