Practice exercise for Quantum Mechanics 3

Extra 1: **Polarization and elastic scattering**

Consider elastic collisions between spin-1/2 particles (such as nucleons) and spin-0 particles (such as pions). The corresponding scattering potential is in general spin dependent. This implies that the scattering amplitude becomes a scattering operator $\hat{F}$ that acts as a $2{\times}2$ matrix F on the spin-1/2 spin space. For a given pure initial spin state $|\chi^i\rangle$ of the spin-1/2 beam particles and a pure final spin state $|\chi^f\rangle$ of the scattered spin-1/2 particles, the differential cross section is given by

$$\frac{\mathrm{d}\sigma_{i\to f}}{\mathrm{d}\Omega} \;=\; |\langle\chi^f|\hat{F}|\chi^i\rangle|^2 \;.$$

Scenario 1: sometimes the polarization of the scattered spin-1/2 particles is not measured. As a result, the differential cross section can be summed over the two independent final spin states $|\chi^f\rangle = |\chi_{\frac{1}{2},\frac{1}{2}}\rangle$ *and* $|\chi^f\rangle = |\chi_{\frac{1}{2},-\frac{1}{2}}\rangle$.

(i) Suppose the incoming spin-1/2 beam is in the pure spin state $|\chi^i\rangle$. Show that the differential cross section is in that case equal to the expectation value of the operator $\hat{F}^\dagger\hat{F}$ for this spin state $|\chi^i\rangle$.

(ii) Next we switch to an incoming spin-1/2 beam with polarization vector $\vec{P}$. Demonstrate that the spin-ensemble average of the differential cross section is given by

$$\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\Omega} \;=\; \tfrac{1}{2}\,\mathrm{Tr}(F^\dagger F) \,+\, \tfrac{1}{2}\,\vec{P}\cdot\mathrm{Tr}(\vec{\sigma}\,F^\dagger F) \;.$$

Scenario 2: an interesting scattering phenomenon is that the scattering process itself may give rise to polarization, irrespective of the initial polarization of the incoming beams.

(iii) The scattering process turns a pure initial spin state $|\chi^i\rangle$ into a pure final spin state $|\chi^{\mathrm{sc}}\rangle = c\,|\hat{F}\chi^i\rangle$ for the scattered spin-1/2 particles, where c is a spin-independent complex normalization factor that takes into account that only part of the incoming particles is actually scattered. Why is $|\chi^{\mathrm{sc}}\rangle$ of this form?

(iv) Consider an incoming spin-1/2 beam with density operator

$$\hat{\rho} \;=\; \sum_i W_i\,|\chi^i\rangle\langle\chi^i| \;.$$

Derive the following density operator for the scattered spin-1/2 particles:

$$\hat{\rho}' \;=\; |c|^2\,\hat{F}\hat{\rho}\hat{F}^\dagger \;, \qquad \text{with} \qquad |c|^2 \;=\; 1/\mathrm{Tr}(\hat{F}\hat{\rho}\hat{F}^\dagger) \;.$$

(v) Finally, assume the incoming spin-1/2 beam to be unpolarized. Determine the polarization vector $\vec{P}'$ of the scattered spin-1/2 particles.