Quantum Field Theory Exercises week 11

Exercise 14 (continued)
Complete exercise 14.

Exercise 15: generic finite-dimensional representations of the Lorentz group

Do exercise 3.1 parts (a) and (b) from Peskin and Schroeder, using that

$$\epsilon_{lmn}\,\epsilon_{lm'n'} \;=\; \delta_{mm'}\,\delta_{nn'} - \delta_{mn'}\,\delta_{nm'}$$

with summation over the repeated index l implied. The real infinitesimal parameters $\vec{\theta}$ and $\vec{\beta}$ coincide with the parameters $\delta\vec{\alpha}$ and $\delta\vec{v}$ that were used in Ex. 14.

Exercise 16: trace technology for gamma-matrices

The γ-matrices γ^μ for $\mu = 0,1,2,3$ and $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ have the following properties:

1. $\{\gamma^\mu,\gamma^\nu\} \;=\; 2g^{\mu\nu}I_4 \quad\Rightarrow\quad (\gamma^0)^2 \;=\; -(\gamma^1)^2 \;=\; -(\gamma^2)^2 \;=\; -(\gamma^3)^2 \;=\; (\gamma^5)^2 \;=\; I_4$,
2. $\{\gamma^\mu,\gamma^5\} \;=\; 0$,
3. $(\gamma^\mu)^\dagger \;=\; \gamma^0\gamma^\mu\gamma^0$ and $(\gamma^5)^\dagger \;=\; \gamma^5$.

Use the first two properties to answer a few questions about trace identities.

(a) – Show that for an odd number of γ-matrices $\mathrm{Tr}(\gamma^{\mu_1}\cdots\gamma^{\mu_{2n+1}}) = 0$.
Hint: multiply the argument of the trace by $(\gamma^5)^2 = I_4$ and use one of the above-given identities as well as the cyclic property of traces.
– Why does that automatically imply that $\mathrm{Tr}(\gamma^{\mu_1}\cdots\gamma^{\mu_{2n+1}}\gamma^5) = 0$?

(b) Reason that similar tricks can be applied to prove that $\mathrm{Tr}(\gamma^5) \;=\; \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^5) \;=\; 0$.

(c) Show that $\mathrm{Tr}(\gamma^{\mu_1}\cdots\gamma^{\mu_{2n}})$ can be expressed as a sum where each term is of the form $\mathrm{Tr}\big([2n-2]\ \gamma\text{-matrices}\big) \times$ a metric tensor.

(d) Use this method to derive the following trace identities:

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) \;=\; 4g^{\mu\nu} \ ,$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) \;=\; 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) \ .$$

(e) – Argue that $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5) = 0$ if $(\mu\nu\rho\sigma) \neq$ permutation of (0123).
– Determine $\mathrm{Tr}(\gamma^0\gamma^1\gamma^2\gamma^3\gamma^5)$ and argue that $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5) = -4i\epsilon^{\mu\nu\rho\sigma}$ with

$$\epsilon^{\mu\nu\rho\sigma} \;=\; \begin{cases} +1 & \text{if } (\mu\nu\rho\sigma) \;=\; \text{even permutation of } (0123) \\ -1 & \text{if } (\mu\nu\rho\sigma) \;=\; \text{odd permutation of } (0123) \\ \ \ 0 & \text{else.} \end{cases}$$