A Special relativity: conventions and definitions

In this appendix we specify a few conventions that will allow us to deal with flat spacetime (Minkowski space). The time coordinate t and spatial coordinates $\vec{r}$ of a particle are combined into a contravariant position 4-vector

$$\boxed{x^\mu \ : \qquad x^0 \ \equiv \ ct \qquad \text{and} \qquad \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \ \equiv \ \vec{x} \ \equiv \ \vec{r}}\ , \tag{A.1}$$

with $\mu = 0, 1, 2, 3$ the Minkowski index. Also the energy E and momentum $\vec{p}$ are combined into a contravariant momentum 4-vector

$$\boxed{p^\mu \ : \qquad p^0 \ \equiv \ E/c \ = \ \frac{mc}{\sqrt{1-\vec{v}^{\,2}/c^2}} \qquad \text{and} \qquad \begin{pmatrix} p^1 \\ p^2 \\ p^3 \end{pmatrix} \ \equiv \ \vec{p} \ = \ \frac{m\vec{v}}{\sqrt{1-\vec{v}^{\,2}/c^2}}}\ , \tag{A.2}$$

corresponding to a particle with rest mass m and velocity $\vec{v} = \mathrm{d}\vec{r}/\mathrm{d}t$.

The metric tensor in Minkowski space is given by

$$g_{\mu\nu} \ = \ g^{\mu\nu} \ = \ \begin{pmatrix} 1 & & & \emptyset \\ & -1 & & \\ & & -1 & \\ \emptyset & & & -1 \end{pmatrix} . \tag{A.3}$$

From the contravariant vector a^μ the corresponding covariant vector a_μ can be derived as

$$a_\mu \ \equiv \ g_{\mu\nu}\, a^\nu \ . \tag{A.4}$$

Here we made use of the Einstein summation convention, which states that a repeated Minkowski index automatically implies summation of that index. Finally we introduce the inner product (scalar product)

$$x \cdot y \ \equiv \ x^\mu y_\mu \ = \ x_\mu y^\mu \ = \ g_{\mu\nu}\, x^\mu y^\nu \ = \ x^0 y^0 - \vec{x} \cdot \vec{y} \ , \tag{A.5}$$

which is not positive definite as a result of the spacetime metric $g_{\mu\nu}$ in Minkowski space.

Inertial systems:	reference frames in which force-free particles travel along straight lines are referred to as inertial systems.
Relativity principle:	all inertial systems are physically equivalent and the speed of light is the same in each inertial system.

The relativity principle is reflected in the transformation characteristic of position vectors while going from inertial system S with origin O to inertial system S' with origin O'.

Assume the origins to coincide (i.e. $O = O'$) at time $t = 0$ and indicate the corresponding coordinate transformation as follows:

$$S \; : \quad x^\mu \qquad \rightarrow \qquad S' \; : \quad x'^\mu \; = \; \Lambda^\mu{}_\nu \, x^\nu \; . \tag{A.6}$$

The coordinate transformation (Lorentz transformation) Λ has to be linear to be able to map linear motions on linear motions in the absence of forces. It can be defined in accordance with the relativity principle by demanding $x^2 \equiv x \cdot x$ to be invariant under the transformation. After all, in that case a particle that moves at the speed of light starting from the origin O at $t = 0$, which implies $c^2 = \vec{v}^{\,2} = \vec{x}^{\,2}/t^2 \;\Rightarrow\; x^2 = 0$, automatically travels with the speed of light in any other inertial system S', since $x'^2 = x^2 = 0$ implies $\vec{v}\,'^2 = \vec{x}\,'^2/t'^2 = c^2$. From this condition we can derive that

$$\forall_x \; x^2 = x'^2 \quad \xrightarrow{(A.5),(A.6)} \quad \forall_x \; g_{\rho\sigma}\, x^\rho x^\sigma \; = \; \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \, g_{\mu\nu}\, x^\rho x^\sigma \quad \Rightarrow \quad \Lambda^\mu{}_\rho \Lambda_{\mu\sigma} \; = \; g_{\rho\sigma} \; , \tag{A.7}$$

which implies that $(\Lambda^0{}_0)^2 = 1 + (\Lambda^1{}_0)^2 + (\Lambda^2{}_0)^2 + (\Lambda^3{}_0)^2 \geq 1$. The inverse of the Lorentz transformation then becomes

$$\Lambda^{\mu\rho}\Lambda_{\mu\sigma} \; = \; g^\rho{}_\sigma \; = \; \begin{pmatrix} 1 & & & \emptyset \\ & 1 & & \\ & & 1 & \\ \emptyset & & & 1 \end{pmatrix} \quad \Rightarrow \quad \Lambda^{\mu\rho} \; = \; (\Lambda^{-1})^{\rho\mu} \quad , \quad \det(\Lambda) \; = \; \pm 1 \; . \tag{A.8}$$

The Lorentz transformations can be subdivided as follows. First of all there are the so-called proper orthochronous Lorentz transformations (or short: "the Lorentz transformations"), for which $\det(\Lambda) = +1$ and $\Lambda^0{}_0 \geq 1$. These transformations comprise:

- spatial rotations, about three independent spatial axes.
- Lorentz boosts, in three independent spatial directions. Example: S' moves relative to S at constant velocity $v = \beta c$ in the x^1-direction. In that case

$$\Lambda^\mu{}_\nu \; = \; \begin{pmatrix} 1/\sqrt{1-\beta^2} & -\beta/\sqrt{1-\beta^2} & 0 & 0 \\ -\beta/\sqrt{1-\beta^2} & 1/\sqrt{1-\beta^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} \cosh\eta & -\sinh\eta & 0 & 0 \\ -\sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{A.9}$$

$$\xrightarrow{\beta \ll 1,\, x^0 = ct} \quad \text{non-relativistic Galilei transformation :} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ -v/c & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

Two more classes of Lorentz transformations can be added to this: transformations with $\det(\Lambda) = -1$ and $\Lambda^0{}_0 \geq 1$, such as the parity transformation that describes spatial inversion, and transformations with $\Lambda^0{}_0 \leq -1$, such as the time-reversal transformation. All transformations can be combined into the so-called homogeneous Lorentz transformations, for which an arbitrary transformation can be written as a product of rotations, boosts, spatial inversion and time reversal. Spatial inversion and time reversal have a special status:

- free-particle systems are invariant under these transformations, but this invariance gets lost once the weak (nuclear) force enters the game;
- these transformations cannot be connected continuously with the identity transformation, as opposed to the proper orthochronous Lorentz transformations which form a continuous group and for which a finite transformation can be written as an infinite series of infinitesimal transformations. Spatial inversion and time reversal in fact incorporate the possible discrete Lorentz transformations.

Finally, we also have to consider the possibility that the origin O' is shifted with respect to the origin O:

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu} + a^{\mu} \qquad (\text{short: } x' = \Lambda x + a)\ , \qquad \text{with } a^{\mu} \text{ a constant 4-vector}$$

$$\Rightarrow \quad x^{\mu} = (\Lambda^{-1})^{\mu}{}_{\nu}\, (x' - a)^{\nu} \qquad (\text{short: } x = \Lambda^{-1}[x' - a]\,)\ . \tag{A.10}$$

All transformations together form the so-called inhomogeneous Lorentz transformations, which are also known as the Poincaré transformations. They are defined by demanding the invariance $(x-y)^2 = (x'-y')^2$.

For the construction of a relativistic wave equation, which should have the same form in all inertial systems, we need several types of quantities (independent of space and time) and fields (dependent on space and time).

- Scalar quantities: quantities that remain the same in each inertial system, such as $p^2 = p \cdot p$ and the speed of light c.
- Scalar fields: fields with the following transformation characteristic

  $$S\ :\ \phi(x) \quad \rightarrow \quad S'\ :\ \phi'(x') = \phi(x) \overset{(A.10)}{=\!=\!=} \phi\big(\Lambda^{-1}[x'-a]\big)\,.$$
- Contravariant vector quantities: 4-vectors that transform as $A'^{\mu} = \Lambda^{\mu}{}_{\nu}\, A^{\nu}$, such as the momentum 4-vector p^{μ}.
- Contravariant vector fields: fields with the following transformation characteristic

  $$S\ :\ A^{\mu}(x) \quad \rightarrow \quad S'\ :\ A'^{\mu}(x') = \Lambda^{\mu}{}_{\nu}\, A^{\nu}(x) \overset{(A.10)}{=\!=\!=} \Lambda^{\mu}{}_{\nu}\, A^{\nu}\big(\Lambda^{-1}[x'-a]\big)\,.$$
- Covariant vector quantities: 4-vectors that transform as

  $$A'_{\mu} = g_{\mu\rho} A'^{\rho} = g_{\mu\rho}\Lambda^{\rho}{}_{\sigma}A^{\sigma} = g_{\mu\rho}\Lambda^{\rho}{}_{\sigma}\, g^{\sigma\nu}A_{\nu} = \Lambda_{\mu}{}^{\nu}\, A_{\nu} \overset{(A.8)}{=\!=\!=} (\Lambda^{-1})^{\nu}{}_{\mu}\, A_{\nu}\,.$$
- Covariant vector fields: fields with the following transformation characteristic

  $$S\ :\ A_{\mu}(x) \quad \rightarrow \quad S'\ :\ A'_{\mu}(x') = g_{\mu\rho}A'^{\rho}(x') = g_{\mu\rho}\Lambda^{\rho}{}_{\sigma}A^{\sigma}(x) = g_{\mu\rho}\Lambda^{\rho}{}_{\sigma}\, g^{\sigma\nu}A_{\nu}(x)$$

  $$\overset{(A.10)}{=\!=\!=} \Lambda_{\mu}{}^{\nu}A_{\nu}\big(\Lambda^{-1}[x'-a]\big) \overset{(A.8)}{=\!=\!=} (\Lambda^{-1})^{\nu}{}_{\mu}A_{\nu}\big(\Lambda^{-1}[x'-a]\big)\,.$$

Examples: the gradient in Minkowski space,

$$\partial_\mu \;\equiv\; \frac{\partial}{\partial x^\mu} \quad\Rightarrow\quad \partial_0 \;=\; \frac{1}{c}\frac{\partial}{\partial t} \qquad \text{and} \qquad \begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix} \;=\; \vec{\nabla}\;, \tag{A.11}$$

is a covariant vector quantity since

$$\partial'_\mu = \frac{\partial}{\partial x'^\mu} \;=\; \Big(\frac{\partial x^\nu}{\partial x'^\mu}\Big)\frac{\partial}{\partial x^\nu} \;\overset{(A.10)}{=\!=\!=}\; (\Lambda^{-1})^\nu{}_\mu\,\frac{\partial}{\partial x^\nu} \;\overset{(A.8)}{=\!=\!=}\; \Lambda_\mu{}^\nu\,\frac{\partial}{\partial x^\nu} \;=\; \Lambda_\mu{}^\nu\,\partial_\nu\;.$$

Consequently, $\partial^\mu \;=\; g^{\mu\nu}\partial_\nu$ is automatically a contravariant vector quantity. Furthermore

- $A_\mu(x) \equiv \partial_\mu\phi(x)$ is a covariant vector field if $\phi(x)$ is a scalar field.
  Proof: $A'_\mu(x') \;=\; \partial'_\mu\phi'(x') \;=\; \Lambda_\mu{}^\nu\partial_\nu\phi(x) \;=\; \Lambda_\mu{}^\nu A_\nu(x)$.

- $\phi(x) \equiv A_\mu(x)B^\mu(x)$ is a scalar field if A and B are vector fields.
  Proof: $\phi'(x') \;=\; A'_\mu(x')B'^\mu(x') \;=\; A_\nu(x)\,(\Lambda^{-1})^\nu{}_\mu\Lambda^\mu{}_\rho B^\rho(x) \;=\; A_\nu(x)B^\nu(x) = \phi(x)$.

- $\phi(x) \equiv \partial_\mu A^\mu(x)$ is a scalar field if $A^\mu(x)$ is a contravariant vector field (see previous example).