Exercises for Quantum Mechanics 3
Set 11 (module 1)

Exercise 22: **Klein−Gordon equation versus probability interpretation**
Identifying the reason why the Klein−Gordon theory is not a suitable starting point for setting up relativistic 1-particle quantum mechanics

Consider the plane-wave solutions to the Klein−Gordon equation for a free particle with rest mass m:

$$\psi_p(x) \;=\; \exp(-ip\cdot x/\hbar)\ ,$$

where the contravariant momentum vector p^μ is defined to satisfy

$$p\cdot p \;=\; p^2 \;=\; m^2c^2 \quad\Rightarrow\quad p^0 \;=\; E/c \;=\; \pm\sqrt{m^2c^2+\vec{p}^{\,2}}\ .$$

(i) Prove that $\psi_p(x)$ is indeed a solution to the Klein−Gordon equation if the condition $p^2 = m^2c^2$ is satisfied.

(ii) Show that the "probability density"

$$\rho(x) \;=\; \frac{i\hbar}{2mc^2}\Big[\,\psi^*(x)\,\frac{\partial}{\partial t}\,\psi(x) \;-\; \psi(x)\,\frac{\partial}{\partial t}\,\psi^*(x)\Big] \;=\; \mathrm{Re}\Big[\,\frac{i\hbar}{mc^2}\,\psi^*(x)\,\frac{\partial}{\partial t}\,\psi(x)\Big]$$

takes on negative values for the plane-wave solutions with negative energy.

(iii) Interpret these negative-energy solutions as being unphysical and remove them from the quantum theory. Next consider two physically acceptable plane-wave solutions $\psi_{p_1}(x)$ and $\psi_{p_2}(x)$ with $p_1^0 > p_2^0 > 0$. Demonstrate that for a linear combination of these solutions the "probability density" $\rho(x)$ is still not automatically positive definite.

Hint: show that it is possible to construct a linear combination with real coefficients such that $\rho(x)$ can still become negative for certain values of x.

(iv) What goes wrong if we would remove such special linear combinations with $\rho(x) < 0$ from the quantum mechanical theory?

As a result of these problems with the probabilistic interpretation, it had to be concluded that the Klein−Gordon equation is not a suitable starting point for setting op relativistic 1-particle quantum mechanics.

Exercise 23: **Dirac equation ... getting to know the corresponding matrices**

In the Dirac equation the matrices β and α^j $(j = 1,2,3)$ feature. These matrices are required to satisfy the matrix identities

$$\beta^2 = I_4 \quad , \quad \beta = \beta^\dagger \quad , \quad \{\alpha^j, \beta\} = 0 \quad , \quad \{\alpha^j, \alpha^k\} = 2\delta^{jk} I_4 \qquad \text{and} \qquad \alpha^j = (\alpha^j)^\dagger \ ,$$

where the symbol I_N is used to denote the identity matrix of rank N.

(i) Show that the following set of 4×4 matrices in spinor space satisfy these matrix identities:

$$\beta = \begin{pmatrix} I_2 & \emptyset \\ \emptyset & -I_2 \end{pmatrix} \qquad \text{and} \qquad \alpha^j = \begin{pmatrix} \emptyset & \sigma^j \\ \sigma^j & \emptyset \end{pmatrix} \qquad (j = 1,2,3) \ ,$$

where each 4×4 matrix is subdivided into 2×2 blocks.

Hint: wherever necessary you may use the properties of the Pauli spin matrices σ^j that are summarized in appendix B of the lecture notes.

(ii) Next we introduce the γ-matrices $\gamma^0 \equiv \beta$ and $\vec{\gamma} \equiv \beta\vec{\alpha}$. Use the matrix identities for β and $\vec{\alpha}$ to verify that

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I_4 \qquad \text{and} \qquad (\gamma^\mu)^\dagger = \begin{cases} \gamma^0 & \text{if} \quad \mu = 0 \\ -\gamma^j & \text{if} \quad \mu = j \end{cases} \qquad (j = 1,2,3) \ .$$

(iii) Derive from this that

$$(\gamma^0)^2 = I_4 \qquad \text{and} \qquad (\gamma^j)^2 = -I_4 \qquad (j = 1,2,3) \ ,$$

as well as

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0 \qquad \text{and} \qquad \text{Tr}(\gamma^\mu) = 0 \ .$$

(iv) In order to keep the expressions in spinor space as compact as possible, the so-called Feynman slash notation is used:

$$\not{a} \equiv \gamma^\mu a_\mu = \gamma_\mu a^\mu \ ,$$

where a^μ is an arbitrary 4-vector. Use part (ii) to prove that

$$\not{a}^2 = a^2 I_4 \ .$$

(v) Employ the latter identity to find an operator that applied to the Dirac operator $(i\hbar\not{\partial} - mc)$ produces the Klein–Gordon operator $(\Box + m^2c^2/\hbar^2)$.