label the independent transformations

vectors $T^1, \ldots, T^n$ for this vector space are called the <u>generators of the Lie algebra</u>. These generators satisfy a characteristic set of fundamental commutation relations: $[T^a, T^b] = i c^{abc} T^c$. ← reflects the group structure

structure constants

*) A finite group element is obtained by the repeated action of infinitesimal elements:

$$g(\varphi^1, \ldots, \varphi^n) = [g(\varphi^1/N, \ldots, \varphi^n/N)]^N \xrightarrow{N\to\infty} e^{i\varphi^a T^a},$$

using that $\lim_{N\to\infty} (1 + x/N)^N = \lim_{N\to\infty} \left(1 + N\frac{x}{N} + \frac{N(N-1)}{2!}\frac{x^2}{N^2} + \ldots\right) = e^x$.

Example: finite rotations of a spin-½ system about the $\vec{e}_K$-axis, $\vec{e}_K^{\,2} = 1$

Ex. 2 → $g(\underbrace{-\Theta^1, -\Theta^2, -\Theta^3}_{\equiv -\Theta \vec{e}_K}) = e^{-i\Theta \vec{e}_K \cdot \vec{\sigma}/2} = I_2 \cos(\Theta/2) - i(\vec{e}_K \cdot \vec{\sigma}) \sin(\Theta/2)$,

with $\sigma^1, \sigma^2, \sigma^3$ the Pauli spin matrices which satisfy the normalization condition $\{\sigma^a/2, \sigma^b/2\} = \frac{1}{2}\delta^{ab} I_2$ and the SU(2) commutation relations $[\sigma^a/2, \sigma^b/2] = i\epsilon^{abc}\sigma^c/2$. Here

SU(2) generators

$$\epsilon^{abc} = \begin{cases} +1 & \text{even perm. of } (123) \\ -1 & \text{odd perm. of } (123) \\ 0 & \text{else} \end{cases}$$ are the SU(2) structure constants.

*) Most important classes of gauge groups used in particle physics:

- rotation group in N dimensions SO(N).

  Properties: real N×N matrices with $OO^T = 1$, $\det O = 1$

  Ex. 1 → ⇒ • $\frac{1}{2}N(N-1)$ generators = # planes in N dimensions,

  • purely imaginary generators with $T^a = -(T^a)^T$.

- unitary transformations of N-dimensional vectors: contains overall phase transf. [Abelian U(1) subgroup] and SU(N) transformations.

  cf QED gauge group

Properties of SU(N≥2): complex N×N matrices with $UU^\dagger = 1$, $\det U = 1$

⇒ • $N^2 - 1$ generators,

• $\mathrm{Tr}(T^a) = 0$,

Ex. 1 → • $\mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$ (normalization),

• $[T^a, T^b] = i f^{abc} T^c$, $f^{abc} \in \mathbb{R}$ totally antisymmetric.

≠ 0: non-Abelian gauge group

These groups have been specified in their natural, defining representation. This is called the fundamental representation: it acts on N-dimensional vectors, with the generators represented by $N\times N$ matrices. In an $SU(N)$ gauge theory the fermionic matter fields live in the fundamental representation, i.e. they form

N-component multiplets of Dirac fields $\psi = \begin{pmatrix}\psi_1\\ \vdots \\ \psi_N\end{pmatrix}$, $\bar\psi = (\bar\psi_1, \ldots, \bar\psi_N)$.

For instance (see later): $SU(2)$ doublets such as $\begin{pmatrix}\nu_e\\ e^-\end{pmatrix}_L$,

$SU(3)$ quark colour triplets $\begin{pmatrix}q_R\\ q_G\\ q_B\end{pmatrix}$.

Another representation of the Lie algebra that will feature prominently is the adjoint representation: it acts on (N^2-1)-dimensional vectors, with the generators $(T^a)^{bc} = -if^{abc}$ being $(N^2-1)\times(N^2-1)$ matrices.

For instance for $SU(2)$: $(T^a)^{bc} = -i\epsilon^{abc}$ are 3×3 matrices in the adjoint representation. Whereas the fundamental representation involves doublets, the adjoint representation deals with triplets.

The gauge fields live in the adjoint representation, e.g. $W_\mu^{1,2,3}$ triplet in $SU(2)$ or gluon octet $G_\mu^{1,\ldots,8}$ in $SU(3)$.

Non-Abelian gauge theories: consider the Lagrangian

$$\mathcal{L}(x) = i\bar\psi(x)\gamma^\mu\partial_\mu\psi(x) - m\bar\psi(x)\psi(x) \quad , \quad \psi(x) = \begin{pmatrix}\psi_1(x)\\ \vdots \\ \psi_N(x)\end{pmatrix}$$

for an N-component multiplet of Dirac fields. This Lagrangian is invariant under global continuous $SU(N)$ transformations

$$\psi(x) \to \psi'(x) = U\psi(x) \quad , \quad \bar\psi(x) \to \bar\psi'(x) = \bar\psi(x)U^\dagger$$

with $U = e^{i\vartheta^a T^a}$ ($\vartheta^1, \ldots, \vartheta^{N^2-1} \in \mathbb{R}$, independent of x). fund. repr.

Generalized gauge principle: demand the $SU(N)$ invariance to also hold locally [i.e. $\vartheta^a \to \vartheta^a(x)$] in order to describe the interaction between matter and gauge bosons. Here N refers to the number of components in the multiplets linked by the gauge interactions.

experiments have to guide us towards the correct multiplet description !

Replace to this end the derivative ∂_μ by the <u>covariant derivative</u>
$\boxed{D_\mu \equiv \partial_\mu + ig W_\mu^a T^a \equiv \partial_\mu + ig W_\mu}$, acting on ψ in the fundamental repr.
Here g is the $SU(N)$ <u>gauge coupling</u> (like $|e|$ in QED) and the <u>gauge field</u> $W_\mu = W_\mu^1 T^1 + \ldots + W_\mu^{N^2-1} T^{N^2-1}$ lives in the $SU(N)$ Lie algebra

N^2-1 independent transf.
$\Rightarrow$ N^2-1 gauge-field components

$\Rightarrow$ $W_\mu^1, \ldots, W_\mu^{N^2-1}$ form a multiplet in the <u>adjoint representation</u>!

To derive the transformation rule for W_μ, consider the local $SU(N)$ transformation $\psi(x) \to \psi'(x) = U(x)\psi(x)$, $\bar\psi(x) \to \bar\psi'(x) = \bar\psi(x) U^\dagger(x)$, with $U(x) = e^{i\vartheta^a(x) T^a}$ and impose that

$$D_\mu \psi(x) \to D'_\mu \psi'(x) = U(x) D_\mu \psi(x) \quad \forall_\psi \quad \Rightarrow \quad D'_\mu U \psi = U D_\mu \psi \quad \forall_\psi \quad \Rightarrow \quad \boxed{D'_\mu = U D_\mu U^{-1}}.$$

This implies that $(\partial_\mu + ig W'_\mu) = U(\partial_\mu + ig W_\mu)U^{-1} = \partial_\mu + U(\partial_\mu U^{-1}) + ig U W_\mu U^{-1}$

$$\Rightarrow \boxed{W'_\mu = U W_\mu U^{-1} - \frac{i}{g} U(\partial_\mu U^{-1})}.$$

Infinitesimally: $(1 + i\vartheta^b T^b) W_\mu^a T^a (1 - i\vartheta^{b'} T^{b'}) - \frac{i}{g}(1 + i\vartheta^b T^b)(-i\partial_\mu \vartheta^a T^a)(1 - i\vartheta^{b'} T^{b'})$

$$\overset{\text{inf.}}{\approx} W_\mu^a T^a - i\vartheta^b W_\mu^a \overbrace{[T^b, T^a]}^{if^{abc}T^c} - \frac{1}{g}\partial_\mu \vartheta^a T^a = \left(W_\mu^a - \underbrace{\frac{1}{g}\partial_\mu \vartheta^a}_{\text{QED style}} - \underbrace{f^{abc}\vartheta^b W_\mu^c}_{\text{non-Abelian}}\right) T^a = W_\mu'^a T^a.$$

relabel: $a \leftrightarrow c$

This transformation characteristic of the gauge field contains $W_\mu^a - f^{abc}\vartheta^b W_\mu^c = [\delta^{ac} + i(T^b)^{ac}\vartheta^b] W_\mu^c$, corresponding to the infinitesimal $SU(N)$ transformation $e^{i\vartheta^b(x) T^b}$ acting on the adjoint representation.

Next we generalize the "kinetic term" for the gauge fields. To this end we define the <u>$SU(N)$ field tensor</u> $W_{\mu\nu}$:

$$\underbrace{ig W_{\mu\nu} \equiv [D_\mu, D_\nu]}_{\text{Ricci identity}} \Rightarrow W_{\mu\nu} = -\frac{i}{g}(\partial_\mu + ig W_\mu)(\partial_\nu + ig W_\nu) + \frac{i}{g}(\mu \leftrightarrow \nu)$$

$$= ig \overbrace{[W_\mu, W_\nu]}^{\text{involves } [T^b, T^c] \neq 0} + \partial_\mu W_\nu - \partial_\nu W_\mu$$

$$= \boxed{\left(\underbrace{\partial_\mu W_\nu^a - \partial_\nu W_\mu^a}_{\text{cf. QED}} - \underbrace{g f^{abc} W_\mu^b W_\nu^c}_{\text{non-Abelian}}\right) T^a \equiv W_{\mu\nu}^a T^a}.$$

Transformation property: $ig W'_{\mu\nu} = [D'_\mu, D'_\nu] = [U D_\mu U^{-1}, U D_\nu U^{-1}] = U [D_\mu, D_\nu] U^{-1}$

$= ig U W_{\mu\nu} U^{-1} \Rightarrow \boxed{W'_{\mu\nu} = U W_{\mu\nu} U^{-1}}$ just like D_μ

The so-called Yang-Mills Lagrangian

scalar, no open labels

$$\mathcal{L}_{YM} = -\frac{1}{2} Tr(W_{\mu\nu} W^{\mu\nu}) = -\frac{1}{2} W^a_{\mu\nu} W^{b,\mu\nu} Tr(T^a T^b) = -\frac{1}{4} W^a_{\mu\nu} W^{a,\mu\nu} \leftarrow \text{cf QED}$$

is then gauge invariant.

Proof: $Tr(W'_{\mu\nu} W'^{\mu\nu}) = Tr(U W_{\mu\nu} \underbrace{U^{-1} U}_{1} W^{\mu\nu} U^{-1}) \overset{\text{cyclic}}{=} Tr(\underbrace{U^{-1} U}_{1} W_{\mu\nu} W^{\mu\nu}) = Tr(W_{\mu\nu} W^{\mu\nu})$.

This results in the following Lagrangian that is invariant under local SU(N) gauge transformations:

$$\boxed{\mathcal{L}_{SU(N)} = i\bar{\psi}(x) \gamma^\mu D_\mu \psi(x) - m\bar{\psi}(x)\psi(x) - \frac{1}{2} Tr(W_{\mu\nu}(x) W^{\mu\nu}(x))},$$

$\psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_N \end{pmatrix}$: N-plet in fundamental repr., $\psi_1 \cdots \psi_N$ are N Dirac fields,

W^a_μ $(a = 1, \ldots, N^2-1)$: N^2-1 gauge fields, which form a multiplet in the adjoint repr.,

T^a $(a = 1, \ldots, N^2-1)$: N^2-1 generators, acting on ψ in the fundamental repr. and on the gauge fields in the adjoint repr.

Interactions: *) gauge interactions between matter fermions and gauge bosons (like in QED): the term $i\bar{\psi}\gamma^\mu D_\mu \psi$ contains the interaction $-g\bar{\psi}(x)\gamma^\mu T^a \psi(x) W^a_\mu(x) = -g j^{a,\mu}(x) W^a_\mu(x)$, which involves N^2-1 gauge fields coupled to N^2-1 currents. These currents

Ex. 3 → are not conserved as the gauge bosons also interact amongst themselves. Remark: in the case of QED we had the freedom to change the interaction strength by scaling. The non-Abelian case is more restrictive: in order to leave the transformation property of the gauge bosons unaltered, also the interactions have to remain unchanged under scaling!

*) Yang-Mills interactions between three or four gauge bosons: the kinetic gauge-boson term $-\frac{1}{2}Tr(W_{\mu\nu}W^{\mu\nu}) = -\frac{1}{4}W^a_{\mu\nu}W^{a,\mu\nu}$ contains three distinct terms involving two, three and four gauge fields $-\frac{1}{4}(\partial_\mu W^a_\nu - \partial_\nu W^a_\mu)(\partial^\mu W^{a,\nu} - \partial^\nu W^{a,\mu})$,

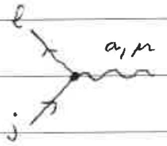
just like in QED

triple gauge couplings (TGC) $\rightarrow$ $+g(\partial_\mu W^a_\nu)W^{b,\mu}W^{c,\nu}f^{abc}$,

absent in QED: $Q_\gamma = 0$

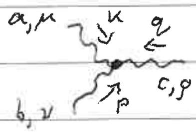
quartic gauge couplings (QGC) $\rightarrow$ $-\frac{g^2}{4}f^{abe}f^{cde}W^a_\mu W^b_\nu W^{c,\mu}W^{d,\nu}$

Feynman rules for interaction vertices: ——— = fermion, ∿∿∿ = gauge boson

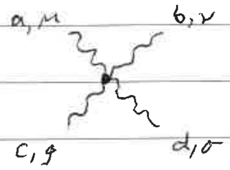


$= -ig\gamma^\mu(T^a)_{\ell j}$ (ℓ, j = multiplet label ; a = gauge-boson label),

used: $\partial_\lambda \rightarrow -i*$(incoming momentum)$_\lambda$

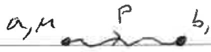


$$= \underbrace{-gf^{abc}}_{-ig(T^a)^{bc}}\left[g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\mu + g^{\rho\mu}(q-k)^\nu\right],$$



$$= -ig^2\left[f^{abe}f^{cde}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) + f^{ace}f^{bde}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}) + f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma})\right],$$

$= \frac{i(\not p + m)}{p^2 - m^2 + i\epsilon}\delta_{\ell j}$,



$= \frac{-ig_{\mu\nu}}{p^2 + i\epsilon}\delta_{ab}$.

*) SU(2): matter doublets and gauge triplets, $T^a = \sigma^a/2$ ($a = 1,2,3$)

Pauli spin matrices

$$\Rightarrow W_\mu = \frac{1}{2}W^a_\mu\sigma^a \equiv \frac{1}{2}\vec{W}_\mu\cdot\vec{\sigma} = \frac{1}{2}\begin{pmatrix} W^3_\mu & \overbrace{W^1_\mu - iW^2_\mu}^{\equiv\sqrt{2}W^+_\mu} \\ \underbrace{W^1_\mu + iW^2_\mu}_{\equiv\sqrt{2}W^-_\mu} & -W^3_\mu \end{pmatrix} = \underbrace{T^3W^3_\mu}_{\text{diagonal}} + \frac{\overbrace{T^+W^+_\mu}^{\text{raising}} + \overbrace{T^-W^-_\mu}^{\text{lowering}}}{\sqrt{2}},$$

$$U(x) = e^{\frac{i}{2}\theta^a(x)\sigma^a} \equiv e^{\frac{i}{2}\vec{\theta}(x)\cdot\vec{\sigma}} = \cos(\theta(x)/2)\,I_2 + i\frac{\vec{\theta}(x)\cdot\vec{\sigma}}{\theta(x)}\sin(\theta(x)/2),$$

with $\theta(x) \equiv |\vec{\theta}(x)| = \sqrt{\theta^a(x)\theta^a(x)}$.

This gauge group will be needed for the gauge-theory description of the weak interactions and the associated $W^\pm$ and Z gauge bosons [although only part of the Z gauge boson belongs to SU(2)].

*) SU(3) : matter triplets and gauge octets , $T^a = \lambda^a/2 \quad (a=1,\cdots,8)$
in terms of the Gell-Mann λ-matrices

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

This gauge group is used in the gauge-theory description of the strong interactions, called Quantum Chromodynamics (QCD). The associated gauge bosons are called gluons.
indicated by

Chapter 2 : Building towards the Standard Model

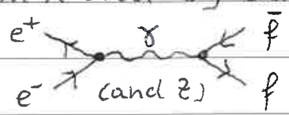
In the next step we discuss the experimental input that has led to the gauge-group ingredients of the Standard Model, which is a gauge-theory description of the strong, weak and hypercharge interactions based on the gauge group $\underbrace{SU(3)_c}_{\text{strong int.}} \times \underbrace{SU(2)_L \times U(1)_Y}_{\text{electroweak int.}}$. The electromagnetic interactions are automatically contained in this description, hidden in the electroweak $SU(2)_L \times U(1)_Y$ sector.

$SU(3)_c$, c = colour : matter triplets (quarks) and gauge octets (gluons)
$\begin{pmatrix} \psi_R \\ \psi_G \\ \psi_B \end{pmatrix}$ $\qquad G_\mu^a \ (a=1,\cdots,8)$

Quantum Chromo-dynamics (QCD)

Gauge coupling : g_s = strong interaction strength.

Why? : *) spin-½ quarks confined inside hadrons have an intrinsic property (quantum number) called colour ⇒ colour multiplets!
This follows from the existence of the Δ^{++} baryon (= uuu bound state with L = 0 and S = 3/2) $\xrightarrow{\text{fermions}}$ at least three colours!
spatial + spin symmetric

*) Consider the reaction $e^+e^- \to f\bar{f}$ ($f\bar{f} = \mu^-\mu^+, \tau^-\tau^+, q\bar{q}$) for unpolarized e^+e^- beams, which is dominated by virtual photon exchange at $\mathcal{O}$(GeV) energies : 

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quarks