Lecture 10: Giving mass to fermions and quark mixing

As mentioned on p. 25, a mass term $-m\bar\psi\psi = -m(\bar\psi_L\psi_R + \bar\psi_R\psi_L)$ would break gauge invariance in the Standard Model. We will therefore use the Higgs mechanism to also give mass to fermions. To this end we introduce $SU(2)_L\times U(1)_Y$ invariant interactions between the fermions and scalar doublets, which compensate the $SU(2)_L$ transformations of $\bar\psi_L$. Moreover, $U(1)_Y$ gauge invariance requires Higgs doublets with hypercharge $-Y(\bar\psi_L\psi_R) = Y(\psi_L) - Y(\psi_R) = 2(Q-I_3) - 2Q = -2I_3$

$\Rightarrow$ use Φ, $Y(\Phi) = -Y(\bar\psi_L\psi_R) = +1$ to give mass to fermions with $I_3 = -\tfrac{1}{2}$,
use $\tilde\Phi$, $Y(\tilde\Phi) = -Y(\bar\psi_L\psi_R) = -1$ to give mass to fermions with $I_3 = +\tfrac{1}{2}$.

For one generation of fermions (ν_e, e, u, d) these so-called <u>Yukawa interactions</u> read

$$\mathcal{L}_{Yu} = -f_e\, \underset{(\bar\psi_{\nu_{e_L}},\, \bar\psi_{e_L})}{\bar L_{1_L}}\, \Phi\, \overset{\psi_{e_R}}{e_R} - f_d\, \underset{(\bar\psi_{u_L},\, \bar\psi_{d_L})}{\bar Q_{1_L}}\, \Phi\, \overset{\psi_{d_R}}{d_R} - f_{\nu_e}\, \bar L_{1_L}\, \tilde\Phi\, \overset{\psi_{\nu_{e_R}}}{\nu_{e_R}} - f_u\, \bar Q_{1_L}\, \tilde\Phi\, \overset{\psi_{u_R}}{u_R} + \text{h.c.},$$

with $\tilde\Phi \equiv i\tau^2\Phi^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\Phi^* = \begin{pmatrix} \phi^{0*} \\ -\phi^{+*} \end{pmatrix} \equiv \begin{pmatrix} \phi^{0*} \\ -\phi^- \end{pmatrix} \xrightarrow[\text{gauge}]{\text{unitary}} \frac{1}{\sqrt{2}}\begin{pmatrix} v+h \\ 0 \end{pmatrix}$,

$I(\tilde\Phi) = \tfrac{1}{2}$, $Y(\tilde\Phi) = -1 \Rightarrow \tilde\Phi$ has the same $SU(2)_L$ and opposite $U(1)_Y$ transformation properties as Φ,

$f_e, f_d, f_{\nu_e}, f_u \in \mathbb{R}$ (called Yukawa couplings).

f_{ν_e}: $= 0$ in the "standard" version of the Standard Model

The fermion mass terms now follow directly from the non-zero vacuum expectation values $\langle\Phi\rangle_0 = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}$ and $\langle\tilde\Phi\rangle_0 = \begin{pmatrix} v/\sqrt{2} \\ 0 \end{pmatrix} \Rightarrow m_i = v f_i/\sqrt{2}$ for $i = e, d, \nu_e, u$.

$\langle\Phi\rangle_0$: <u>mass lower doublet comp.</u> $\langle\tilde\Phi\rangle_0$: <u>mass upper doublet comp.</u>

The mass generation for the other fermion species proceeds in a similar way.

<u>Gauge eigenstates need not be identical to mass eigenstates</u>: eigenstates with the same quantum numbers can mix, as observed in the weak decays

$\underset{(udd)}{n} \to \underset{(uud)}{p}\, e^-\bar\nu_e$ as well as $\underset{(uds)}{\Lambda} \to \underset{(uud)}{p}\, e^-\bar\nu_e$!

($d \to u$ transition) ($s \to u$ transition)

Hence, we need a more general version of the Yukawa sector in the actual Standard Model.

Introduce therefore several generations of leptons and quarks, and indicate the gauge eigenstates (which feature in/are produced by weak interactions) as

generation label

$\nu'_{A_R} = (\nu'_{e_R}, \nu'_{\mu_R}, \dots)$, $e'_{A_R} = (e'_R, \mu'_R, \dots)$, $u'_{A_R} = (u'_R, c'_R, \dots)$, $d'_{A_R} = (d'_R, s'_R, \dots)$,

gauge singlets

$L'_{A_L} = \begin{pmatrix} \nu'_{A_L} \\ e'_{A_L} \end{pmatrix}$, $Q'_{A_L} = \begin{pmatrix} u'_{A_L} \\ d'_{A_L} \end{pmatrix}$ ← gauge doublets

The matter part of the Lagr., including electroweak covariant derivatives, then reads

$$\mathcal{L}_F = \bar{L}'_{A_L} i\gamma^\mu [\partial_\mu + \tfrac{i}{2} g \vec{\tau}\cdot\vec{W}_\mu - \tfrac{i}{2} g' B_\mu] L'_{A_L} + \bar{e}'_{A_R} i\gamma^\mu [\partial_\mu - i g' B_\mu] e'_{A_R} + \dots$$

Furthermore we allow for Yukawa int. with non-diagonal complex couplings:

$$\mathcal{L}_{Yu} = -(f_e)_{AB} \bar{L}'_{A_L} \Phi e'_{B_R} - (f_d)_{AB} \bar{Q}'_{A_L} \Phi d'_{B_R} - (f_{\nu_e})_{AB} \bar{L}'_{A_L} \tilde{\Phi} \nu'_{B_R} - (f_u)_{AB} \bar{Q}'_{A_L} \tilde{\Phi} u'_{B_R} + h.c.,$$

with $(f_i)_{AB}$ $(i = e, d, \nu_e, u)$ complex $n \times n$ matrices in family space with n generations of leptons and quarks. This results in mass terms

$$-\frac{v}{\sqrt{2}} \left[(f_e)_{AB} \bar{e}'_{A_L} e'_{B_R} + (f_d)_{AB} \bar{d}'_{A_L} d'_{B_R} + (f_{\nu_e})_{AB} \bar{\nu}'_{A_L} \nu'_{B_R} + (f_u)_{AB} \bar{u}'_{A_L} u'_{B_R} \right] + h.c.,$$

involving mass matrices $\boxed{M^{(i)}_{AB} = \frac{v}{\sqrt{2}} (f_i)_{AB} \quad (i = e, d, \nu_e, u)}$ for the gauge eigenstates!

Next we want to bring this into diagonal form with positive entries on the diagonal, thereby switching to the mass eigenstates. For an arbitrary complex matrix $M^{(i)}$ we know from linear algebra that

$$\underset{S_i, T_i \text{ unitary}}{\exists} \quad S_i^\dagger M^{(i)} T_i = M^{(i)}_{diag} = \text{diagonal matrix with } (M^{(i)}_{diag})_{AA} \geq 0.$$

Consequence: gauge eigenstates contain a mixture of mass eigenstates

$$\left.\begin{aligned} u'_{A_L} &= (S_u)_{AB}\, u_{B_L} \quad \text{etc.} \\ u'_{A_R} &= (T_u)_{AB}\, \underset{\uparrow \text{ mass eigenstates}}{u_{B_R}} \quad \text{etc.} \end{aligned}\right\} \Rightarrow \bar{u}'_{A_L} M^{(u)}_{AB} u'_{B_R} = \bar{u}_{A_L} (S_u^\dagger M^{(u)} T_u)_{AB} u_{B_R} = \bar{u}_{A_L} \underset{\uparrow \text{ properly diagonalized}}{(M^{(u)}_{diag})_{AB}} u_{B_R} \quad \text{etc.}$$

Next we check whether the electroweak int. can mix the generations (flavours).

Flavour-changing mixing effects: we start at lowest-order level

- No lowest order flavour-changing neutral currents (FCNC):

$$\bar{u}'_{A_L}\gamma_\mu u'_{A_L} = \bar{u}_{A_L}\gamma_\mu \underbrace{(S_u^\dagger S_u)}_{I}{}_{AB}\, u_{B_L} = \bar{u}_{A_L}\gamma_\mu u_{A_L}$$ etc., and similarly for R modes.

- Mixing does occur in the charged-current (CC) interactions:

$$\bar{u}'_{A_L}\gamma_\mu d'_{A_L} = \bar{u}_{A_L}\gamma_\mu (S_u^\dagger S_d)_{AB}\, d_{B_L} \equiv \bar{u}_{A_L}\gamma_\mu (V_{CKM})_{AB}\, d_{B_L},$$

V_{CKM} = unitary quark-mixing matrix (Cabibbo-Kobayashi-Maskawa, 1973) ↳ Nobel Prize 2008

$$\equiv \begin{pmatrix} V_{ud} & V_{us} & \cdots \\ V_{cd} & V_{cs} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix},$$ which can be assigned to the down-type quarks.

A similar thing can be done in the lepton sector, although that is not seen as a standard procedure in many texts on the Standard Model. We will come back to this in lecture 12.

How many independent parameters will determine V_{CKM}, bearing in mind that some phases can be absorbed into a redefinition of the quark fields?

Concept: $\psi \to e^{i\varphi}\psi \Rightarrow \pi_\psi = i\psi^\dagger \to e^{-i\varphi}\pi_\psi$ and therefore all fundamental anticommutation relations stay the same $\Rightarrow$ physics unaltered!

In expressions like $\bar{u}_{A_L}\gamma_\mu (V_{CKM})_{AB}\, d_{B_L}$ one phase can be absorbed for each quark flavour, with the exception of an overall phase. Assuming V_{CKM} to be a complex $n\times n$ matrix, the d.o.f. counting goes as follows:

$2n^2$ d.o.f. $\xrightarrow{V^\dagger V = I}$ $2n^2 - n^2 = n^2$ d.o.f. $\xrightarrow[\text{quark phases}]{\text{absorb unphysical}}$ $n^2 - (2n-1) = (n-1)^2$ d.o.f., out of which $\binom{n}{2}$ d.o.f. correspond to real n-dim. rotations (i.e. angles) and $(n-1)^2 - \binom{n}{2} = \frac{1}{2}(n-1)(n-2)$ d.o.f. correspond to independent physical phases.

Hence:
- for $n=2$ one rotation (Cabibbo) angle determines V_{CKM}, which is effectively real since all phases are absorbable;
- for $n=3$ three rotation angles and one unabsorbable phase determine V_{CKM}, opening up the possibility that $V_{CKM} \notin \mathbb{R}$

$\Rightarrow$ at least three generations are needed for having $V_{CKM} \notin \mathbb{R}$ in the CC int.!

The experimental results on V_{CKM} can be summarized in the Wolfenstein parametrization:

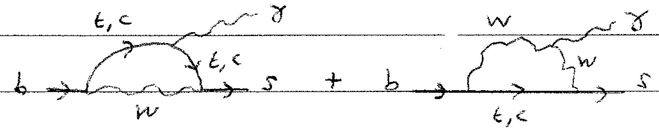
good approximation

$$V_{CKM} \approx \begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3(\rho-i\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix},$$

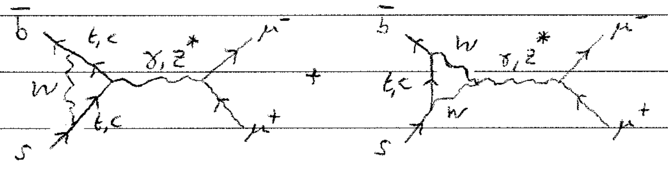
with $\lambda = 0.226 \pm 0.001$, $A = 0.81 \pm 0.02$, $\rho = 0.14 \pm 0.02$, $\eta = 0.35 \pm 0.02$.

The mixing in the quark sector gives rise to some interesting phenomena.

*) FCNC occur in higher loop order, inducing rare decays such as $b \to s\gamma$ and $B_s^0 \to \mu^+\mu^-$:

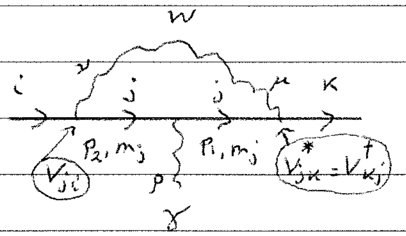


$\leftarrow b \to s\gamma$



$\leftarrow B_s^0 \to \mu^+\mu^-$

The associated amplitudes are small due to the GIM mechanism (Glashow-Iliopoulos-Maiani, 1970):



$$\propto \sum_j V^\dagger_{kj} V_{ji}\, \bar{u}_k \gamma_\mu (1-\gamma^5) \frac{\not p_1 + m_j}{p_1^2 - m_j^2} \gamma_\rho \frac{\not p_2 + m_j}{p_2^2 - m_j^2} \gamma_\nu (1-\gamma^5) u_i$$

$$= \sum_j \frac{2 V^\dagger_{kj} V_{ji}}{(p_1^2 - m_j^2)(p_2^2 - m_j^2)}\, \bar{u}_k \gamma_\mu \left[\not p_1 \gamma_\rho \not p_2 + m_j^2 \gamma_\rho \right] \gamma_\nu (1-\gamma^5) u_i$$

needed: additional j dependence in order to avoid that $\sum_j V^\dagger_{kj} V_{ji} = \delta_{ki}$ (i.e. no FCNC)

$\Rightarrow$ suppression factors $V^\dagger_{kj} V_{ji}$ for $k \neq i$, $\frac{g^2}{m_W^2} m_j^2$ if j light, and $\frac{g^2}{m_j^2} m_i^2$ if $j = t$.

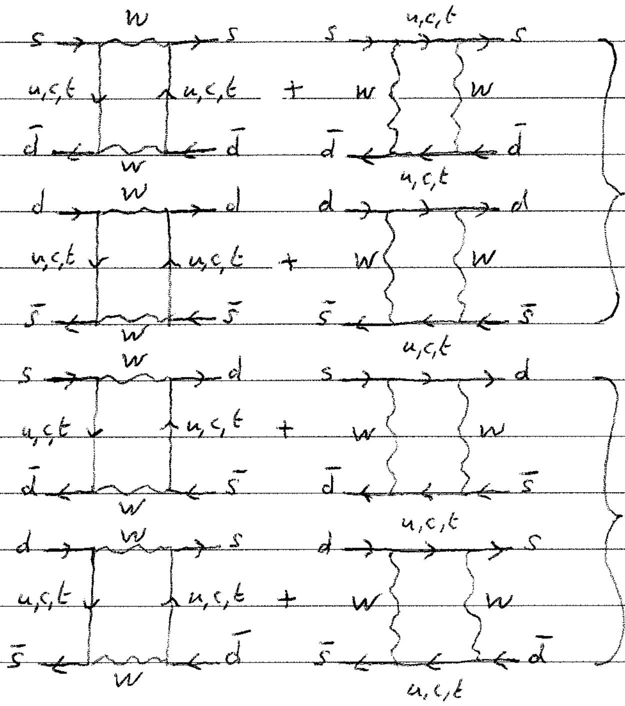
Such FCNC processes put severe constraints on physics beyond the SM: models beyond the SM might give rise to additional generational mixing, involving new (non-SM) particles that feature in the FCNC loop diagrams

$\Rightarrow$ constraints on the associated mixing matrices and/or the masses of the new particles of the model, in order not to exceed the amount of FCNC effects that we observe in experiment!

*) <u>Meson mixing effects + CP violation</u>: let's consider the mixing effects in the $\bar{K}^0 - K^0$ (kaon) system, which give rise to energy eigenstates $K_{S/L}$ (short/long lived).
$\bar{K}^0 = (s\bar{d})$, $K^0 = (d\bar{s})$

Under a CP transf. $K^0 \overset{CP}{\longleftrightarrow} \bar{K}^0$. Imposing <u>CP symmetry</u> would imply $[\hat{CP}, \hat{H}] = 0 \Rightarrow \exists$ simultaneous set of eigenfunctions of $\hat{H}$ and $\hat{CP}$.

Consequence: <u>K_S = CP-even</u> $\Rightarrow$ <u>can decay into CP-even $\pi^0\pi^0, \pi^+\pi^-$ final states</u>,
<u>K_L = CP-odd</u> $\Rightarrow$ <u>cannot decay into CP-even $\pi^0\pi^0, \pi^+\pi^-$ final states</u>, the CP-odd $\pi^0\pi^0\pi^0, \pi^0\pi^+\pi^-$ final states are <u>possible</u>!



$\boxed{\bar{K}^0 \to \bar{K}^0}$:
$\boxed{K^0 \to K^0}$:

matrix elements (e.g. $|V_{us}V_{ud}^*|^2$) are real and equal to each other (e.g. $|V_{ud}V_{us}^*|^2$)

$\boxed{\bar{K}^0 \to K^0}$:
$\boxed{K^0 \to \bar{K}^0}$:

matrix elements (e.g. $(V_{us}V_{ud}^*)^2$) potentially non-real due to V_{td}, c.c. of each other (e.g. $(V_{ud}V_{us}^*)^2$)

The mixing matrix $\begin{pmatrix} A & B \\ B^* & A \end{pmatrix} = \begin{pmatrix} A & |B|e^{i\varphi_B} \\ |B|e^{-i\varphi_B} & A \end{pmatrix}$ ($A \in \mathbb{R}$) has eigenvalues $A \pm |B|$ and

eigenvectors $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm e^{-i\varphi_B} \end{pmatrix} = \frac{1}{\sqrt{2}}\left(|\bar{K}^0\rangle \pm e^{-i\varphi_B}|K^0\rangle\right)$.

<u>Scenario 1</u>: <u>$B \in \mathbb{R}$</u>, $\varphi_B = 0 \Rightarrow$ eigenstates $K_S = \underbrace{\tfrac{1}{\sqrt{2}}(|\bar{K}^0\rangle + |K^0\rangle)}_{\text{CP-even}} \equiv K_+$, $K_L = \underbrace{\tfrac{1}{\sqrt{2}}(|\bar{K}^0\rangle - |K^0\rangle)}_{\text{CP-odd}} \equiv K_-$
$\Rightarrow$ <u>CP symmetry, $K_L \not\to \pi^0\pi^0, \pi^+\pi^-$</u>.

<u>Scenario 2</u>: <u>$B \notin \mathbb{R}$</u>, $\varphi_B \neq 0$ (small) $\Rightarrow$ eigenstates $K_S \approx (1 - \frac{i}{2}\varphi_B)K_+ + \frac{i}{2}\varphi_B K_-$, (almost CP-even)
$K_L \approx \frac{i}{2}\varphi_B K_+ + (1 - \frac{i}{2}\varphi_B)K_-$ (almost CP-odd)
$\Rightarrow$ <u>no CP symmetry</u>, <u>suppressed $K_L \to \pi^0\pi^0, \pi^+\pi^-$ allowed</u>.

<u>Scenario 2 observed in 1964</u>! Consequence: $V_{CKM} \notin \mathbb{R} \overset{\text{p. 31}}{\Longrightarrow}$ <u>the SM should involve at least three generations of quarks</u>!