Standard Model and Beyond Exercises week 3

Exercise 4: See you later, propagator

The Proca theory describes free, massive gauge bosons. It gives the following lagrangian

$$\mathcal{L}_{\text{Proca}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu, \tag{1}$$

where $m \neq 0$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field tensor for the gauge field A_μ. We ignore for the moment the gauge invariance problems that massive gauge fields have.

(a) Calculate the equation of motion for the lagrangian and show that it contains, alongside a Klein-Gordon term, a new term.

(b) Contract the free index in the e.o.m. with a derivative ∂ and show that this leads to the Lorenz condition $\partial_\mu A^\mu = 0$.

(c) What does the fact that the Lorenz condition holds imply for the equation of motion and its solutions A^μ?

For the polarization vectors belonging to the particles in the Proca theory it holds that

$$v(p,\lambda) = \bar{v}_c(p,\lambda) = \epsilon_\mu^\lambda(p) \qquad \text{and} \tag{2}$$

$$v_c(p,\lambda) = \bar{v}(p,\lambda) = \epsilon_\mu^{\lambda^*}(p). \tag{3}$$

(d) Use this information on the polarization and the plane wave expansion

$$\hat{A}_\mu(x) = \sum_\lambda \int \frac{\mathrm{d}\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left(\epsilon_\mu^\lambda(p)\hat{a}(\vec{p},\lambda)\mathrm{e}^{-ip\cdot x} + \epsilon_\mu^{\lambda^*}(p)\hat{a}_c^\dagger(\vec{p},\lambda)\mathrm{e}^{+ip\cdot x}\right)|_{p^0=E_{\vec{p}}} \tag{4}$$

to argue that $p^\mu \epsilon_\mu^\lambda(p) = 0$.

Now that we have a bit of a mathematical understanding of the theory, let us try to find the mathematical formulation of the propagator in this theory. This object should encode the creation of a particle/antiparticle at one spacetime point ("source") and its subsequent destruction at another ("sink"). The probability for this to happen can be expressed as

$$\langle 0|T(\hat{A}_{\mu I}(x)\hat{A}_{\nu}{}_I^\dagger(y))|0\rangle\,, \tag{5}$$

where $T(\ldots)$ orders its arguments according to their time component. If we were to substitute the plane wave expansion (4) into this expression, we would end up with a sum over the polarizations, which we will call $R_{\mu\nu}$:

$$R_{\mu\nu} = \sum_\lambda \epsilon_\mu^\lambda(p)\epsilon_\nu^{\lambda^*}(p) \tag{6}$$

Let us try to see what we can already deduce from this polarization sum alone.

(e) Argue that the generic result for the sum in equation (6) should be

$$R_{\mu\nu} \propto c_1 g_{\mu\nu} + c_2 p_\mu p_\nu, \tag{7}$$

where c_1 and c_2 are scalar factors. What is the mass dimension of c_1 and c_2? Explain why we can simplify this equation to $R_{\mu\nu} \propto c_1\left(g_{\mu\nu} - \frac{1}{m^2}p_\mu p_\nu\right)$.

In order to find the full expression of the propagator, we notice that equation (7) is given in momentum space. Since the propagator describes the motion of a particle, you might have already noticed a conceptual link between the propagator and the equation of motion. The equation of motion you derived earlier is however given in position space.

(f) Transform the equation of motion to momentum space. Write your result as $D^{\mu\nu}_{\text{Proc}}(p)A_\mu = 0$.

Since we know that the tensorial structure of the propagator is fully given by the polarization sum of equation (7), we know that the full propagator can be written as

$$P_{\nu\lambda} = c_3(p^2)g_{\mu\nu} + c_4(p^2)p_\mu p_\nu, \tag{8}$$

(g) Determine the values for c_3 and c_4 if we would demand that

$$D^{\mu\nu}_{\text{Proc}}(p)P_{\nu\lambda}(p) = i\delta^\mu_\lambda. \tag{9}$$

What does this imply for the expression for the full propagator?

> In this course we will encounter a whole series of different propagators. They all share that they can be calculated via the approach you took here: inverting the equation of motion. *Any* propagator can be calculated in this way!
> If you would do this rigorously, you'd find that the denominator will always be the same (for all *stable* particles), because it essentially originates from the Klein-Gordon theory (or, to speak in more physical terms, from the energy-momentum relation). The only thing that changes every time is the numerator, which describes the polarization sum.

Exercise 5: The science behind the black magic of equation (9)

In the previous exercise we introduced equation (9), without which you would not be able to find an expression for the propagator. This equation states that the propagator $P_{\mu\nu}(p)$ should be the inverse of the equation of motion $D^{\mu\nu}_{\text{Proc}}(p)$. In this exercise you will learn why this should hold by looking at a free complex scalar theory with field $\phi(x)$. The propagator is given by

$$D_F(x-y) \equiv \langle 0|T(\hat{\phi}_I(x)\hat{\phi}^\dagger_I(y))|0\rangle \,. \tag{10}$$

(a) Show, by inserting the scalar plane wave expansion at the appropriate places in (10), that this propagator can be written as

$$\int \frac{\mathrm{d}\vec{p}}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\left(\theta(x^0-y^0)\mathrm{e}^{-ip\cdot(x-y)} + \theta(y^0-x^0)\mathrm{e}^{+ip\cdot(x-y)}\right)|_{p^0=E_{\vec{p}}}, \tag{11}$$

where $\theta(\ldots)$ is the step function.

In order to check if equation (9) indeed holds, we replicate it by applying the differential operator from the Klein-Gordon theory to equation (11):

$$\left(\Box_x + m^2\right)D_F(x-y). \tag{12}$$

This requires us to perform a d'Alembertian to terms like $\theta(x^0-y^0)\mathrm{e}^{-ip\cdot(x-y)}$. This requires some working knowledge on applying derivatives to delta functions. To avoid this we will provide you with the following equation that simplifies the calculations[1]:

$$\left(\Box_x + m^2\right)\left(\theta(x^0-y^0)\mathrm{e}^{-ip\cdot(x-y)}\right)|_{p^0=E_{\vec{p}}} = \left(\frac{\partial}{\partial x^0}\theta(x^0-y^0)\right)\left(\frac{\partial}{\partial x^0}\mathrm{e}^{-ip\cdot(x-y)}\right)|_{p^0=E_{\vec{p}}}. \tag{13}$$

(b) Show that $\left(\Box_x + m^2\right)D_F(x-y) = -i\delta^{(4)}(x-y)$.

(c) In one of the steps you took in (b) you lost the four-vector notation of p and x and replaced it with $\vec{p}$ and $\vec{x}$ instead. Explain what this step implied physically.

[1] You are of course free to do the full calculation nevertheless!

In momentum representation the Klein-Gordon differential operator can be written as $(m^2 - p^2)$ The propagator is nevertheless most of the time expressed as

$$\text{propagator} = \frac{i}{p^2 - m^2 + i\epsilon}, \tag{14}$$

instead of $\frac{i}{p^2 - m^2}$, where we let $\epsilon \downarrow 0$ after performing all calculations.

(d) Explain why this is done. How could you have seen from the beginning, based on the differential operator of the Klein-Gordon theory, that you would have to do something like adding the $i\epsilon$ in the end?

> You might have already noted that you lost the energy-momentum relation for your propagating particle. The integral over p that you have to perform over the propagator therefore also results in contributions for particles with an 'incorrect' mass. We call such particles off-shell. As you can see from the propagator (14) such particles are suppressed. You can also turn this statement around: if you provide exactly the right amount of energy, the propagator *enhances* the matrix element, which will lead – as we will see in the next lecture – to higher probabilities for the process to happen. Such enhanced probabilities are called *resonances*. Creating colliders operating at exactly such a resonance allows you to produce the associated particle in relatively high abundance. A possible new collider, the ILC, will most likely also operate around 250 GeV, placing it at the Higgs resonance. Because of this, it is called a 'Higgs factory'.

Exercise 6: Feynman diagrams

(a) Draw all lowest order Feynman diagrams of the following QED processes, i.e. the continuous diagrams with the lowest number of interaction vertices.

$$e^+e^- \to \mu^+\mu^- \qquad e^-\mu^- \to e^-\mu^- \qquad e^-e^- \to e^-e^-$$

(b) Calculate the matrix element $\mathcal{M}$ for the three processes in (a), using the Feynman rules of QED. When multiple diagrams contribute to the same process, you have to sum appropriately over them to get the matrix element for that process:

$$i\mathcal{M} = \sum \text{diagrams} \tag{15}$$

(c) Draw the lowest order diagram that contains 1 fermion loop for the process $e^+(k_+, r_+)e^-(k_-, r_-) \to \mu^+(p_+, s_+)\mu^-(p_-, s_-)$.

(d) Show that the corresponding matrix element $\mathcal{M}$ equals the following expression

$$i\mathcal{M} = -\frac{|e|^2 q_f^2}{(q^2 + i\epsilon)^2} \left[\bar{v}^{r_+}(k_+)\gamma^\mu u^{r_-}(k_-)\right] \cdot \left[\bar{u}^{s_-}(p_-)\gamma^\nu v^{s_+}(p_+)\right] \cdot \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \frac{\mathrm{Tr}\left[\gamma_\mu[\not{l} + m_f]\gamma_\nu[\not{l} + \not{q} + m_f]\right]}{\left[l^2 - m_f^2 + i\epsilon\right]\left[(l+q)^2) - m_f^2 + i\epsilon\right]} \tag{16}$$

Where $q_e = q_\mu = -|e|$ is the charge of the incoming and outgoing particles, q_f is the charge of the fermion in the internal loop, m_f its mass and q the momentum of the internal photon propagators.

> You have written the mathematical expressions for a couple of diagrams, but the matrix elements themselves represent nothing that is directly measurable. Next lecture we will see how to calculate quantities that are experimentally verifiable. This is an important step! You can calculate anything you want, but you always have to verify your results with experiments to know if you are really describing nature.