Master course "The Standard Model and Beyond"

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Lecture 1: Lagrange formalism and Noether theorem

Conventions: in the following we will use so-called natural units ($\hbar = c = \mu_0 = \epsilon_0 = 1$) by absorbing these constants in the relevant fields/quantities ⇒ a single scale remains: mass.

For example: $E \to E * 1/c^2$ (cf $mc^2 \to m$),
$p \to p * 1/c$ (cf $mc \to m$),
$t \to t * c^2/\hbar$ (cf $\lambda_{compton}/c = \hbar/mc^2 \to 1/m$),
$r \to r * c/\hbar$ (cf $\lambda_{compton} = \hbar/mc \to 1/m$).

We will be dealing with a flat spacetime. For the flat spacetime metric we will use the signature $g_{\mu\nu} = diag(+1,-1,-1,-1)$, with the Minkowski indices μ, ν running from 0 to 3. Repeated indices (of any kind) are implicitly summed over, unless stated otherwise.

In case you need to refresh your knowledge concerning the use of 4-vectors in Special Relativity and Lorentz transformations to switch between inertial systems, please read the Special Relativity appendix.

Classical physics: experiment has taught us that there should be no "action at a distance" ⇒ forces should not be felt everywhere instantaneously.
Consequence: the instantaneous laws of Newton and Coulomb had to be replaced by the local laws of nature of Einstein and Maxwell, based on field theories!

Guided by this requirement of having local laws of nature, particles and their mutual interactions will be described by continuous (quantum) fields. These fields satisfy classical equations of motion, which can be best formulated in terms of Lagrangians for continuous systems. Such Lagrangians are particularly suitable for dealing with relativistic theories and for discussing symmetries, the cornerstones of the Standard Model and its extensions.

Lagrange formalism: going from discrete to continuous.

Discrete case: for a finite number of degrees of freedom (d.o.f.) the Lagrangian reads $L(\{q_j(t)\}, \{\underbrace{\dot{q}_j(t)}_{dq_j(t)/dt}\}, t) = T - V$,

with q_j the generalized coordinates, T the kinetic energy and V the potential energy.

Hamilton's variation principle: classical solutions to the equations of motion (called classical paths) are obtained by finding the extrema of the action $S = \int_{t_1}^{t_2} dt\, L$ under synchronous variations of the paths while keeping the endpoints fixed.

"evolution between fixed coordinate configurations"

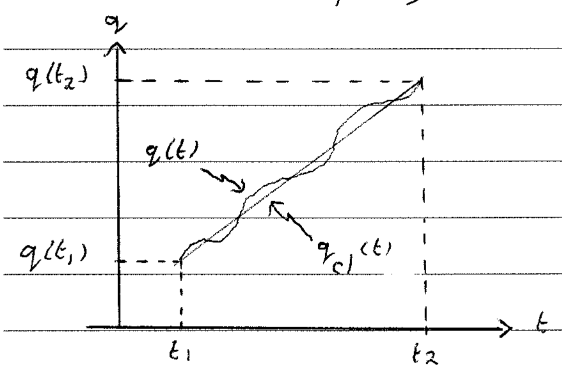

Illustration: variation around the classical path for a free particle.

Proof: the condition for a stationary action reads

$$\delta S = \delta\Big(\int_{t_1}^{t_2} dt\, L\Big) = 0 \quad \text{for } q_j(t) \to q_j(t) + \delta q_j(t) \text{ such that } \delta q_j(t_{1,2}) = 0$$

$$\Rightarrow \forall_{\delta q_j} \int_{t_1}^{t_2} dt \Big(\frac{\partial L}{\partial q_j}\delta q_j + \frac{\partial L}{\partial \dot{q}_j}\overbrace{(\delta \dot{q}_j)}^{\frac{d}{dt}\delta q_j}\Big) = \cancel{\Big[\frac{\partial L}{\partial \dot{q}_j}\delta q_j\Big]_{t=t_1}^{t=t_2}} + \int_{t_1}^{t_2} dt \Big(\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}\Big)\delta q_j = 0$$

summation implied

$\Rightarrow$ Lagrange equations of motion $\forall_j \; \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}_j}\Big) = \frac{\partial L}{\partial q_j}$

for a system without boundary conditions.

Continuous case: now we switch from a discrete set of particles and associated coordinates to a set of fields $\{F_K(x)\}$, with the argument x referring to the 4-vector $x^\mu = (t, \vec{x})$. In fact the discrete label j is replaced here by both the field label κ and the continuous spatial variable $\vec{x}'$ $\Rightarrow$ we are dealing with an infinite number of degrees of freedom! For a proper relativistic treatment we need to treat t and $\vec{x}$ on equal footing. This is manifest in the use of fields $F_\kappa(x)$, but this disqualifies the use of "velocities" $\partial F_\kappa/\partial t$ without simultaneously adding the spatial gradients $\vec{\nabla} F_\kappa = \partial F_\kappa/\partial \vec{x}$ $\Rightarrow$ use $F_\kappa(x)$, $\underbrace{\frac{\partial F_\kappa(x)}{\partial x^\mu} \equiv \partial_\mu F_\kappa(x)}_{\text{4-velocity}}$.

Hence, $L(\{q_j(t)\}, \{\dot{q}_j(t)\}, t) \to \int d\vec{x}\, \mathcal{L}(\{F_\kappa(x)\}, \{\partial_\mu F_\kappa(x)\})$, with $\mathcal{L}$ referred to as the Lagrangian density (or sloppily "the Lagrangian").

Deriving the equations of motion from the action $S = \int_{t_1}^{t_2} dt \int d\vec{x}\, \mathcal{L}$ $\left(= \int_{x_1}^{x_2} d^4x\, \mathcal{L}\right)$: using a generalized version of Hamilton's variation principle, the equations of motion are obtained by finding the extrema of S under the continuous synchronous field variations $\leftarrow$ "evolution between fixed field configurations at $t_{1,2}$ and $|\vec{x}| \to \infty$"

$$F_\kappa(x) \to F_\kappa(x) + \delta F_\kappa(x) \quad \text{such that} \quad \forall_{\vec{x}}\ \delta F_\kappa(t_{1,2}, \vec{x}) = 0 \quad \text{and} \quad \delta F_\kappa(x) \xrightarrow{|\vec{x}| \to \infty} 0$$

$$\Rightarrow \forall_{\delta F_\kappa} \quad \delta S = \int_{x_1}^{x_2} d^4x \left\{ \frac{\partial \mathcal{L}}{\partial F_\kappa} \delta F_\kappa + \frac{\partial \mathcal{L}}{\partial(\partial_\mu F_\kappa)} \overbrace{\delta(\partial_\mu F_\kappa)}^{\partial_\mu(\delta F_\kappa)} \right\}$$

$$= \cancel{\underbrace{\int_{x_1}^{x_2} d^4x\, \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu F_\kappa)} \delta F_\kappa\right)}_{\text{Gauss}}} + \int_{x_1}^{x_2} d^4x \left\{ \frac{\partial \mathcal{L}}{\partial F_\kappa} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu F_\kappa)}\right) \right\} \delta F_\kappa = 0$$

$\Rightarrow$ Euler-Lagrange equations: $\boxed{\forall_\kappa \ \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu F_\kappa)}\right) = \frac{\partial \mathcal{L}}{\partial F_\kappa}}$.

Example: complex free Klein-Gordon field $\phi(x)$

$$\hookrightarrow \mathcal{L}(\phi, \phi^*, \partial_\mu\phi, \partial_\mu\phi^*) = \underbrace{(\partial_\mu\phi)(\partial^\mu\phi^*)}_{\text{kinetic + elastic terms}} - \underbrace{m^2\phi\phi^*}_{\text{mass term}},$$

with Euler-Lagrange equations $\partial_\mu(\partial^\mu\phi^*) + m^2\phi^* \equiv (\Box + m^2)\phi^* = 0$ and $\partial_\mu(\partial^\mu\phi) + m^2\phi \equiv (\Box + m^2)\phi = 0$. The independent fields ϕ and ϕ^* satisfy Klein-Gordon equations, allowing for an expansion in terms of plane waves $e^{-ip\cdot x}$ with $p\cdot p \equiv p^2 = m^2$

$\uparrow$ particle-wave duality $\rightarrow$ free particles in relativistic quantum mechanics $[p^\mu = (E_{\vec{p}} = \sqrt{\vec{p}^{\,2} + m^2}, \vec{p})]$.

Remarks

1) switching from the Lagrangian to the Hamiltonian involves replacing the field "velocities" $\partial F_K/\partial t$ by the associated conjugate momenta $\Pi_K \equiv \partial \mathcal{L}/\partial(\partial F_K/\partial t)$. This implies that
   - the Hamiltonian is crucial for setting up the quantum version of field theory (QFT),
   - t and $\vec{x}$ are not treated on equal footing anymore, making the Hamiltonian less suitable for dealing with relativistic theories.

2) Adding a F_K-dependent 4-divergence to the Lagrangian, $\mathcal{L} \to \mathcal{L} + \partial_\mu G^\mu(\{F_K\})$, leaves the equations of motion invariant. This is a direct consequence of the fact that this term merely adds a boundary contribution to S. Such a boundary contribution remains unaffected by a field variation with fixed boundaries. This aspect is crucial for the topic of symmetries.

3) The relativity principle states that in each inertial frame the physics should be same. This implies that Lorentz transformed solutions to the Euler-Lagrange equations should be solutions as well. To this end we will use Lorentz-scalar Lagrangian densities in relativistic field theories, since the action will in that case be Lorentz invariant and therefore an extremum of the action will indeed yield another extremum upon Lorentz transformation.

short: $x \to x' = \Lambda x$ — describes rotations and boosts

Proof: consider the Lorentz transformation $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$ and a Lorentz-scalar Lagrangian density $\mathcal{L}(x) \to \mathcal{L}'(x) = \mathcal{L}(\Lambda^{-1}x) \equiv \mathcal{L}(y)$

$$\Rightarrow \quad S = \int d^4x\, \mathcal{L}(x) \to S' \equiv \int d^4x\, \mathcal{L}'(x) = \int d^4x\, \mathcal{L}(y)$$

$$\xrightarrow[\text{Jacobian}=1]{x = \Lambda y} \int d^4y\, \mathcal{L}(y) = S,$$

i.e. the action is indeed invariant.

This immediately explains why the covariant 4-velocity $\partial_\mu \phi$ had to be multiplied by another (contravariant) 4-vector to yield a scalar Lagrangian density for the complex Klein-Gordon theory. The Klein-Gordon field $\phi(x)$ should be scalar as well, of course.

<u>Noether's theorem for continuous symmetries</u>: consider a set of fields $\{F_k(x)\}$ that satisfies the Euler-Lagrange equations of $\mathcal{L}(\{F_k\}, \{\partial_\mu F_k\})$ and apply the infinitesimal continuous transformation

$$F_k(x) \to F_k'(x) = F_k(x) + \vartheta \Delta F_k(x) \qquad (\vartheta \text{ independent of } x \text{, infinitesimal}).$$

We speak of a symmetry under this transformation if $\mathcal{L}(x)$ changes by a 4-divergence, which leaves the equations of motion invariant. In that case

$$\vartheta \partial_\mu G^\mu \equiv \vartheta \Delta \mathcal{L} = \vartheta \left(\frac{\partial \mathcal{L}}{\partial F_k} \Delta F_k + \frac{\partial \mathcal{L}}{\partial(\partial_\mu F_k)} \overbrace{\Delta(\partial_\mu F_k)}^{\partial_\mu \Delta F_k} \right)$$

$$= \vartheta \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu F_k)} \Delta F_k \right) + \vartheta \underbrace{\left[\frac{\partial \mathcal{L}}{\partial F_k} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu F_k)} \right) \right]}_{\text{E.-L. eqns.}:\, 0} \Delta F_k$$

$$\Rightarrow \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu F_k)} \Delta F_k - G^\mu \right) \equiv \partial_\mu j^\mu = 0 \,,$$

i.e. j^μ is a <u>conserved Noether current</u> and $\int\limits_V d\vec{x}\, j^0(x)$ is a <u>locally conserved "charge"</u> since $\frac{d}{dt} \int\limits_V d\vec{x}\, j^0 = - \int\limits_V d\vec{x}\, \vec{\nabla} \cdot \vec{j} \overset{\text{Gauss}}{=} - \int\limits_{S(V)} d\vec{s} \cdot \vec{j}$.

This forms the basis for the charge conservation laws that will feature prominently in a symmetry-based description of the electromagnetic, strong and weak interactions (and any interactions beyond the Standard Model). Noether's theorem can also be used to derive conserved quantities such as 4-momentum (translation symmetry) and angular momentum (rotation symmetry), which feature prominently in the quantum mechanical treatment of field theories and the corresponding particle interpretation.

Examples: *) Complex Klein-Gordon theory (spin 0): $\mathcal{L}(\phi, \phi^*, \partial_\mu \phi, \partial_\mu \phi^*) = (\partial_\mu \phi)(\partial^\mu \phi^*) - m^2 \phi \phi^*$.

This Lagrangian is invariant under the continuous phase transformation

$$\phi \to \phi' = e^{i\vartheta} \phi \overset{\text{inf.}}{\approx} \phi + \vartheta \underbrace{(i\phi)}_{\Delta\phi} \,, \quad \phi^* \to \phi'^* = e^{-i\vartheta} \phi^* \overset{\text{inf.}}{\approx} \phi^* + \vartheta \underbrace{(-i\phi^*)}_{\Delta\phi^*}$$

$$\Rightarrow \Delta \mathcal{L} = 0, \text{ i.e. } G^\mu = 0, \text{ and } j^\mu = i\phi \partial^\mu \phi^* - i\phi^* \partial^\mu \phi \text{ is conserved.}$$

spin 1/2

*) Free Dirac theory (see exercise): $\mathcal{L}(\psi, \bar{\psi}, \partial_\mu \psi) = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$, with $\psi(x)$ a 4-dimensional Dirac spinor, $\bar{\psi}(x) = \psi^\dagger(x)\gamma^0$ and γ^μ $(\mu = 0, \ldots, 3)$ the Dirac γ-matrices. Phase symmetry this time leads to the conserved current $j_V^\mu = -\bar{\psi}\gamma^\mu\psi$.

also: Abelian

These two phase transformations are referred to as "global U(1) gauge transformations" [global: φ independent of x; U(1): the same $e^{i\varphi}$ for each component of the field]. Later on we will also encounter a generalization of this: global unitary transformations that mix different fields of the same type (such as 2 or 3 Dirac fields), referred to as non-Abelian gauge transf.

weak int. strong int.

Canonical quantization: the fields $F_j(x)$ and associated conjugate momenta $\pi_k(x) = \partial\mathcal{L}/\partial(\partial_0 F_k)$ become operators that satisfy canonical commutation/anticommutation relations for bosonic/fermionic fields F_j.

- Bosonic case, fields $\phi_j(x)$: $[\hat{\phi}_j(t,\vec{x}), \hat{\pi}_k(t,\vec{y})] = i\delta_{jk}\,\delta(\vec{x}-\vec{y})\,\hat{1}$,
  $[\hat{\phi}_j(t,\vec{x}), \hat{\phi}_k(t,\vec{y})] = [\hat{\pi}_j(t,\vec{x}), \hat{\pi}_k(t,\vec{y})] = 0$,
  cf. $[\hat{q}_j, \hat{p}_k] = i\delta_{jk}\hat{1}$, etc. — equal-time commutation relations.
- Fermionic case, fields $\hat{\psi}_{j,\alpha}(x)$ (spinor label α): $\{\hat{\psi}_{j,\alpha}(t,\vec{x}), \hat{\pi}_{k,\beta}(t,\vec{y})\} = i\delta_{jk}\,\delta_{\alpha\beta}\,\delta(\vec{x}-\vec{y})\,\hat{1}$,
  $\{\hat{\psi}_{j,\alpha}(t,\vec{x}), \hat{\psi}_{k,\beta}(t,\vec{y})\} = \{\hat{\pi}_{j,\alpha}(t,\vec{x}), \hat{\pi}_{k,\beta}(t,\vec{y})\} = 0$,
  equal-time anticommutation relations.

This can also be formulated in terms of creation and annihilation operators, from which the particle interpretation of the fields follow: each field annihilates a certain type of "particle" and creates the corresponding type of "antiparticle".

Conjugate momenta: *) Free complex Klein-Gordon theory: $\dfrac{\partial\mathcal{L}}{\partial(\partial_0\phi)} = \partial_0\phi^* \equiv \pi$, $\dfrac{\partial\mathcal{L}}{\partial(\partial_0\phi^*)} = \partial_0\phi \equiv \pi^*$.

2 real d.o.f. $\rightarrow$

*) Free Dirac theory: $\dfrac{\partial\mathcal{L}}{\partial(\partial_0\psi)} = i\bar{\psi}\gamma^0 = i\psi^\dagger \equiv \pi_\psi$, $\dfrac{\partial\mathcal{L}}{\partial(\partial_0\bar{\psi})} = 0$

$\Rightarrow$ out of the 8 real d.o.f. of the Dirac spinor in fact 4 belong to the conjugate momentum!

effectively 4 real d.o.f. $\rightarrow$