Lecture 2: Local gauge invariance and Feynman rules.

Quantum Electrodynamics (QED) and the gauge principle

Consider the electromagnetic field in "vacuum" with charge density $\rho(x) = j_c^0(x) \in \mathbb{R}$ and current density $\vec{j}_c(x) \in \mathbb{R}^3$: $A^\mu(x) = (A^0(x), \vec{A}(x))$

4-vector potential ; scalar potential ; vector potential

This e.m. field satisfies the electromagnetic wave equation $\partial_\mu F^{\mu\nu}(x) = j_c^\nu(x)$, with $F^{\mu\nu}(x) = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x)$ the antisymmetric electromagnetic field tensor. This wave equation summarizes the Maxwell equations in 4-vector language.

Gauge freedom: the physics described by the e.m. wave equation is unaffected by the following transformation of the e.m. field

$A^\mu(x) \to A'^\mu(x) = A^\mu(x) + \partial^\mu \chi(x)$ ($\chi(x) \in \mathbb{R}$ arbitrary scalar fct.).

e.m. gauge transformation ; but sufficiently smooth

Proof: $F^{\mu\nu}(x) \to F'^{\mu\nu}(x) = \partial^\mu A'^\nu(x) - \partial^\nu A'^\mu(x) = F^{\mu\nu}(x) + (\partial^\mu\partial^\nu - \partial^\nu\partial^\mu)\chi(x) = F^{\mu\nu}(x)$.

The associated freedom to choose the e.m. field is called the gauge freedom.

Local charge conservation: $\partial_\nu j_c^\nu(x) = \partial_\nu \partial_\mu F^{\mu\nu}(x) = 0$, since $F^{\mu\nu} = -F^{\nu\mu}$.
Hence, the current density $j_c^\mu(x)$ is a conserved current and the electric charge $\int_V d\vec{x}\, j_c^0(x) = \int_V d\vec{x}\, \rho(x)$ is conserved locally. Just as we have seen for Noether currents.

Electromagnetic Lagrangian: the Lagr. density belonging to the e.m. wave equation is given by $\mathcal{L}_{e.m.}(x) = -\frac{1}{4} F^{\mu\nu}(x) F_{\mu\nu}(x) - j_c^\mu(x) A_\mu(x)$.

Proof: $\frac{\partial \mathcal{L}_{e.m.}}{\partial(\partial_\mu A_\nu)} = -\frac{1}{2}\left(\frac{\partial}{\partial(\partial_\mu A_\nu)} F_{\rho\sigma}\right) F^{\rho\sigma} = -F^{\mu\nu} \Rightarrow$ E.-L. eqn.: $-\partial_\mu F^{\mu\nu}(x) = -j_c^\nu(x)$.

We see that the e.m. field couples to the conserved current density caused by matter. The charged particles that matter is comprised of are all of the Dirac type (electrons, up quarks, down quarks). For a Dirac-type particle with charge q the e.m. current density is given by the conserved current $j^\nu_{c,\text{Dirac}}(x) = q\bar{\psi}(x)\gamma^\nu\psi(x)$.

see Ex. 2

In that case the complete Lagrangian density takes the form

$$\mathcal{L}_{QED}(x) = \overbrace{\bar\psi(x)\,(i\partial_\mu\gamma^\mu - m)\,\psi(x)}^{\text{free Dirac Lagr.}} \overbrace{- \tfrac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x)}^{\text{free e.m. Lagr.}} \overbrace{- q\,\bar\psi(x)\gamma^\mu\psi(x)A_\mu(x)}^{\text{interaction}}$$

$$= \bar\psi(x)\,(iD_\mu\gamma^\mu - m)\,\psi(x) - \tfrac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) = \mathcal{L}_{QED}(x),$$

with $D_\mu = \partial_\mu + iqA_\mu \leftarrow$ (cf. $-i\hat{p}_\mu + iqA_\mu = -i(\hat{p}_\mu - qA_\mu)$ in quantum mechanics)

This last step we recognize as the concept of minimal substitution: the interaction between matter particles and e.m. fields is obtained by inserting $p^\mu \to p^\mu - qA^\mu$ in the free Lagrangian density.

QED from a symmetry principle (gauge invariance and the gauge principle): let's turn around the argument and alternatively start with the Dirac Lagr. $\mathcal{L}(x) = i\bar\psi(x)\gamma^\mu\partial_\mu\psi(x) - m\bar\psi(x)\psi(x)$. As we have seen in Ex. 2, this Lagr. is invariant under the global $U(1)$ gauge transformation $\psi(x) \to e^{i\vartheta}\psi(x)$, $\bar\psi(x) \to e^{-i\vartheta}\bar\psi(x)$ ($\vartheta \in \mathbb{R}$, independent of x). According to Noether's theorem this global gauge symmetry can be associated with a conserved current and charge, precisely what we need for e.m. interactions.

Non-relativistic quantum mechanics: this global gauge invariance of a free-fermion system simply underlines the unobservability of the absolute phase of a wave function (i.e. only relative phases are observable through interference).

Relativistic quantum field theory (postulate): demand this to hold locally, since relativistic quantum mechanics should be a local theory. As charges interact while being at non-zero distance, the local interactions should be mediated by force carriers.

$\Rightarrow$ The gauge principle: demand the gauge invariance to hold locally!

Local $U(1)$ gauge transformation: $\psi(x) \to e^{i\vartheta(x)}\psi(x)$, $\bar\psi(x) \to e^{-i\vartheta(x)}\bar\psi(x)$, with $\vartheta(x)$ a real scalar field

$$\Rightarrow i\bar\psi(x)\gamma^\mu\partial_\mu\psi(x) \to i\bar\psi(x)\gamma^\mu\partial_\mu\psi(x) - \underbrace{\bar\psi(x)\gamma^\mu\psi(x)\overbrace{[\partial_\mu\vartheta(x)]}^{\text{covariant vector}}}_{\neq 0 \text{ in general} \Rightarrow \text{ not locally invariant}}.$$

Remedy: replace the ordinary derivative ∂_μ by a gauge covariant derivative (or short: covariant derivative) D_μ such that
$D_\mu \psi(x) \to D'_\mu \psi'(x) = e^{i\vartheta(x)} D_\mu \psi(x)$,
causing $D_\mu \psi(x)$ and $\psi(x)$ to transform similarly under local gauge transformations! This can be achieved by

gauge field

$$D_\mu \equiv \partial_\mu + i g A_\mu(x), \quad \text{with } A_\mu(x) \to A'_\mu(x) = A_\mu(x) - \frac{1}{g}\partial_\mu \vartheta(x).$$

cf. ~~minimal substitution~~ gauge coupling cf. e.m. gauge transf. with $\chi(x) = -\frac{\vartheta(x)}{g}$

Proof: $D'_\mu \psi'(x) = (\partial_\mu + i g [A_\mu(x) - \frac{1}{g}\partial_\mu \vartheta(x)])\, e^{i\vartheta(x)} \psi(x)$
$= e^{i\vartheta(x)} (\partial_\mu + \cancel{[i\partial_\mu \vartheta(x)]} + i g A_\mu(x) - \cancel{[i\partial_\mu \vartheta(x)]})\, \psi(x) = e^{i\vartheta(x)} D_\mu \psi(x)$.

The Dirac Lagr. density with $\partial_\mu \to D_\mu$ is now invariant under local gauge transf., but a gauge-invariant kinetic term for a dynamical gauge field still has to be added. To this end we define the gauge-invariant field tensor

$$i g F_{\mu\nu}(x) \equiv [D_\mu, D_\nu] = [\partial_\mu + i g A_\mu(x)][\partial_\nu + i g A_\nu(x)] - (\mu \leftrightarrow \nu)$$
$$= i g (\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x))$$

and add the term $-\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x)$ to the Lagrangian, resulting in the same Lagr. density as obtained from electromagnetism and minimal substitution. This locally invariant theory describes the fundamental e.m. interactions between matter fermions as being mediated by photons (gauge bosons):

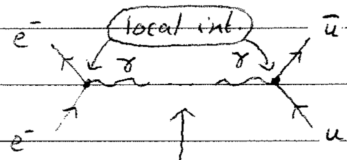
$$\mathcal{L}^{int.}_{e.m.} = -g\, \bar{\psi}(x) \gamma^\mu A_\mu(x) \psi(x) \leftarrow$$ this is called a gauge interaction

The transformation properties of $\psi(x)$, $\bar{\psi}(x)$ and $A_\mu(x)$ are collectively referred to as a "U(1) gauge transformation", where $A_\mu(x)$ does not transform in the global case.

Scaling the coupling strength: different matter particles can (and do) have different charges and therefore should have a different coupling strength to the e.m. field. That can be taken into account by scaling the transf. parameter $\vartheta(x)$ and gauge coupling g according to $\vartheta(x) \to Q\vartheta(x)$ and $g \to Qg$.

$\Rightarrow$ • the transformation property of $A_\mu(x)$ is unaltered (because of q/g), so one and the same gauge field can couple to particles of different charge and can therefore mediate the interaction.

Graphically:



with the photon coupling to conserved currents.

a single gauge field mediates the finite-distance interaction between (differently) charged particles !

• the covariant derivative changes to $D_\mu \to \partial_\mu + iQgA_\mu(x)$, thereby changing the interaction strength. Setting $g = |e|$ (unit charge) and $q = Q|e|$ (particle charge) we retrieve the e.m. interaction as given on p. 8.

Massless gauge fields: a massive gauge field would require a mass term $+\frac{1}{2} m_A^2 A_\mu(x) A^\mu(x)$ in the Lagr. density, which is not gauge invariant $\Rightarrow$ without an additional trick local gauge invariance implies $m_A = 0$!
This is fine for the e.m. field, which we know to be massless, but it will come back to haunt us once we discuss the Standard Model.

Motivated by the success of describing QED through the gauge principle, turning a global gauge symmetry (phase symmetry) into a local one, we will later on generalize this idea to other types of gauge transf. in an attempt to describe other fundamental interactions in nature.

Transition amplitudes: before addressing the Feynman rules, we first have to figure out what should be calculated to obtain the probability amplitude for the transition from a given initial state $|i\rangle$ to a final state $|f\rangle$. Here $|i\rangle$ will consist of a single particle for a decay process and two particles for a scattering reaction. Due to the negligible wave-function overlap, we will treat the initial and final states long before/after the transition (i.e. at asymptotic times $t \to t_\mp = \mp\infty$) as asymptotic free (non-interacting) situations involving free particles. In between the interaction takes place.

in QFT this has to be treated carefully (interactions cannot be switched off)