Standard Model and Beyond Exercises week 1

Exercise 1: Transforming the real Klein-Gordon field

This exercise leans heavily on the appendix on special relativity, which can be found on the course's website. Read this appendix before starting this exercise (especially the part on the *relativity principle* and *Lorentz transformations*).

The lecture notes cover the following proof for a Lorentz-scalar lagrangian density $\mathcal{L}(x)$ and associated action $S = \int_{t_1}^{t_2} \int \mathrm{d}x\ \mathcal{L}(x) \equiv \int \mathrm{d}^4x\ \mathcal{L}(x)$:

$$S = \int \mathrm{d}^4x\ \mathcal{L}(x) \to S' \equiv \int \mathrm{d}^4x\ \mathcal{L}'(x) = \int \mathrm{d}^4x\ \mathcal{L}\left(\Lambda^{-1}x\right) \tag{1}$$

$$\xlongequal{x=\Lambda y} \int \mathrm{d}^4x\ \mathcal{L}(y) = \int \mathrm{d}^4y\ \mathcal{L}(y) = S \tag{2}$$

where the arrow indicates the transition from one inertial system to another by means of a Lorentz transformation.

(a) Consider in this exercise a boost as Lorentz transformation. Explain why the last step in this proof is justified. What happens to the integral and its boundaries during this step?

(b) What does the proof above imply for the physics observed in both intertial systems?

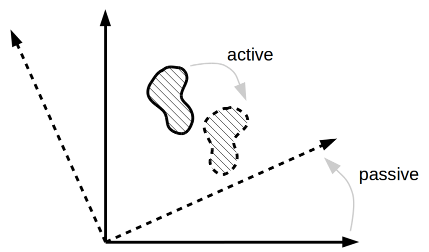
In the lecture we looked at the complex Klein-Gordon theory, here we will look at the lagrangian density for the real KG theory

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \qquad (\phi(x) \in \mathbb{R}) \tag{3}$$

(c) Given that we want the lagrangian density to be scalar, what does that imply for $\phi(x)$?

(d) Calculate the equation of motion for this lagrangian and write this as $D\phi(x) = 0$, where D is some differential operator.

Having derived the equation of motion from the lagrangian density, it is interesting to see how this behaves under Lorentz transformations. There are two types of transformations that we can apply here: active and passive transformations (see image). In passive transformations you transform the coordinate system you are working in (i.e. $x^\mu \to x'^\mu$), whereas in active transformations you transform the field itself.



(e) Perform both the active and passive Lorentz transformation on the equation of motion you found. Show that these yield the same physics.
Hint: in the passive case D and $\phi(x)$ transform into D' and $\phi'(x')$ w.r.t. the new coordinates x', whereas in the active case D and x stay the same but $\phi(x)$ becomes $\phi'(x)$.

Particles described by the lagrangian density given above are spin-0 bosons. In this exercise you have just proven the fundamental property of such particles that they have no "directionality" whatsoever. In the Standard Model, there is only one such spin-0 particle: the Higgs boson. We should note however that for the actual Higgs boson there is an additional potential term in the lagrangian density, but this extra term does not influence the fundamental property you proved.

Exercise 2: Describing the phase symmetry of fermions

For describing free fermions you have to use the Dirac theory. In this theory, the lagrangian density is given by

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi}\,(i\gamma^\mu \partial_\mu - m)\,\psi, \tag{4}$$

in which $\psi(x)$ is a 4-dimensional Dirac spinor, $\bar{\psi} = \psi^\dagger(x)\gamma^0$ its adjoint and $\gamma^\mu(\mu = 0, \ldots, 3)$ the 4×4 Dirac gamma matrices.

(a) Calculate the equations of motion for ψ and $\bar{\psi}$ from this lagrangian density. The equation of motion you find for $\bar{\psi}$ is called the Dirac equation.

(b) Consider the continuous phase transformation $\psi \to e^{i\alpha}\psi$, where α is a real constant. Prove that $\mathcal{L}$ is invariant under this transformation. Can we speak of a symmetry under this phase transformation?

(c) Given that ψ and $\bar{\psi}$ are solutions to their respective equation of motion, derive the conserved Noether current j^μ and charge Q.

(d) We took α to be a real constant, but what if it is dependent on x^μ? Which problem do you run into? Why is this an interesting question to ask, phenomenologically speaking?

(e) Can you give a reason why a phase transformation, even one with a constant real α, is problematic in the real Klein-Gordon theory of exercise 1? What does this imply for the charge Q of the field ϕ?

This exercise might look to you like an exercise in bare mathematics, but the conserved current you have derived here is an essential part of the description of electromagnetic interactions. Without knowing it, you already touched upon one of the most essential parts of the Standard Model!