Quantum Field Theory 2 : the Standard Model

Wim Beenakker, www.hef.ru.nl/~wimb

Conventions: in the following we will use so-called natural units ($\hbar = c = \mu_0 = \epsilon_0 = 1$) by absorbing these constants in the relevant fields/quantities
⇒ a single scale remains: mass.
For example: $E \to E * 1/c^2$ (cf $mc^2 \to m$),
$p \to p * 1/c$ (cf $mc \to m$),
$t \to t * c^2/\hbar$ (cf $\lambda_{\text{compton}}/c = \hbar/mc^2 \to 1/m$),
$r \to r * c/\hbar$ (cf $\lambda_{\text{compton}} = \hbar/mc \to 1/m$).
For the flat spacetime metric we will use the signature $g_{\mu\nu} = \text{diag}(+1,-1,-1,-1)$, with the Minkowski indices μ,ν running from 0 to 3. Repeated indices (of any kind) are implicitly summed over, unless stated otherwise.

Guided by the requirement of having local laws of nature, particles and their mutual interactions will be described by continuous (quantum) fields. These fields satisfy classical equations of motion, which can be best formulated in terms of Lagrangians for continuous systems. Such Lagrangians are particularly suitable for discussing symmetries, the cornerstones of relativistic quantum field theory. In order to have the relativity principle built in, we will be working with scalar Lagrangians to describe the physics that we observe.

set of fields $F_j(x)$, x = spacetime 4-vector

Generic: Lagrangian (density) $\mathcal{L}(\{F_j\},\{\partial_\mu F_j\})$, with $\partial_\mu F_j = \frac{\partial}{\partial x^\mu} F_j$
→ equations of motion for F_j (Euler-Lagrange eqns.): $\boxed{\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu F_j)}\right) = \frac{\partial \mathcal{L}}{\partial F_j}}$.

Examples: *) Real free Klein-Gordon field $\phi(x)$: $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2$
(1 d.o.f.) → ⇒ ϕ satisfies the Klein-Gordon eqn. $\partial_\mu(\partial^\mu\phi) + m^2\phi = (\Box + m^2)\phi = 0$.
*) Complex free Klein-Gordon field $\phi(x)$: $\mathcal{L} = (\partial_\mu\phi)(\partial^\mu\phi^*) - m^2\phi\phi^*$
(2 d.o.f.) → ⇒ ϕ and ϕ^* satisfy Klein-Gordon eqns. $(\Box+m^2)\phi = (\Box+m^2)\phi^* = 0$.
*) Free Dirac field $\psi(x)$: $\mathcal{L} = \bar\psi(i\gamma^\mu\partial_\mu - m)\psi$, with γ^μ the γ-matrices of Dirac and $\bar\psi \equiv \psi^\dagger\gamma^0$ the Dirac-adjoint of ψ
(4 d.o.f.) → ⇒ ψ and $\bar\psi$ satisfy the Dirac eqn. and adjoint Dirac eqn.
$(i\gamma^\mu\partial_\mu - m)\psi = 0$ and $\bar\psi(i\overleftarrow{\partial_\mu}\gamma^\mu + m) = 0$.
acting to the left

Noether's theorem for continuous symmetries: consider fields $F_j(x)$ that satisfy the Euler-Lagrange eqns. of $\mathcal{L}(\{F_j\},\{\partial_\mu F_j\})$ and apply the infinitesimal continuous transformations $F_j(x) \to F_j'(x) = F_j(x) + \vartheta \Delta F_j(x)$ (ϑ independent of x, infinitesimal). We speak of a symmetry under this transformation if $\mathcal{L}(x)$ changes by a 4-divergence, which leaves the eqn. of motion invariant. In that case

$$\vartheta \partial_\mu G^\mu \equiv \vartheta \Delta\mathcal{L} = \vartheta\left(\frac{\partial \mathcal{L}}{\partial F_j}\Delta F_j + \frac{\partial \mathcal{L}}{\partial(\partial_\mu F_j)}\underbrace{\Delta(\partial_\mu F_j)}_{\partial_\mu(\Delta F_j)}\right) = \vartheta \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu F_j)}\Delta F_j\right) + \vartheta\underbrace{\left[\frac{\partial \mathcal{L}}{\partial F_j} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu F_j)}\right)\right]}_{\text{E.-L. eqns.: } 0}\Delta F_j$$

$$\Rightarrow \boxed{\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu F_j)}\Delta F_j - G^\mu\right) \equiv \partial_\mu j^\mu = 0},$$

"charge"

i.e. j^μ is a conserved current (Noether current) and $\int_V d\vec{x}\, j^0(x)$ is conserved locally since $\frac{d}{dt}\int_V d\vec{x}\, j^0 = -\int_V d\vec{x}\, \vec{\nabla}\cdot\vec{j} \overset{\text{Gauss}}{=} -\int_{S(V)} d\vec{S}\cdot\vec{j}$.

This forms the basis for the charge conservation laws that will feature prominently in a symmetry-based description of the electromagnetic, strong and weak interactions (and any interactions beyond the Standard Model).

Examples: *) Complex Klein-Gordon theory: $\mathcal{L}(\phi,\phi^*,\partial_\mu\phi,\partial_\mu\phi^*) = (\partial_\mu\phi)(\partial^\mu\phi^*) - m^2\phi\phi^*$.

This Lagrangian is invariant under a continuous phase transformation

$$\phi \to \phi' = e^{i\vartheta}\phi \overset{\text{inf.}}{\approx} \phi + \vartheta\underbrace{(i\phi)}_{\Delta\phi}, \quad \phi^* \to \phi'^* = e^{-i\vartheta}\phi^* \overset{\text{inf.}}{\approx} \phi^* + \vartheta\underbrace{(-i\phi^*)}_{\Delta\phi^*}$$

$\Rightarrow \Delta\mathcal{L} = 0$, i.e. $G^\mu = 0$, and $j^\mu = i\phi\,\partial^\mu\phi^* - i\phi^*\partial^\mu\phi$ is conserved.

*) Free Dirac theory: $\mathcal{L}(\psi,\bar\psi,\partial_\mu\psi) = \bar\psi(i\gamma^\mu\partial_\mu - m)\psi$.

Also this Lagrangian is invariant under the transformation

$$\psi \to \psi' = e^{i\vartheta}\psi \overset{\text{inf.}}{\approx} \psi + \vartheta\underbrace{(i\psi)}_{\Delta\psi}, \quad \bar\psi \to \bar\psi' = e^{-i\vartheta}\bar\psi \overset{\text{inf.}}{\approx} \bar\psi + \vartheta\underbrace{(-i\bar\psi)}_{\Delta\bar\psi}$$

$\Rightarrow \Delta\mathcal{L} = 0$, i.e. $G^\mu = 0$, and $j_V^\mu = \bar\psi i\gamma^\mu(i\psi) + 0 = -\bar\psi\gamma^\mu\psi$ is conserved.

also: Abelian

These two phase transformations are referred to as "global U(1) gauge transformations" [global: ϑ independent of x; U(1): the same $e^{i\vartheta}$ is used for each component of the field]. Later on we will also encounter a generalization of this: global unitary transformations that mix different fields of the same type (such as 2 or 3 Dirac fields), referred to as non-Abelian gauge transf.

weak int.

strong int.

Conjugate momenta : $\pi_j(x) = \partial\mathcal{L}/\partial(\partial_0 F_j)$ conjugate momentum associated with F_j.

*) Free complex Klein-Gordon theory : $\dfrac{\partial\mathcal{L}}{\partial(\partial_0\phi)} = \partial_0\phi^* \equiv \pi$, $\dfrac{\partial\mathcal{L}}{\partial(\partial_0\phi^*)} = \partial_0\phi \equiv \pi^*$.

*) Free Dirac theory : $\dfrac{\partial\mathcal{L}}{\partial(\partial_0\psi)} = i\bar\psi\gamma^0 = i\psi^\dagger \equiv \pi_\psi$, $\dfrac{\partial\mathcal{L}}{\partial(\partial_0\bar\psi)} = 0$

⇒ out of the 8 real d.o.f. of the Dirac spinor in fact 4 belong to the conjugate momentum !

Canonical quantization : the fields $F_j(x)$ and associated conjugate momenta $\pi_j(x)$ become operators that satisfy canonical commutation relations for bosonic fields F_j or anticommutation relations for fermionic fields F_j.

- Bosonic case, fields $\hat\phi_j(x)$: $[\hat\phi_j(t,\vec{x}), \hat\pi_k(t,\vec{y})] = i\delta_{jk}\,\delta(\vec{x}-\vec{y})\,\hat{1}$,
  $[\hat\phi_j(t,\vec{x}), \hat\phi_k(t,\vec{y})] = [\hat\pi_j(t,\vec{x}), \hat\pi_k(t,\vec{y})] = 0$.
  equal-time commutation relations
- Fermionic case, fields $\hat\psi_{j,\alpha}(x)$: $\{\hat\psi_{j,\alpha}(t,\vec{x}), \hat\pi_{k,\beta}(t,\vec{y})\} = i\delta_{jk}\,\delta_{\alpha\beta}\,\delta(\vec{x}-\vec{y})\,\hat{1}$,
  spinor label
  $\{\hat\psi_{j,\alpha}(t,\vec{x}), \hat\psi_{k,\beta}(t,\vec{y})\} = \{\hat\pi_{j,\alpha}(t,\vec{x}), \hat\pi_{k,\beta}(t,\vec{y})\} = 0$.
  equal-time anticommutation relations

This can also be formulated in terms of creation and annihilation operators, from which the particle interpretation of the fields follow: each field annihilates a certain type of "particle" and creates the corresponding type of "antiparticle".

Chapter 1 : Abelian and non-Abelian gauge theories

We will see that the Standard Model is built on the gauge principle, which was introduced in the course Quantum Field Theory in the context of Quantum Electrodynamics (QED).

§1.1 QED and the gauge principle

Consider the electromagnetic field in "vacuum" with charge density $\rho(x) = j_c^0(x) \in \mathbb{R}$ and current density $\vec{j}_c(x) \in \mathbb{R}^3$: $A^\mu(x) = (A^0(x), \vec{A}(x))$.
vector potential
4-vector potential
scalar potential

This e.m. field satisfies the electromagnetic wave equation $\partial_\mu F^{\mu\nu}(x) = j_c^\nu(x)$, with $\boxed{F^{\mu\nu}(x) = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x)}$ the antisymmetric electromagnetic field tensor. This wave equation summarizes the Maxwell equations in 4-vector language.

Gauge freedom: the physics described by the e.m. wave equation is unaffected by the following transformation of the e.m. field, $A^\mu(x) \to A'^\mu(x) = A^\mu(x) + \partial^\mu \chi(x)$, with $\chi(x) \in \mathbb{R}$ an arbitrary sufficiently smooth scalar function.

e.m. gauge transformation

Proof: $F^{\mu\nu}(x) \to F'^{\mu\nu}(x) = \partial^\mu A'^\nu(x) - \partial^\nu A'^\mu(x) = F^{\mu\nu}(x) + (\partial^\mu\partial^\nu - \partial^\nu\partial^\mu)\chi(x) \overset{!}{=} F^{\mu\nu}(x)$.

The associated freedom to choose the e.m. field is called the gauge freedom.

Local charge conservation: $\partial_\nu j_c^\nu(x) = \partial_\nu \partial_\mu F^{\mu\nu}(x) = 0$, since $F^{\mu\nu} = -F^{\nu\mu}$. Hence, the current density $j_c^\mu(x)$ is a conserved current and the electric charge $\int_V d\vec{x}\, j_c^0(x) = \int_V d\vec{x}\,\rho(x)$ is conserved locally. Just as we have seen for Noether currents.

Electromagnetic Lagrangian: the Lagrangian density belonging to the e.m. wave equation is given by
$\mathcal{L}_{e.m.}(x) = -\frac{1}{4}F^{\mu\nu}(x)F_{\mu\nu}(x) - j_c^\mu(x)A_\mu(x)$.

Proof: $\frac{\partial \mathcal{L}_{e.m.}}{\partial(\partial_\mu A_\nu)} = -\frac{1}{2}\left(\frac{\partial}{\partial(\partial_\mu A_\nu)}F_{\rho\sigma}\right)F^{\rho\sigma} = -F^{\mu\nu} \Rightarrow$ E.-L. eqn.: $-\partial_\mu F^{\mu\nu}(x) = -j_c^\nu(x)$.

We see that the e.m. field couples to the conserved current density caused by matter. The charged particles that matter is comprised of are all of the Dirac type (electrons, up quarks, down quarks). For a Dirac-type particle with charge q, the e.m. current density is given by the conserved current $j^\nu_{c,\,Dirac}(x) = q\bar{\psi}(x)\gamma^\nu\psi(x)$. In that case the complete Lagrangian density takes the form

$$\mathcal{L}_{QED}(x) = \overbrace{\bar{\psi}(x)(i\partial_\mu\gamma^\mu - m)\psi(x)}^{\text{free Dirac Lagr.}} \overbrace{- \frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x)}^{\text{free e.m. Lagr.}} \overbrace{- q\bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x)}^{\text{interaction}}$$

$$\equiv \boxed{\bar{\psi}(x)(iD_\mu\gamma^\mu - m)\psi(x) - \frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) = \mathcal{L}_{QED}(x)},$$

with $D_\mu = \partial_\mu + iqA_\mu$. ← cf $-i\hat{p}_\mu + iqA_\mu = -i(\hat{p}_\mu - qA_\mu)$ in quantum mechanics

This last step we recognize as the concept of <u>minimal substitution</u>: the interaction between matter particles and e.m. fields is obtained by inserting $p^\mu \to p^\mu - qA^\mu$ in the free Lagrangian density.

<u>QED from a symmetry principle</u> (gauge invariance and the gauge principle): let's turn around the argument and alternatively start with the Dirac Lagrangian $\mathcal{L}(x) = i\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x) - m\bar{\psi}(x)\psi(x)$. As we have seen, this Lagrangian is invariant under the global $U(1)$ gauge transf. $\psi(x) \to e^{i\varphi}\psi(x)$, $\bar{\psi}(x) \to e^{-i\varphi}\bar{\psi}(x)$ ($\varphi \in \mathbb{R}$, independent of x). According to Noether's theorem this global gauge symmetry can be associated with a conserved current and charge, precisely what we need for e.m. interactions.

<u>Non-relativistic quantum mechanics</u>: this global gauge invariance of a free-fermion system simply underlines the unobservability of the absolute phase of a wave function (i.e. only relative phases are observable through interference).

<u>Relativistic quantum field theory</u> (postulate): demand this to hold locally, since relativistic quantum mechanics should be a local theory. As charges interact while being at non-zero distance, the local interactions should be mediated by force carriers.

⇒ <u>The gauge principle</u>: demand the gauge invariance to hold locally !

Local $U(1)$ gauge transf.: $\psi(x) \to e^{i\varphi(x)}\psi(x)$, $\bar{\psi}(x) \to e^{-i\varphi(x)}\bar{\psi}(x)$, with $\varphi(x)$ a real scalar field

$$\Rightarrow i\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x) \to i\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x) - \underbrace{\bar{\psi}(x)\gamma^\mu\psi(x)\,[\overbrace{\partial_\mu\varphi(x)}^{\text{covariant vector}}]}_{\neq 0 \text{ in general} \Rightarrow \text{not locally invariant}}.$$

Remedy: replace the ordinary derivative ∂_μ by a gauge covariant derivative (or short: <u>covariant derivative</u>) D_μ such that

$$D_\mu \psi(x) \to D'_\mu \psi'(x) = e^{i\varphi(x)} D_\mu \psi(x),$$

causing $D_\mu\psi(x)$ and $\psi(x)$ to transform similarly under local gauge transformations! This can be achieved by

gauge field

=0 in global case

$$D_\mu \equiv \partial_\mu + ig A_\mu(x), \quad \text{with } A_\mu(x) \to A'_\mu(x) = A_\mu(x) - \frac{1}{g}\partial_\mu \varphi(x).$$

cf minimal substitution

gauge coupling

cf e.m. gauge transf. with $\chi(x) = -\frac{\varphi(x)}{g}$

we are exploiting the gauge freedom here!

Proof: $D'_\mu\psi'(x) = (\partial_\mu + ig[A_\mu(x) - \frac{1}{g}\partial_\mu\varphi(x)])e^{i\varphi(x)}\psi(x)$

$= e^{i\varphi(x)}(\partial_\mu + [i\partial_\mu\varphi(x)] + igA_\mu(x) - [i\partial_\mu\varphi(x)])\psi(x) = e^{i\varphi(x)} D_\mu\psi(x).$

The Dirac Lagrangian density with $\partial_\mu \to D_\mu$ is now invariant under the collective local "U(1) gauge transformations" of $\psi(x)$, $\bar\psi(x)$ and $A_\mu(x)$. However, a gauge-invariant kinetic term for a dynamical gauge field still has to be added. To this end we define the gauge-invariant field tensor

$$igF_{\mu\nu}(x) \equiv [D_\mu, D_\nu] = (\partial_\mu + igA_\mu(x))(\partial_\nu + igA_\nu(x)) - (\mu \leftrightarrow \nu)$$

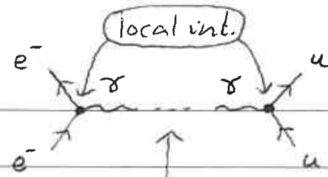
$$= ig\,(\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x))$$

and add the term $-\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x)$ to the Lagrangian, resulting in the same Lagrangian density as obtained from electromagnetism and minimal substitution. This locally invariant theory describes the fundamental e.m. interactions between matter fermions as being mediated by photons (gauge bosons):

$$\mathcal{L}^{int.}_{e.m.} = -g\bar\psi(x)\gamma^\mu A_\mu(x)\psi(x).$$ ← this is called a gauge interaction

Scaling the coupling strength: different matter particles can (and do) have different charges and therefore should have a different coupling strength to the e.m. field. This can be taken into account by scaling $\varphi(x) \to Q\varphi(x)$ and $g \to Qg$:

⇒ • the transformation property of $A_\mu(x)$ is unaltered (because of $\varphi(x)/g$). so one and the same gauge field can couple to particles of different charge and can therefore mediate the interaction.

Graphically: 

with the photon coupling to conserved currents!

a single gauge field mediates the finite-distance interaction between (differently) charged particles

!

- the covariant derivative changes to $D_\mu \to \partial_\mu + iQgA_\mu(x)$, thereby changing the interaction strength. Setting $g = |e|$ (unit charge) and $q = Q|e|$ (particle charge) we retrieve the e.m. interaction given on p. 4.

<u>Massless gauge fields</u>: a massive gauge field would require a mass term $+\frac{1}{2}m_A^2 A_\mu(x)A^\mu(x)$ in the Lagrangian, which is not gauge invariant $\Rightarrow$ <u>without an additional trick local gauge invariance implies $m_A = 0$</u>!
This is fine for the e.m. field, which we know to be massless, but it will come back to haunt us once we discuss the Standard Model.

§1.2 <u>Non-Abelian gauge theories</u>

Motivated by the success of describing QED through the gauge principle, turning a global gauge symmetry (phase symmetry) into a local one, we now generalize this idea to other types of gauge transformations in an attempt to describe other fundamental interactions in nature. As in non-relativistic quantum mechanics we will use continuous unitary transformations to describe such extended symmetries, belonging to the class of <u>Lie groups</u>.

<u>Some relevant aspects of Lie groups</u>:

*) Transformations that lie infinitesimally close to the identity element,

$$g(\vartheta^1, \ldots, \vartheta^n) = 1 + i\vartheta^a T^a + \mathcal{O}(\vartheta^2) \qquad (\vartheta^1, \ldots, \vartheta^n \in \mathbb{R} \text{ infinitesimal}\,;\; T^a \text{ hermitian}),$$

define a vector space called the <u>Lie algebra</u> of the group. The basis

(label the independent transformations)

vectors $T^1, \dots, T^n$ for this vector space are called the <u>generators of the Lie algebra</u>. These generators satisfy a characteristic set of fundamental commutation relations: $[T^a, T^b] = i\underbrace{C^{abc}}_{\text{structure constants}} T^c$. ← (reflects the group structure)

*) A finite group element is obtained by the repeated action of infinitesimal elements:

$$g(\varphi^1, \dots, \varphi^n) = [\,g(\varphi^1/N, \dots, \varphi^n/N)\,]^N \xrightarrow{N \to \infty} e^{i\varphi^a T^a},$$

using that $\lim\limits_{N\to\infty} (1 + x/N)^N = \lim\limits_{N\to\infty} \left(1 + N\frac{x}{N} + \frac{N(N-1)}{2!}\frac{x^2}{N^2} + \dots\right) = e^x$.

Example: finite rotations of a spin-½ system about the $\vec{e}_K$-axis, ($\vec{e}_K^{\,2} = 1$)

$\boxed{\text{Ex. 2}}$ → $g(\underbrace{-\theta^1, -\theta^2, -\theta^3}_{= -\theta \vec{e}_K}) = e^{-i\theta \vec{e}_K \cdot \vec{\sigma}/2} = I_2 \cos(\theta/2) - i(\vec{e}_K \cdot \vec{\sigma}) \sin(\theta/2)$,

with $\sigma^1, \sigma^2, \sigma^3$ the Pauli spin matrices which satisfy the normalization condition $\{\sigma^a/2, \sigma^b/2\} = \frac{1}{2}\delta^{ab} I_2$ and the SU(2) commutation relations $[\sigma^a/2, \sigma^b/2] = i\epsilon^{abc}\sigma^c/2$. Here (SU(2) generators)

$$\epsilon^{abc} = \begin{cases} +1 & \text{even perm. of } (123) \\ -1 & \text{odd perm. of } (123) \\ 0 & \text{else} \end{cases}$$ are the SU(2) structure constants.

*) Most important classes of gauge groups used in particle physics:

- rotation group in N dimensions SO(N).

  Properties: real $N \times N$ matrices with $OO^T = 1$, $\det O = 1$

  $\boxed{\text{Ex. 1}}$ → ⇒ • $\frac{1}{2}N(N-1)$ generators = # planes in N dimensions,

  • purely imaginary generators with $Tr(T^a) = 0$.

- unitary transformations of N-dimensional vectors: contains overall phase transf. [Abelian U(1) subgroup] and SU(N) transformations.

  (cf QED gauge group)

  Properties of SU(N≥2): complex $N \times N$ matrices with $UU^\dagger = 1$, $\det U = 1$

  ⇒ • $N^2 - 1$ generators,

  • $Tr(T^a) = 0$,

  $\boxed{\text{Ex. 1}}$ → • $Tr(T^a T^b) = \frac{1}{2}\delta^{ab}$ (normalization),

  • $[T^a, T^b] = i f^{abc} T^c$, $f^{abc} \in \mathbb{R}$ totally antisymmetric.

  (≠ 0: non-Abelian gauge group)