Exercises for Quantum Mechanics 3
Set 12 (module 1)

Exercise 24: Lorentz transformations in the free Dirac theory

Determining the infinitesimal form of the spinor transformation matrix $S(\Lambda)$

Consider the Dirac equation for free spin-1/2 particles. In the associated spinor space, the infinitesimal Lorentz transformation $\Lambda^{\mu}{}_{\nu} \approx g^{\mu}{}_{\nu} + \delta\omega^{\mu}{}_{\nu}$ is characterized by the transformation matrix

$$S(\Lambda) \;\approx\; I_4 + s(\delta\omega) \; .$$

As derived in the lecture notes, $s(\delta\omega)$ is required to satisfy the conditions

$$\big[\gamma^{\rho}, s(\delta\omega)\big] \;=\; \delta\omega^{\rho}{}_{\nu}\gamma^{\nu} \qquad (\rho = 0,1,2,3) \; , \tag{1}$$

$$s(-\delta\omega) \;=\; -\,s(\delta\omega) \; . \tag{2}$$

In order to determine the precise form of the 4×4 matrix $s(\delta\omega)$, we use the decomposition

$$s(\delta\omega) \;\equiv\; c I_4 \,+\, c_{\alpha}\gamma^{\alpha} \,+\, c_{\alpha\beta}\sigma^{\alpha\beta} \,+\, \bar{c}_{\alpha}\gamma^{\alpha}\gamma^5 \,+\, \bar{c}\gamma^5$$

in terms of the basis (283) of 4×4 matrices, with the coefficients $c, c_{\alpha}, c_{\alpha\beta}, \bar{c}_{\alpha}$ and $\bar{c}$ being complex numbers.

(i) Use the defining relation $\big\{\gamma^{\mu}, \gamma^{\nu}\big\} = 2g^{\mu\nu} I_4$ for the γ-matrices to derive that

$$\gamma^{\mu}\gamma^{\nu} \;=\; \begin{cases} -\,\gamma^{\nu}\gamma^{\mu} & \text{if } \mu \neq \nu \\ \gamma^{\nu}\gamma^{\mu} \;=\; g^{\mu\nu} I_4 & \text{if } \mu = \nu \end{cases} \;\Rightarrow\; \begin{cases} \text{vanishing anticommutators} \\ \text{vanishing commutators} \end{cases} ,$$

$$\gamma^{\mu}\gamma^5 \;=\; -\,\gamma^5\gamma^{\mu} \qquad \text{for} \qquad \gamma^5 \;\equiv\; i\gamma^0\gamma^1\gamma^2\gamma^3$$

and

$$\sigma^{\mu\nu} \;\equiv\; \frac{i}{2}\big[\gamma^{\mu}, \gamma^{\nu}\big] \;=\; \begin{cases} i\gamma^{\mu}\gamma^{\nu} & \text{if } \mu \neq \nu \\ 0 & \text{if } \mu = \nu \end{cases} .$$

Consequence of these identities: upon multiplication by a particular γ-matrix the basis matrices change one-to-one into a specific permutation of the basis matrices (up to constant factors), with the number of γ-matrices raised/lowered by one if the particular γ-matrix was not/already present in the basis matrix. As a result, both left- and right-hand side of equation (1) involve basis matrices, making it possible to equate the corresponding coefficients.

(ii) Consider the basis matrices γ^α, $\gamma^\alpha\gamma^5$ and γ^5.

- Find for each of these basis matrices a value for ρ such that the left-hand side of equation (1) does not vanish.
- Argue that this implies that c_α, $\bar{c}_\alpha$ and $\bar{c}$ should all vanish.

(iii) Use the commutator identity $\left[\gamma^\rho, \gamma^\alpha\gamma^\beta\right] = 2g^{\rho\alpha}\gamma^\beta - 2g^{\rho\beta}\gamma^\alpha$ to prove that

$$c_{\alpha\beta} \;=\; -\,i\,\delta\omega_{\alpha\beta}/4\ ,$$

bearing in mind that both $c_{\alpha\beta}$ and $\delta\omega_{\alpha\beta}$ are antisymmetric in their indices.

(iv) Employ condition (2) to eliminate the term proportional to the identity matrix.

Exercise 25: Rotations in spinor space

Dirac spin operator = generator of rotations in spinor space

Consider the Dirac equation for free spin-1/2 particles and a spatial rotation by an infinitesimal angle $\delta\alpha$ about the axis oriented along the unit vector $\vec{e}_n$. Under this infinitesimal spatial rotation the spatial components of the position 4-vector x^μ transform according to

$$\vec{x} \;\;\to\;\; \vec{x}\,' \;=\; \vec{x} + \delta\alpha\,(\vec{e}_n\times\vec{x})\ .$$

(i) Rewrite this infinitesimal transformation in the form $x'^\mu = \Lambda^\mu_{\ \nu}\,x^\nu \approx x^\mu + \delta\omega^\mu_{\ \nu}\,x^\nu$ and demonstrate that

$$\delta\omega^\mu_{\ \nu} \;=\; \begin{cases} 0 & \text{if } \mu=0 \,\vee\, \nu=0 \\[2ex] -\,\delta\alpha \displaystyle\sum_{l=1}^{3}\epsilon^{jkl}\,e^l_n & \text{if } \mu=j \,\wedge\, \nu=k \end{cases} \qquad (j,k=1,2,3)\ ,$$

in terms of the completely antisymmetric coefficient ϵ^{jkl} that is defined in equation (274) of the lecture notes.

Hint: you may use the (very handy) identity $(\vec{e}_n\times\vec{x})^j = \sum_{k,l=1}^{3}\epsilon^{jlk}\,e^l_n\,x^k$.

(ii) Derive from this that in the Dirac representation the corresponding infinitesimal transformation characteristic of the Dirac spinors is given by

$$\psi'(x') \;=\; \Big(I_4 - \frac{i}{\hbar}\,\delta\alpha\sum_{l=1}^{3}e^l_n\,\hat{S}^l\Big)\,\psi(x)\ , \qquad \text{with} \quad \hat{S}^l \;=\; \frac{\hbar}{2}\begin{pmatrix}\sigma^l & \emptyset\\ \emptyset & \sigma^l\end{pmatrix}.$$

Hint: use equations (291) and (284) of the lecture notes, as well as $\sum_{j,k=1}^{3}\epsilon^{jkl}\epsilon^{jkl'} = 2\delta^{ll'}$.

(iii) The transformation characteristic for rotations by a finite angle α then becomes

$$\psi'(x') \;=\; \exp\bigl(-i\alpha\,\vec{e}_n \cdot \hat{\vec{S}}/\hbar\bigr)\,\psi(x) \;.$$

Take the rotation angle to be $\alpha = 2\pi$ and show that the spinor transformation matrix $S(\Lambda)$ equals $-I_4$ in that case. (Does this result look familiar to you?)
Hint: expand the exponent by means of the identity $(\vec{\sigma}\cdot\vec{e}_n)^2 = I_2$ and use that

$$\cos(z) \;=\; \sum_{k=0}^{\infty} (-1)^k\, z^{2k}/(2k)! \quad , \quad \sin(z) \;=\; \sum_{k=0}^{\infty} (-1)^k\, z^{2k+1}/(2k{+}1)! \;.$$

<u>Exercise 26</u>: Mutually commuting observables in the free Dirac theory

Free Dirac particles can be described by their energy, momentum and helicity

Consider the Dirac theory for free spin-1/2 particles, with Hamilton operator

$$\hat{H} \;=\; c\vec{\alpha}\cdot\hat{\vec{p}} \,+\, \beta mc^2 \;.$$

The spin operator $\hat{\vec{S}}$ for Dirac spinors satisfies the commutation relations

$$\bigl[\beta, \hat{S}^j\bigr] \;=\; 0 \qquad \text{and} \qquad \bigl[\alpha^k, \hat{S}^j\bigr] \;=\; i\hbar \sum_{n=1}^{3} \epsilon^{kjn}\,\alpha^n \qquad (j,k = 1,2,3)\;,$$

with ϵ^{kjn} as defined in equation (274) of the lecture notes.
Prove that $\hat{H}$, $\hat{\vec{p}}$ and $\hat{\vec{p}}\cdot\hat{\vec{S}}$ are mutually commuting operators.